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Online and size anti-Ramsey numbers

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A graph is properly edge-colored if no two adjacent edges have the same color. The smallest number of edges in a graph, any of whose proper edge coloring contains a totally multicolored copy of a graph H, is the **size anti-Ramsey number** $AR_s(H)$ of H. This number in offline and online setting is investigated here.

1. Introduction

In the coloring problems of this paper, we assume that the edges of graphs are colored with natural numbers. A graph is *rainbow* or *totally multicolored* if all its edges have distinct colors. For a graph H, let $AR(n, H)$, the **classical anti-Ramsey number**, be the largest number of colors used on the edges of a complete n-vertex graph without containing a rainbow copy of H. This function was introduced by Erdős, Simonovits and Sós [14] and it is shown to be finite. In particular, having sufficiently many colors on the edges of a complete graph forces the existence of a rainbow copy of H and this number of colors is closely related to the Turán, or extremal, number for H.

An important and natural class of colorings are the proper colorings. A *proper coloring* of a graph assigns distinct colors to adjacent edges. Here, we are concerned with the most efficient way to **force a rainbow copy** of H in properly colored graphs. It is clear that to force a rainbow copy of a graph one does not necessarily need to color a complete graph. For that, we investigate the smallest number of edges in a graph such that any proper coloring of that graphs contains a rainbow copy of H. We introduce a size and online version of anti-Ramsey numbers for graphs, denoted AR_s and AR_o, respectively.

Note that these very natural variants were studied for Ramsey numbers, see [10, 12, 13, 22], to name a few. There are interesting connections between classical, size and online Ramsey numbers. However, in the anti-Ramsey

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setting, these functions have not yet been considered. This paper initiates this study.

Formally, we define the **size anti-Ramsey number**, denoted by $AR_s(H)$, to be the smallest number of edges in a graph G such that any of its proper edge-colorings contains a rainbow copy of H. We shall write $G \to H$ if every proper edge-coloring of G contains a rainbow copy of H.

The function $AR_o(H)$, the **online anti-Ramsey number**, is defined in the following game setting. There are two players, Builder and Painter. Builder creates vertices and edges, one edge at a time. After each new edge has been built by Builder, Painter assigns a color to it such that no two adjacent edges have the same color. Both Builder and Painter have complete information, that is, at all times the so-far built graph and its coloring are known to both players. Builder wins after t steps if after the t-th edge has been built and colored there is a rainbow copy of a graph H. We shall show below that Builder can always win and we will denote the smallest such t by $AR_o(H)$.

In studying size and online anti-Ramsey numbers, we make use of two additional functions: AR_{FF} and AR_{loc}. Here, $AR_{FF}(H)$ is the **first-fit anti-Ramsey number** and it is defined as $AR_o(H)$ with the exception that Painter must follow the first-fit strategy, i.e., Painter must use the smallest feasible color. In addition, $AR_{loc}(H)$ is the **local anti-Ramsey number**, the smallest n such that any proper edge coloring of the complete graph K_n on n vertices contains a rainbow copy of H, see for example Babai [5], Alon, Lefmann, and Rödl [2].

We observe an easy chain of inequalities for graphs H without isolated vertices, which we prove formally in Theorem 1:

$$\frac{AR_{loc}(H)^{1/3}}{2} \leq AR_{FF}(H) \leq AR_o(H) \leq AR_s(H) \leq \binom{AR_{loc}(H)}{2}. \tag{1}$$

For the above anti-Ramsey numbers, we consider only simple graphs unless stated differently. Otherwise, let $AR_s^*(H)$ be the smallest number of edges in a *not necessarily simple* graph G with $G \to H$, that is, every proper edge-coloring of G contains a rainbow copy of H. Clearly, we have $AR_s^*(H) \leq AR_s(H)$ for every graph H. Indeed, we can prove slightly better upper bounds when graphs with multiple edges are allowed.

Our Results We provide bounds on $AR_s(H), AR_o(H)$ and $AR_{FF}(H)$ for general graphs H, as well as for some specific graphs including K_n, the path P_k on k edges, the cycle C_k on k edges, and the matching M_k on k edges.

Table 1: Overview of online, first-fit and size anti-Ramsey numbers for paths, cycles, matchings, and complete graphs. Upper bounds marked with * are derived from graphs G with multiple edges that force rainbow H. Here c denotes an absolute positive constant, τ the vertex cover number and Δ the maximum degree of H

H	$AR_{FF}(H)$	$AR_o(H)$	$AR_s(H)$
n-vertex k-edge graph	$\leq (\tau+1)\cdot k$ $\geq \frac{k^2}{4(\Delta+1)}$	$\leq k^2$	$\leq 8k^4/n^2$ $\leq \frac{k(k+1)}{2}^*$ $\geq \frac{(k-1)^2}{2\Delta}$
P_k	$\frac{k^2}{8}(1+o(1))$	$\leq \frac{k^2}{2}$	$\leq \frac{8}{9}k^2$ [17] $\leq \frac{k^2}{2}^*$ $\geq \frac{k(k+1)}{4}$
C_k	$\frac{k^2}{8}(1+o(1))$	$\leq \frac{k^2}{2}$	$\leq k^2$ $\leq \frac{k^2}{2}^*$ $\geq \frac{k(k-1)}{4}$
M_k	$\left\lceil\frac{k}{2}\right\rceil \cdot \left\lfloor\frac{k}{2}\right\rfloor + 1$	$\leq \frac{k^2}{3}(1+o(1))^*$	$\frac{k(k+1)}{2}$
K_n	$\frac{n^3}{6}(1+o(1))$	$\leq \frac{n^4}{4}$	$\leq cn^6/\log^2 n$ [1] $\geq cn^5/\log n$

For the terminology used in this paper we refer to the standard textbook of West [31].

We summarize our main results in Table 1 and provide the corresponding theorems in respective sections.

Related Work The anti-Ramsey numbers and their variations were studied in a number of papers, see [16] for a survey. The problem of finding rainbow subgraphs in properly-colored graphs was originally motivated by the transversals of Latin squares, see [29]. The largest number of edges in a graph on n vertices that has a proper coloring without rainbow H is referred to as *rainbow Turán number* and denoted $\mathrm{ex}^*(n,H)$. It was studied by Das, Lee and Sudakov [11] and Keevash, Mubayi, Sudakov and Verstraëte [21]. Clearly, we have $AR_s(H) \leq \mathrm{ex}^*(n,H)+1$ if n is big enough since if an n-vertex graph G has $\mathrm{ex}^*(n,H)+1$ edges, then *any* of its proper edge-colorings will contain a rainbow copy of H. However, $AR_s(H)$ can be arbitrarily smaller than $\mathrm{ex}^*(n,H)$ when n is not fixed. To see this, consider $H = M_2$, a matching with two edges. Any star does not contain M_2, so

$\mathrm{ex}^*(n, H) \geq n - 1$, however, $AR_s(H) = 3$ as we shall see in Section 3. So, these two functions are completely different in nature.

Finding rainbow copies of a given graph in a not-necessarily properly colored graph has been investigated as well. The conditions on the colorings that force rainbow copies include large total number of colors, large number of colors incident to each vertex, small number of edges in each color class, and bounded number of edges in each monochromatic star, see [1, 4, 8, 14], to name a few. Among some, perhaps unexpected, results on anti-Ramsey type numbers is the fact that rainbow cycles can be forced in properly colored graphs of arbitrarily high girth, see [28], and that rainbow subgraphs in edge-colored graphs are related to matroid theory, see [9, 30].

In this work, we are concerned with size and online anti-Ramsey numbers, not studied before. In comparison, the online Ramsey number has been introduced by Beck [7] in 1993 and independently by Friedgut *et al.* [15] in 2003 and Kurek and Ruciński [24] in 2005. Here, Builder presents some graph one edge at a time, and Painter colors each edge as it appears either red or blue. The goal of Builder is to force a monochromatic copy of a fixed target graph H and the goal of Painter is to avoid this as long as possible. The online Ramsey number of H is then the minimum number of edges Builder has to present to force the desired monochromatic copy of H, no matter how Painter chooses to color the edges, see [10, 18, 19, 27]. The size Ramsey number for a graph H is the smallest number of edges in a graph G such that any 2-coloring of the edges of G results in a monochromatic copy of H, see [6, 13]. One of the fundamental questions studied in size Ramsey and online Ramsey theory is how these numbers relate to each other and to the classical Ramsey numbers and how they are expressed in terms of certain graph parameters.

Here, we initiate the study of size and online anti-Ramsey numbers answering similar questions: *"How do the various anti-Ramsey numbers relate to each other?"*, *"What are the specific values for classes of graphs?"*, *"What graph parameters play a determining role in studying these anti-Ramsey type functions?"*

Organization of the Paper

- In Section 2, we prove the inequalities (1). We also investigate the relation between the local anti-Ramsey number AR_{loc} and the size anti-Ramsey number AR_s. By recalling known upper bounds for AR_{loc} we obtain some upper bounds for AR_s using (1).

- In Section 3, we only consider the size anti-Ramsey numbers. We prove upper and lower bounds on $AR_s(K_n)$, and provide general bounds on $AR_s(H)$ in terms of the chromatic index $\chi'(H)$ and the vertex cover number $\tau(H)$. We give upper and lower bounds for $AR_s(P_k)$ and $AR_s(C_k)$ and determine $AR_s(M_k)$ exactly for every k. Moreover, we show how rainbow copies can be forced using fewer edges if multiple edges are allowed.
- In Section 4, we consider the first-fit anti-Ramsey number AR_{FF}. We show that it is determined up to a factor of 4 by a certain matching parameter of H and prove asymptotically tight bounds for paths, cycles, matchings and complete graphs.
- In Section 5, we consider the online anti-Ramsey number AR_o. We present a strategy for Builder that gives a general upper bound on $AR_o(H)$ in terms of $|E(H)|$. We also show how to improve this bound in case when H is a matching.

We drop floors and ceilings when appropriate not to congest the presentation. We also assume, without loss of generality, that in all online settings the graph H that is to be forced has no isolated vertices.

2. Comparison of anti-Ramsey numbers

In this section, we investigate the relation between the four concepts of anti-Ramsey numbers: first-fit anti-Ramsey number AR_{FF}, online anti-Ramsey number AR_o, size anti-Ramsey number AR_s and local anti-Ramsey number AR_{loc}. We present several immediate inequalities that are valid for all (non-degenerate) graphs H, and conclude upper bounds on the new anti-Ramsey numbers for complete graphs, general graphs, paths, cycles and matchings.

Theorem 1. *For any graph H without isolated vertices we have*

- $\frac{AR_{loc}(H)^{1/3}}{2} \leq AR_{FF}(H) \leq AR_o(H) \leq AR_s(H) \leq \binom{AR_{loc}(H)}{2}$
- $AR_s(H) \geq \frac{1}{2} AR_{loc}(H)\delta(H)$,

where $\delta(H)$ denotes the minimum degree of H.

Proof. For the first inequality, note that $\delta(H) \geq 1$ implies $|E(H)| \geq \frac{1}{2}|V(H)|$. Then clearly $AR_{FF}(H) \geq |E(H)| \geq |V(H)|/2$. On the other hand, Alon *et al.* [1] proved that $AR_{loc}(H) \leq |V(H)|^3$, which proves the claimed inequality. Let us remark that taking the better upper bound $AR_{loc}(K_n) \leq cn^3/\log(n)$ from [1] we can obtain the slightly better bound $AR_{FF}(H) \geq C(AR_{loc}(H) \log AR_{loc}(H))^{1/3}$ for some constant $C > 0$.

To see the second inequality, observe that Builder has an advantage if she knows the strategy of Painter. More precisely, if for *every* strategy of Painter Builder can create a graph G and an order of the edges of G that forces a rainbow H, in particular she can do so for first-fit. Hence $AR_{FF}(H) \leq AR_o(H)$.

For the third inequality, consider G on $AR_s(H)$ edges such that $G \to H$. Clearly, if Builder exposes the edges of G in any order, then every coloring strategy of Painter will produce a proper coloring of G and thus a rainbow H. In other words, $AR_o(H) \leq AR_s(H)$.

For the last inequality, recall that $AR_s(H)$ is the minimum number of edges in any graph G with $G \to H$. Since G could always be taken as a complete graph, $AR_s(H) \leq \binom{AR_{loc}(H)}{2}$.

For the inequality in the second item, let $G \to H$ and G have the smallest number of edges with that property. Then $\delta(G) \geq \delta(H)$. Otherwise, if v were a vertex of G with degree less than $\delta(H)$, coloring $G - v$ properly without a rainbow copy of H and coloring the edges incident to v arbitrarily and properly gives a coloring of G without a rainbow copy of H. Observe also that $|V(G)| \geq AR_{loc}(H)$ because otherwise there is a complete graph on $AR_{loc}(H) - 1$ vertices all of whose proper colorings contain a rainbow H. Together with the lower bound on the degree this yields the claim. □

Remark. All of the inequalities in Theorem 1 except for the very first one are also valid if H contains isolated vertices. The first inequality is not valid in this case because $AR_{FF}(H)$ measures the number of edges in a graph G containing rainbow H and $AR_{loc}(H)$ measures the number of vertices in G containing rainbow H. So, when $H = I_n$ is an n-element independent set, then clearly $AR_{FF}(I_n) = 0$ while $AR_{loc}(I_n) = n$.

The following proposition shows that $AR_s(H)$ can differ significantly from $\binom{AR_{loc}(H)}{2}$.

Proposition 2.

$$AR_s(K_4) = 15 \quad \text{and} \quad \binom{AR_{loc}(K_4)}{2} = \binom{7}{2} = 21.$$

Proof. To see that $AR_{loc}(K_4) > 6$, observe that K_6 can be properly colored with 5 colors, but a rainbow K_4 requires 6 colors. Let G be a graph on 7 vertices with vertex set $X \cup Y$, X induces a triangle, Y induces an independent set on 4 vertices and (X, Y) is a complete bipartite graph. Consider an arbitrary proper coloring of G and assume that X induces colors $1, 2, 3$. Since there are at most three edges of colors from $\{1, 2, 3\}$ between X and

Y, there is a vertex $v \in Y$ sending 3 edges of colors different from $1, 2$, and 3 to X. Thus $\{v\} \cup X$ induces a rainbow K_4. So, we have that $G \to K_4$ and thus $AR_{loc}(K_4) = 7$ and $AR_s(K_4) \leq 15$.

To see that any graph with at most 14 edges can be properly colored without a rainbow K_4, we use induction on the number of vertices and edges of G with the basis $|V(G)| = 6$ shown above. Now, we assume that G has at least 7 vertices and any of its proper subgraphs can be properly colored without containing a rainbow K_4, however G itself can not be properly colored avoiding a rainbow K_4. It is easy to see that G has minimum degree at least 3 and each edge belongs to at least 2 copies of K_4. Consider the copies of K_4 in G. Assume first that there are two such copies sharing three vertices. Then together they span 9 edges and 5 vertices. Thus, the remaining set of at least 2 vertices is incident to at most 5 edges. Since these vertices are of degree at least 3, there must be exactly 2 of them, they both must be of degree exactly 3, and they must be adjacent. Thus, they belong to at most one copy of K_4, a contradiction. Thus, any 2 copies of K_4 share at most two vertices. Let K be the graph whose vertices are the copies of K_4 in G and 2 vertices of K are adjacent if and only if the corresponding copies share an edge. There is a component of K with at least 3 vertices, otherwise we can easily color G avoiding rainbow K_4. This component corresponds to at least $6 + 5 + 4 = 15$ edges in G, a contradiction. □

3. Size anti-Ramsey numbers

The inequalities in Theorem 1, together with known upper bounds on the local anti-Ramsey number AR_{loc}, immediately give upper bounds on AR_s, AR_o and AR_{FF}. We shall first collect some known bounds on $AR_{loc}(H)$ for several graphs H, and then conclude corresponding upper bounds on $AR_s(H)$ in Corollary 3 below.

Alon *et al.* [1] prove that there are absolute constants $c_1, c_2 > 0$, such that

$$c_1 n^3 / \log n \leq AR_{loc}(K_n) \leq c_2 n^3 / \log n. \tag{2}$$

See also earlier papers of Alon, Lefmann, Rödl [2] and Babai [5]. In the same paper, [1], the authors prove that for any graph H on n vertices and $k \geq 2$ edges

$$k - 1 \leq AR_{loc}(H) \quad \text{and} \quad \frac{ck^2}{n \log n} \leq AR_{loc}(H) \leq 4k^2/n \quad \text{for some } c > 0.$$

Here, the upper bound is obtained by a greedy counting and the lower bound is given by a probabilistic argument.

The local anti-Ramsey number for the path P_k on k edges was investigated by Gyárfás *et al.* [20] and was conjectured to be equal to $k+c$ for a small constant c. Indeed, it seems to be widely believed that it is $k+2$ (see e.g. [17]), which would be best-possible since Maamoun and Meyniel [26] show that for n being a power of 2 there are proper colorings of K_n admitting no Hamiltonian rainbow path. Frank Mousset gave the best currently known upper bound in his bachelor's thesis [17], so that together we have $k+2 \le AR_{loc}(P_k) \le \frac{4}{3}k + o(k)$. Gyárfás *et al.* [20] considered the cycle C_k on k edges and showed that $k+1 \le AR_{loc}(C_k) \le \frac{7}{4}k + o(k)$. For a matching M_k on k edges, Kostochka and Yancey [23], see also Li, Xu [25], proved that $AR_{loc}(M_k) = 2k$.

From these bounds and the inequality $AR_s(H) \le \binom{AR_{loc}(H)}{2}$ for all graphs H from Theorem 1 we immediately get the following.

Corollary 3. *For an n-vertex, k-edge graph H, we have $AR_s(H) \le 8k^4/n^2$. Moreover, $AR_s(K_n) \le cn^6/\log^2 n$, $AR_s(P_k) \le \binom{4k/3}{2} + o(k^2)$, $AR_s(C_k) \le \binom{7k/4}{2} + o(k^2)$, and $AR_s(M_k) \le 2k^2$.*

Next in this section, we improve some of these bounds.

Theorem 4. *There exist positive constants c_1, c_2 such that*

$$c_1 n^5/\log n \le AR_s(K_n) \le c_2 n^6/\log^2 n.$$

Proof. The upper bound follows from Corollary 3. For the lower bound, we shall show that there is a positive constant c such that any graph G on less than $cn^5/\log n$ edges can be properly colored to avoid a rainbow copy of K_n. Let $\ell = \binom{n/2}{2}$, V_1 be the set of vertices of G of degree at most $\ell - 2$ and $V_2 = V(G) \setminus V_1$. We shall properly color the edges induced by V_1, V_2 and the edges between V_1 and V_2 separately. Color $G[V_1]$ properly with at most $\ell - 1$ colors using Vizing's theorem. Since $\ell - 1 < \binom{n/2}{2}$, there is no rainbow $K_{n/2}$ in $G[V_1]$. Because $|E(G)| < cn^5/\log n$ and every vertex in V_2 has degree at least $\ell - 1$ we have $|V_2| \le \frac{2|E(G)|}{\ell-1} < c' \frac{n^3}{\log n} \le c'' \cdot \frac{(n/2)^3}{\log(n/2)}$. Hence the lower bound in (2) guarantees that there is a proper coloring of $G[V_2]$ without rainbow copy of $K_{n/2}$ for an appropriate constant. Now, coloring the edges between V_1 and V_2 arbitrarily so that the resulting coloring is proper, we see that G has no rainbow copy of K_n under the described coloring because this copy would have at least $n/2$ vertices either in V_1 or V_2. However, neither $G[V_1]$ nor $G[V_2]$ contains a rainbow copy of $K_{n/2}$. □

To get general lower bounds on $AR_s(H)$, we can use lower bounds on $AR_{FF}(H)$, which we present in Section 4 below. For some sparse graphs H, such as matchings and paths, this gives the correct order of magnitude. Note that $AR_{FF}(K_{1,k}) = AR_s(K_{1,k}) = k$, so there is no meaningful lower bound just in terms of the number of edges of H. However, the maximum degree could help.

Lemma 5. *Let H be a graph on k edges with maximum degree Δ. Then $AR_s(H) \geq \frac{(k-1)^2}{2\Delta}$. Moreover, if H has chromatic index $\chi'(H) = \Delta$, then $AR_s(H) \geq \frac{k(k+1)}{2\Delta}$.*

Proof. Let G be a graph with the least number of edges such that $G \to H$. In particular, $|E(G)| = AR_s(H)$. We shall find several copies of H in G one after another and argue that as long as we do not reach a certain number of copies we can color their edges properly with fewer than k colors, i.e., without a rainbow copy of H. Note that for graphs with $\chi' = k$ we have $AR_s(H) = k$ and the claimed inequalities hold. Therefore, without loss of generality, we assume that $\chi'(H) = \chi' < k$, which means that H is neither a star nor a triangle.

Let H_0 be any copy of H in G. Edge-color H_0 properly with χ' colors. Because $\chi' < k$, H_0 is not rainbow. Extending this coloring arbitrarily to a proper coloring of G gives some rainbow copy H_1 of H since $G \to H$. Since H_1 is rainbow it shares at most χ' colors with H_0. Thus $H_1 \setminus H_0$ has at least $k - \chi'$ edges. Color $H_0 \cup H_1$ properly with $\chi'(H_0 \cup H_1)$ colors. Generally, for $i > 1$ after having colored $\hat{H}_i = \cup_{0 \leq j < i} H_j$ with $\chi'(\hat{H}_i)$ colors and extending this to a proper coloring of G, there is a rainbow copy H_i of H in G sharing at most $\chi'(\hat{H}_i)$ edges with $\hat{H}_i$ and thus contributing at least $k - \chi'(\hat{H}_i)$ new edges to G. Since for each $i \geq 1$ the graph $\hat{H}_i$ is the union of i copies of H, we have $\Delta(\hat{H}_i) \leq i\Delta$ and hence $\chi'(\hat{H}_i) \leq i\Delta + 1$.

Let ℓ' be the smallest integer for which $\chi'(\hat{H}_{\ell'}) \geq k$. With $\chi'(\hat{H}_{\ell'}) \leq \ell'\Delta + 1$ we obtain $\ell' \geq \ell = \lceil \frac{k-1}{\Delta} \rceil$ and conclude that

$$\begin{aligned} AR_s(H) = |E(G)| \geq |E(\hat{H}_\ell)| \geq |E(H_0)| + \sum_{i=1}^{\ell-1} (k - \chi'(\hat{H}_i)) \\ = \ell k - \sum_{i=1}^{\ell-1} \chi'(\hat{H}_i) \geq \ell k - \sum_{i=1}^{\ell-1} (i\Delta + 1) \\ = \ell k - \Delta \binom{\ell}{2} - (\ell - 1) > \ell \left(k - \frac{\Delta}{2}(\ell - 1) - 1 \right) \geq \frac{(k-1)^2}{2\Delta}. \end{aligned}$$

This proves the first part of the statement. Now if $\chi'(H) = \Delta$, then $\chi'(\hat{H}_i) \leq i\chi'(H) = i\Delta$ and thus $\ell' \geq \ell = \lceil \frac{k}{\Delta} \rceil$, which gives in a similar calculation that $AR_s(H) \geq k(k+1)/2\Delta$. □

Next we derive several upper bounds on $AR_s(H)$ for some specific graphs H, namely paths, cycles and matchings. From Corollary 3 we obtain that $AR_s(P_k) \leq \binom{4k/3}{2} + o(k^2) = \frac{8}{9}k^2 + o(k^2)$, which simply follows from the best-known bound on $AR_{loc}(P_k)$. Apparently, this remains the best-known upper bound. But we can improve the bound $AR_s(C_k) \leq \binom{7k/4}{2} + o(k^2) = \frac{49}{32}k^2 + o(k^2)$ from Corollary 3.

Theorem 6. *For every $k \geq 2$, $\frac{k(k-1)}{4} \leq AR_s(P_{k-1}) \leq AR_s(C_k) \leq k^2$, and $AR_s(M_k) = \binom{k+1}{2}$.*

Proof. The lower bound on $AR_s(P_{k-1})$ follows directly from Lemma 5. Since P_{k-1} is a subgraph of C_k we have that $AR_s(P_{k-1}) \leq AR_s(C_k)$.

For the upper bound $AR_s(C_k) \leq k^2$ we first consider the case that k is even. We shall construct G such that $G \to C_k$. Let $V(G) = \{v_1, \ldots, v_{k/2}\} \cup W_1 \cup \cdots \cup W_{k/2}$, where the union is disjoint and $|W_i| = 4(i-1)$, for $i = 2, \ldots, k/2$ and $|W_1| = 1$. Let the edge set of G consist of the union of complete bipartite graphs with parts $\{v_i, v_{i+1}\}, W_i$, for $i = 1, \ldots, k/2 - 1$ and $\{v_1, v_{k/2}\}, W_{k/2}$. We refer to Figure 1 for an illustration of the graph G.

We shall identify a rainbow C_k greedily by picking a rainbow path $R_1 = v_1, w_1, v_2,\ w_1 \in W_1$, then a rainbow path $R_2 = v_2, w_2, v_3,\ w_2 \in W_2$, such that the colors of R_2 and R_1 are disjoint and so on. Then $|E(G)| = 2 + \sum_{i=2}^{k/2} 8(i-1) \leq k^2 - 2k + 2$.

This concludes the proof when k is even. Contracting one of the two edges in the only 2-path between v_0 and v_2, we obtain a graph G' with

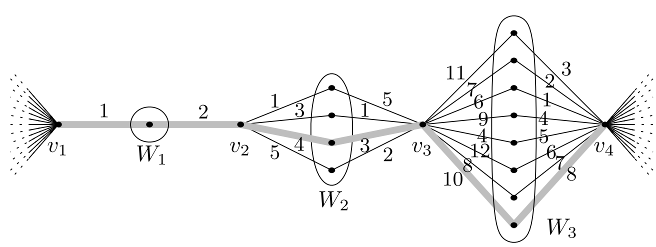

Figure 1: The portion of the graph G with $G \to C_k$ consisting of the first three complete bipartite subgraphs and a proper coloring of it. The rainbow paths R_1, R_2 and R_3 are highlighted.

$G' \to C_{k-1}$, and with at most $k^2 - 2k + 1 = (k-1)^2$ edges. This completes the proof for odd cycles.

To show the upper bound for matchings, consider the graph G_k that is the vertex-disjoint union of k stars on $1, 2, \ldots, k$ edges respectively. It is clear that any proper coloring of G_k contains a rainbow matching constructed by greedily picking an available edge from the stars starting with the smallest star and ending with the star of size k. The lower bound follows from Lemma 5. □

Theorem 7. *For any graph H with maximum degree Δ and k edges we have $AR^*_s(H) \le \binom{k+1}{2} - \binom{\Delta}{2}$. Moreover, $AR^*_s(P_k) \le \binom{k}{2} + 1$, $AR^*_s(C_k) \le \binom{k}{2}$, and $AR^*_s(M_k) = \binom{k+1}{2}$.*

Proof. Let H be any graph with k edges. We label the edges of H by $e_1, \ldots, e_k$, where the first Δ edges are incident to a fixed vertex of maximum degree Δ. Now define G by leaving the first Δ edges unchanged and replacing edge e_i by i parallel edges for all $\Delta + 1 \le i \le k$. We have $|E(G)| = \binom{k+1}{2} - \binom{\Delta}{2}$. Since the first Δ edges have to be rainbow and for the remaining edges there always has to be one with a new color we get $G \to H$.

For the path P_k we take a $P_k = (e_1, \ldots, e_k)$ and define G by replacing edge e_i by $i-1$ parallel edges for $i \ge 2$. The edge e_1 remains as it is. Again, it is straightforward to argue that every proper edge coloring of G contains a rainbow P_k. Additionally, every copy of P_k in G has the same start and end vertices.

For the cycle C_k we first take the graph G' on $\binom{k-1}{2} + 1$ edges that forces a rainbow P_{k-1} with end vertices u and v. Now a graph G is obtained from G' by introducing $k-2$ parallel edges between u and v. Then $|E(G)| = \binom{k-1}{2} + 1 + k - 2 = \binom{k}{2}$ and $G \to C_k$.

Finally, observe that the lower bound on $AR_s(H)$ in Lemma 5 is also a valid lower bound on $AR^*_s(H)$. Together with the first part we conclude that $AR^*_s(M_k) = \binom{k+1}{2}$. □

Next, we shall give a general upper bound on $AR_s(H)$ in terms of the size $\tau(H)$ of a smallest vertex cover. We need the following special graph $K(s,t)$ which is the union of a complete bipartite graph with parts S and T of sizes s and t, respectively, and a complete graph on vertex set S.

Theorem 8. *Let H be an n-vertex graph with a vertex cover of size $\tau = \tau(H)$. Then there exists an absolute constant $c > 0$ such that*

$$AR_s(H) \le AR_s(K(\tau, n-\tau)) \le cn\tau^5/\log\tau + c\tau^6/\log^2\tau.$$

Proof. Observe that $H \subseteq K(\tau, n-\tau)$. Let $G = K(c_2\tau^3/\log\tau,\ n\tau^2+1)$ be properly colored, where c_2 is from (2). We shall show that there is a rainbow copy of $K(\tau, n-\tau)$ in G and thus there is a rainbow copy of H in G.

First, by (2), there is subset S' of S with τ vertices inducing a rainbow clique. Let C be the set of colors used on $G[S']$. Consider the set E' of edges between S' and T having a color from C. Then $|E'| \leq |C| \cdot \tau = \binom{\tau}{2} \cdot \tau \leq \tau^3$. Let $T' \subseteq T$ be the set of vertices not incident to the edges in E'. So, we have that $|T'| \geq |T| - \tau^3 \geq n\tau^2 + 1 - \tau^3$. The set of colors induced by S' and the set of colors on the edges between S' and T' are disjoint.

We shall find vertices $v_1, \dots, v_{n-\tau}$ from the sets $T' = T_1 \supseteq T_2 \supseteq \cdots \supseteq T_{n-\tau}$, respectively, such that the colors used on the edges between $v_1, \dots, v_{i-1}$ and S' are not present on the edges between T_i and S', $i = 2, \dots, n-\tau$. This would imply that the complete bipartite graph with parts S' and $\{v_1, \dots, v_{n-\tau}\}$ is rainbow.

Let $v_1 \in T_1 = T'$. After $v_1, \dots, v_{i-1}$ has been selected from $T_1, \dots, T_{i-1}$ respectively, consider the colors used between v_{i-1} and S'. There are at most τ^2 edges of such colors between S' and T_{i-1}. Delete the endpoints of these edges from T_{i-1} to obtain T_i and choose the next vertex $v_i \in T_i$. Since $|T_i| \geq |T_{i-1}| - \tau^2$, we see that $|T_{n-\tau}| \geq |T'| - \tau^2(n-\tau-1) \geq (n\tau^2 - \tau^3 + 1) - n\tau^2 + \tau^3 + \tau^2 \geq 1$. Thus we can choose $v_1, \dots, v_{n-\tau}$ as desired. The set $S' \cup \{v_1, \dots, v_{n-\tau}\}$ induces a rainbow copy of $K(\tau, n-\tau)$. Since $|E(G)| = \binom{c_2\tau^3/\log\tau}{2} + (c_2\tau^3/\log\tau)(n\tau^2+1) \leq cn\tau^5/\log\tau + c\tau^6/\log^2\tau$ for a constant c, the theorem follows. □

4. First-fit anti-Ramsey numbers

Definition 9. Let G and H be two graphs. A proper edge-coloring $c : E(G) \to \mathbb{N}$ of G is called H*-good* if

(P1) G contains a rainbow copy of H and
(P2) for every edge e of color $i \geq 2$ there are edges $e_1, \dots, e_{i-1}$ incident to e such that $c(e_j) = j$ for $j = 1, \dots, i-1$.

A graph G is called H*-good* if there exists an H-good coloring of G.

Lemma 10 (First-fit Lemma)**.** *Let H be a graph. Then $AR_{FF}(H)$ equals the minimum number of edges in an H-good graph.*

Proof. Clearly, if Painter uses first-fit then at each state of the game the coloring of the current graph is proper and satisfies (P2). Hence if Painter uses first-fit and the game ends then Builder has presented an H-good graph.

On the other hand, let G be an H-good graph and c be an H-good coloring of G. Let Builder present the edges in the order of non-decreasing colors in c. We claim that if Painter uses first-fit she produces the coloring c. This is true for the edges of color 1, which come first in the ordering. Now when an edge e is presented with $c(e) = i \geq 2$, we know that the colors of all previously presented edges coincide with their colors in c. By (P2), there are edges $e_1, \ldots, e_{i-1}$ incident to e that were presented before e and thus got their corresponding colors $1, \ldots, i-1$. Thus e must get color i in the first-fit setting. □

First-fit and weighted matchings

The first-fit anti-Ramsey number of any graph H is very closely related to the maximum weight of a matching in rainbow colorings of H. More precisely, we define the **weight of a matching** M with respect to an edge-coloring c as the sum of colors of edges in M, that is, as $\sum_{e \in M} c(e)$. A **maximum matching** for c is a matching M with largest weight with respect to c, and a **greedy matching** for c is a matching M that is constructed by greedily taking an edge with the highest color that is not adjacent to any edge already picked. It is well-known that for any c, any greedy matching for c has weight at least half the weight of a maximum matching for c [3]. However, we will not use this fact.

Note that if c is a rainbow coloring of H, then the greedy matching for c is unique. For a given graph H we define

$$w_{\max}(H) = \min_{\text{rainbow colorings } c} \sum_{e \in M} c(e), \quad \text{where } M \text{ is a maximum matching for } c$$

and

$$w_{\text{greedy}}(H) = \min_{\text{rainbow colorings } c} \sum_{e \in M} c(e), \quad \text{where } M \text{ is the greedy matching for } c.$$

Lemma 11. *For any graph H on k edges, chromatic index χ' and vertex cover number τ,*

$$\frac{k^2}{4\chi'} \leq \frac{w_{max}(H)}{2} \leq AR_{FF}(H) \leq 2w_{greedy}(H) \leq (\tau + 1)k.$$

Proof. To prove the lower bound on $AR_{FF}(H)$, it suffices by the first-fit Lemma to find a lower bound on $|E(G)|$ for an H-good graph G. We assume G to be minimal, i.e., $|E(G)| = AR_{FF}(H)$, by the first-fit Lemma. Fix any H-good coloring c of G. There is a rainbow coloring of H and thus a matching M in H with $\sum_{e\in M} c(e) \geq w_{\max}(H)$. By (P2) in Definition 9 each $e \in E(H)$ is incident to at least $c(e)-1$ other edges of G. Since every edge in $E(G) \setminus M$ is adjacent to at most two edges in M we get

(3)

$$AR_{FF}(H) = |E(G)| \geq |M| + \sum_{e\in M} \left\lceil \frac{c(e)-1}{2} \right\rceil \geq \frac{|M|}{2} + \frac{1}{2}\sum_{e\in M} c(e) \geq \frac{w_{\max}(H)}{2}.$$

To prove that $w_{\max}(H) \geq k^2/(2\chi')$, consider any rainbow coloring c of H. The total sum of colors in H is at least $1+\cdots+k = \binom{k+1}{2}$. Since H splits into χ' matchings $M_1,\ldots,M_{\chi'}$, at least one of the matchings has weight at least $\binom{k+1}{2}/\chi'$. Thus, $w_{\max}(H) \geq k^2/(2\chi')$, proving the first inequality of the lemma.

To find the upper bound on $AR_{FF}(H)$, it is sufficient by the first-fit Lemma to construct an H-good graph G with a small number of edges. Start with a rainbow coloring c of H and the greedy matching M for c. Every edge e in H is either in M or incident to an edge $e' \in M$ with $c(e') > c(e)$. We construct a graph G with an H-good coloring by taking H and c and adding pendant edges to endpoints of edges in M as follows. For every $e \in M$, every endpoint v of e and every color ℓ, $\ell < c(e)$ that is not present at v, we add a new pendant edge $e_{\ell,v}$ of color ℓ. Since for every $e \in M$ we added at most $2(c(e)-1)$ new pendant edges, we get

$$AR_{FF}(H) \leq |E(G)| \leq |M| + \sum_{e\in M} 2(c(e)-1) \leq 2\sum_{e\in M} c(e) - |M| \leq 2w_{\text{greedy}}(H).$$

Finally, we find an upper bound on $w_{\text{greedy}}(H)$. Let the degree of an edge in a graph be the number of edges adjacent to e including e itself. Let $V(e)$ be the set of endpoints of e. Order the edges $e_1,\ldots,e_k$ of H as follows: $e_1 = e_{i_1}$ is an edge having highest degree in H, $e_{i_1+1},\ldots,e_{i_2-1}$ are the edges adjacent to e_{i_1}, e_{i_2} is an edge of highest degree in $H - V(e_{i_1})$, $e_{i_2+1},\ldots,e_{i_3-1}$ are the edges adjacent to e_2 in $H - V(e_{i_2})$, e_{i_3} is an edge or the highest degree in $H - (V(e_{i_1}) \cup V(e_{i_2}))$, and so on. Assign the colors $k, k-1, \ldots, 1$ to the edges $e_1, e_2, \ldots, e_k$, respectively. Then $e_{i_1}, e_{i_2}, \ldots, e_{i_m}$ is a greedy matching, with the weight w. Denoting by d_j the degree of e_{i_j} in $H - (V(e_{i_1}) \cup \cdots \cup V(e_{i_{j-1}}))$, $j = 2, \ldots, m$, and d_1 the degree of e_{i_1} in H, we have $w = k + (k - d_1) + (k - d_1 - d_2) + \cdots + (k - d_1 - d_2 - \cdots - d_{m-1})$. Note also that $d_1 \geq d_2 \geq \cdots \geq d_m$ and $d_1 + \cdots + d_m = k$. Thus

$$\begin{aligned} w &= mk - \sum_{i=1}^{m-1}(m-i)d_i = mk - m\sum_{i=1}^{m-1} d_i + \sum_{i=1}^{m-1} id_i \\ &= md_m + \sum_{i=1}^{m-1} id_i = \sum_{i=1}^{m} id_i \leq \left(\sum_{i=1}^{m} i\right)\frac{k}{m} = (m+1)k/2. \end{aligned}$$

Since $m \leq \tau$ we have $w_{\text{greedy}}(H) \leq w \leq (\tau+1)k/2$. □

For each of the four inequalities in Lemma 11, it is easy to construct a graph for which this inequality is tight up to a small additive constant. Moreover, as we shall see later, the inequality $k^2/(4\chi') \leq AR_{FF}(H)$ is also asymptotically tight for some graphs like paths or matchings. However, we suspect that the upper bound $AR_{FF}(H) \leq (\tau+1)k$ is not tight. If $\chi'(H) = \Delta(H)$, for example when H is a forest, then by Lemma 11 we have $k^2/(4\Delta) \leq AR_{FF}(H) \leq (\tau+1)k$. Moreover, for every graph H we have $k/\Delta \leq \tau$. Hence, one might hope that $AR_{FF}(H)$ can be sandwiched between $k^2/(4\Delta)$ and ck^2/Δ or between $c(\tau+1)k$ and $(\tau+1)k$ for some absolute constant $c > 0$. However, this is not the case.

Observation 12. *For any constant $c > 0$ there exists forests H_1, H_2 each with k edges, maximum degree Δ, and vertex cover number τ, such that*

$$AR_{FF}(H_1) \geq c\frac{k^2}{\Delta} \quad \textit{and} \quad AR_{FF}(H_2) \leq c(\tau+1)k.$$

Proof. We consider the graph $H(x, y)$ that is the union of a star on x edges and a matching on y edges for sufficiently large x and y. Then $k = |E(H(x, y))| = x + y$, $\Delta = \Delta(H(x, y)) = x$ and $\tau = \tau(H(x, y)) = y + 1$. It is straightforward to argue that $w_{\max}(H(x, y)) = w_{\text{greedy}}(H(x, y)) = x + y + \sum_{i=1}^{y} i$. Thus, by Lemma 11 we have $x/2 + y(y+3)/4 \leq AR_{FF}(H(x, y)) \leq 2x + y(y+3)$. Let $H_1 = H(x, x)$ with $x \geq 16c$. Then $AR_{FF}(H_1) \geq x^2/4 \geq 4cx = ck^2/\Delta$. Let $H_2 = H(y^2, y)$, with $y \geq 3/c$. Then $AR_{FF}(H_2) \leq 3y(y+1) \leq cy^2(y+1) \leq c(\tau+1)k$. □

Next we consider the first-fit anti-Ramsey numbers of some specific graphs H, such as the path P_k, the matching M_k and the complete graph K_n. From Lemma 11 we get $AR_{FF}(P_k) \leq k^2/2 + k$ and $AR_{FF}(M_k) \leq k^2 + k$. However, in both cases we can do better by a factor of 4 (c.f. Theorem 13 and Theorem 15), which is best-possible.

Theorem 13. *For all $k \geq 1$, $AR_{FF}(P_k) = \frac{k^2}{8}(1 + o(1))$.*

Proof. The lower bound follows directly from Lemma 11. For the upper bound we shall construct a P_k-good graph on at most $\frac{k^2}{8} + \mathcal{O}(k)$ edges. The result then follows from the first-fit Lemma. This graph is defined inductively. In particular, for every $i \geq 1$ we construct a P_i-good graph G_i which contains G_{i-1} as a subgraph. The graph G_1 consists of a single edge and for $i > 1$ the graph G_i is constructed from G_{i-1} by adding three new vertices and at most $\lfloor \frac{i-2}{4} \rfloor + 3$ new edges. Thus

$$AR_{FF}(P_k) \leq |E(G_k)| \leq \sum_{i=1}^{k} \left(\left\lfloor \frac{i-2}{4} \right\rfloor + 3 \right) \leq \frac{1}{4} \binom{k+1}{2} + k \leq \frac{k^2}{8} + 2k.$$

We prove the existence of G_i with the following stronger induction hypothesis.

Claim. *For every $i \geq 1$ there exists a graph G_i on at most $\frac{i^2}{8} + \mathcal{O}(i)$ edges together with a fixed proper coloring c_i using colors $1, \ldots, i$ and containing a rainbow path $Q_i = (v_1, \ldots, v_i)$ with respect to these colors, such that the* reverse $\hat{c}_i$ of c_i, *which is defined as $\hat{c}_i(e) = i+1-c_i(e)$ for every $e \in E(G_i)$, is P_i-good. Moreover, for every edge $e = v_s v_t$ not on Q_i but with both endpoints on Q_i, we have $c_i(e) = s + t$.*

We start by defining G_1 to be just a single edge $e_1 = v_1 v_2$ with $c_1(e_1) = 1$. Having defined G_{i-1} we define the graph G_i as being a supergraph of G_{i-1}. For every edge $e \in E(G_{i-1}) \subset E(G_i)$, we set $c_i(e) = c_{i-1}(e)$, i.e., edges in the subgraph G_{i-1} have the same color in c_i as in c_{i-1}. The vertices in $V(G_i) \setminus V(G_{i-1})$ are denoted by u_i, u_i' and v_{i+1}. The edges in $E(G_i) \setminus E(G_{i-1})$ form a matching and are listed below. Each such edge is assigned color i in c_i.

- The edge $e_i = v_i v_{i+1}$ extends the path Q_{i-1} to the longer path Q_i.
- The edges $v_{2j} v_{i-2j}$ for $j = 1, \ldots, \lfloor \frac{i-2}{4} \rfloor$. These edges do not coincide with any edge on Q_i since their endpoints have distance at least two on Q_i. Moreover, since $c_i(v_{2j} v_{i-2j}) = i = 2j + (i - 2j)$ these edges do not coincide with any edge in $E(G_{i-1}) \setminus E(Q_{i-1})$.
- There are up to two edges e_i', e_i'', each incident to a vertex v_j with j even and $2\lfloor \frac{i-2}{4} \rfloor < j < i - 2\lfloor \frac{i-2}{4} \rfloor$. The other endpoint of e_i' and e_i'' is u_i and u_i', respectively.

See Figure 2 for an illustration of how the graph G_i arises from G_{i-1}. It is easy to see that for $j = 1, \ldots, i$ we have $c_i(e_j) = j$, i.e., that Q_i is a rainbow copy of P_i with respect to c_i.

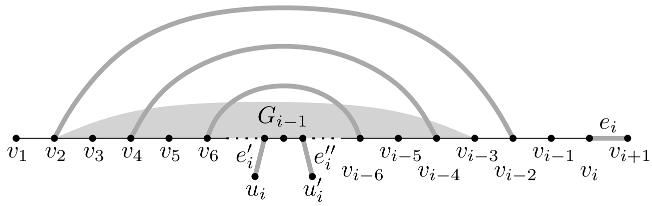

Figure 2: Definition of the graph G_i on basis of G_{i-1}; All edges in $E(G_i) \setminus E(G_{i-1})$ are drawn thick and have color i in c_i.

The rest of the proof is a routine check that the reverse $\hat{c_i}$ of c_i is P_i-good, i.e., that for every edge $f \in E(G_i)$ of color j in c_i there exist edges $e_{j+1}, \ldots, e_i$ adjacent to f with colors $j+1, \ldots, i$ in c_i, respectively. This is clearly the case for G_1, which consists of a single edge. Assuming that the reverse of c_{i-1} in G_{i-1} is P_{i-1}-good, it then suffices to show that every edge in $E(G_{i-1})$ is incident to at least one edge of color i in c_i, that is, at least one edge from $E(G_i) \setminus E(G_{i-1})$.

First consider an edge e_j from Q_{i-1}. If e_j is incident to a vertex v_s with s even and $s \leq 2\lfloor \frac{i-2}{4} \rfloor$, then e_j is incident to the edge $v_s v_{i-s}$ of color $s + (i-s) = i$. If e_j is incident to a vertex v_s with s even and $2\lfloor \frac{i-2}{4} \rfloor < s < i - 2\lfloor \frac{i-2}{4} \rfloor$, then e_j is incident to e or e', both of color i. If $e_j = e_{i-1}$, then it is incident to e_i. If none of the cases applies, then e_j is incident to a vertex v_{i-2s} with $s \in \{1, \ldots, \lfloor \frac{i-2}{4} \rfloor\}$ and hence incident to the edge $v_{2s} v_{i-2s}$ of color $2s + (i-2s) = i$.

Finally, consider any edge $f \in E(G_{i-1}) \setminus E(Q_{i-1})$. Then f is incident to some vertex v_{2j} with $j < c_i(e) - 2\lfloor \frac{c_i(e)-2}{4} \rfloor \leq i - 2\lfloor \frac{i-2}{4} \rfloor$. Hence f is incident to either e, e' or the edge $v_{2j} v_{i-2j}$, all of which have color i. □

Theorem 14. *For all $k \geq 3$, $AR_{FF}(C_k) = \frac{k^2}{8}(1 + o(1))$.*

Proof. The lower bound follows from Theorem 13 and that $AR_{FF}(C_k)$ is at least $AR_{FF}(P_{k-1})$. For the upper bound we first use Theorem 13 to force a rainbow copy P of P_{k-2}. Let its end vertices be u and v. Now we introduce a set A of $2k$ new vertices and all edges between A and $\{u, v\}$. Each vertex in A together with P forms a copy of C_k. There are at most $2(k-2)$ edges between A and $\{u, v\}$ of colors from P. Thus there is a vertex $w \in A$ such that the colors of uw and uv do not appear in P. Thus P and w induce a rainbow C_k. □

Theorem 15. *For all $k \geq 1$, $AR_{FF}(M_k) = \lceil \frac{k}{2} \rceil \cdot \lfloor \frac{k}{2} \rfloor + 1$. Moreover, there is a bipartite M_k-good graph on $\lceil \frac{k}{2} \rceil \cdot \lfloor \frac{k}{2} \rfloor + 1$ edges.*

Proof. For the lower bound we consider the first inequality in (3). It is easy to see that for $H = M_k$ this amounts for

$$AR_{FF}(M_k) \geq |M_k| + \sum_{i=1}^{k} \left\lceil \frac{i-1}{2} \right\rceil = \left\lceil \frac{k}{2} \right\rceil \cdot \left\lfloor \frac{k}{2} \right\rfloor + 1.$$

For the upper bound, which is similar to the one in the proof of Theorem 13, we shall prove the following stronger claim by induction on i.

Claim. *For every $i \geq 1$ there exists a bipartite graph G_i with $\lceil \frac{i^2}{4} \rceil$ edges together with a fixed proper coloring c_i using colors $1, \ldots, i$ and containing a rainbow matching $Q_i = \{e_1, \ldots, e_i\}$, $e_i = u_i v_i$, with respect to these colors, such that the reverse $\hat{c}_i$ of c_i is M_i-good. Moreover, for every edge $e = u_s v_t$ not in Q_i but with both endpoints in Q_i we have $c_i(e) = s + t$.*

The graph G_1 consists of just a single edge $e_1 = u_1 v_1$ with $c_1(e_1) = 1$ and $Q_1 = \{e_1\}$. Having defined G_{i-1} we define the graph G_i to be a supergraph of G_{i-1}, where we set $c_i(e) = c_{i-1}(e)$ for each edge $e \in E(G_{i-1}) \subset E(G_i)$ and $c_i(e) = i$ for each edge $e \in E(G_i) \setminus E(G_{i-1})$. The vertices in $V(G_i) \setminus V(G_{i-1})$ are denoted by u_i, v_i and w_i. The edges in $E(G_i) \setminus E(G_{i-1})$ form a matching and are listed below.

- The edge $e_i = u_i v_i$. It extends the matching Q_{i-1} to the larger matching Q_i.
- The edges $u_j v_{i-j}$ for $j = 1, \ldots, \lfloor \frac{i-1}{2} \rfloor$. These edges do not coincide with any edge of Q_i since the indices of endpoints differ by at least one. Moreover, since $c_i(u_j v_{i-j}) = j + (i - j) = i$ these edges do not coincide with any edge in $E(G_{i-1}) \setminus E(Q_i)$.
- In the case that $\lfloor \frac{i-1}{2} \rfloor < i - \lfloor \frac{i-1}{2} \rfloor - 1$ there is an edge $e = u_j w_i$ with $j = \lfloor \frac{i-1}{2} \rfloor + 1$.

See Figure 3 for an illustration of how the graph G_i arises from G_{i-1}. Similarly to the proof of Theorem 13, we argue that the reverse of c_i is M_i-good by showing that every edge $e \in E(G_{i-1}) \subset E(G_i)$ is incident to at least one edge of color i in c_i.

First consider an edge $e_j = u_j v_j$ from Q_{i-1}. If $j \leq \lfloor \frac{i-1}{2} \rfloor$, then it is incident to the edge $u_j v_{i-j}$ of color i. If $j \geq i - \lfloor \frac{i-1}{2} \rfloor$, then e_j is incident to the edge $u_{i-j} v_j$ of color i. Finally, if $\lfloor \frac{i-1}{2} \rfloor < j < i - \lfloor \frac{i-1}{2} \rfloor$, then e_j is incident to the edge $v_j w_i$ of color i.

Now consider any edge $e \in E(G_{i-1}) \setminus E(Q_i)$. Then e is incident to some vertex u_j with $j < c_i(e) - \lfloor \frac{c_i(e)-1}{2} \rfloor \leq i - \lfloor \frac{i-1}{2} \rfloor$. Hence e is incident to the edge $u_j v_{i-j}$ of color i.

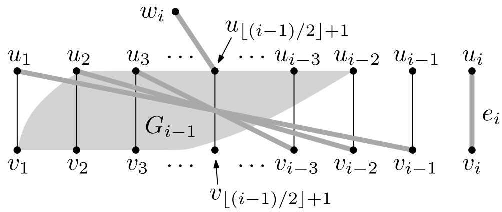

Figure 3: Definition of the graph G_i on basis of G_{i-1}. All edges in $E(G_i) \setminus E(G_{i-1})$ are drawn thick and have color i in c_i.

Finally, note that the number of edges of color j in c_j is exactly $\lceil \frac{j-1}{2} \rceil + 1$. Thus $|E(G_i)| = \sum_{j=1}^{i} (\lceil \frac{j-1}{2} \rceil + 1) = \lceil \frac{k}{2} \rceil \cdot \lfloor \frac{k}{2} \rfloor + 1$. Moreover, every edge in G_i is incident to one vertex with a u-label and one vertex with a v-label or w-label, that is, G_i is bipartite. □

For $H = K_n$ we get from Lemma 11 that $\frac{n^3}{16} \leq AR_{FF}(K_n) \leq \frac{n^3}{3}$. However, we can do better and indeed determine the value up to lower order terms exactly.

Theorem 16. *For all $n \geq 2$, $AR_{FF}(K_n) = \frac{n^3}{6}(1 + o(1))$.*

Proof. By the first-fit Lemma it suffices to prove that every K_n-good graph has at least $\frac{n^3}{6}(1+o(1))$ edges and that there is a K_n-good graph on at most $\frac{n^3}{6}$ edges.

For the lower bound let G be any K_n-good graph and c be any K_n-good coloring of G. Fix Q to be any rainbow copy of K_n in G. We denote the edges of Q by $e_1, \ldots, e_k$, where $k = \binom{n}{2}$, so that $c(e_i) < c(e_j)$ whenever $i < j$. Clearly, $c(e_i) \geq i$ for every $i = 1, \ldots, k$. Next, for any fixed color i we give a lower bound on the number of edges of color i in G.

By (P2) each of the $k - i$ edges $e_{i+1}, \ldots, e_k$ has an adjacent edge of color i. Since only two vertices of Q are joined by an edge of color i, every edge of color i different from e_i is incident to at most one vertex of Q. Moreover, if two vertices $u, v \in V(Q)$ both have *no* incident edge of color i then $c(uv) < i$. In particular, the subset of $V(Q)$ with no incident edge of color i is a clique in the graph $(V(Q), \{e_1, \ldots, e_{i-1}\})$. Thus there are at most $\sqrt{2i}$ such vertices and hence there are at least $n - \sqrt{2i} - 1$ edges of color i in G. Thus

$$|E(G)| \geq \sum_{i=1}^{\binom{n}{2}} n - \sqrt{2i} - 1 = \frac{n^3}{6}(1 + o(1)),$$

which implies $AR_{FF}(K_n) \geq \frac{n^3}{6}(1 + o(1))$.

For the upper bound we use the inequality $AR_{FF}(H) \leq 2w_{\text{greedy}}(H)$ from Lemma 11. That is, we shall find a rainbow coloring c of $H = K_n$ such that for the greedy matching M for c we have $\sum_{e\in M} c(e) \leq n^3/12$. We may assume without loss of generality that n is even. We label the vertices of K_n by $v_1, \dots, v_n$ and define the rainbow coloring c arbitrarily so that for $i = 1, \dots, n/2$

- edge $v_{2i-1}v_{2i}$ has color $\binom{2i}{2}$ and
- the colors $1, \dots, \binom{2i}{2}$ appear in the complete subgraph of K_n on the first $2i$ vertices $v_1, \dots, v_{2i}$.

The greedy matching M for c is given by $M = \{v_{2i-1}v_{2i} \mid i = 1, \dots, n/2\}$ and hence we have

$$\begin{aligned} AR_{FF}(K_n) &\leq 2w_{\text{greedy}}(K_n) \leq 2\sum_{i=1}^{n/2}\binom{2i}{2} \\ &\leq 2\sum_{i=1}^{n/2}\frac{4i^2}{2} \leq 4\cdot\frac{1}{3}\left(\frac{n}{2}\right)^3 + n^2 = \frac{n^3}{6}(1+o(1)). \end{aligned}$$ □

5. Online anti-Ramsey numbers

In this section we consider the online anti-Ramsey number AR_o. In particular, Painter is no longer restricted to use the strategy first-fit. By Theorem 1 lower bounds for AR_{FF} from the previous section are also lower bounds for AR_o, just like upper bounds for AR_s are upper bounds for AR_o. Indeed, these lower bounds are the best we have. However, we can improve on the upper bounds.

Theorem 17. *For any k-edge graph H, we have $AR_o(H) \leq k^2$. Moreover, $AR_o(P_k) \leq \binom{k}{2} + 1$, $AR_o(C_k) \leq \binom{k}{2} + 1$, and $AR_o(K_n) \leq n^4/4$.*

Proof. Let H be an graph on k edges with vertex set $V(H) = \{v_1, \dots, v_n\}$. Let the vertices be ordered $v_1, \dots, v_n$ such that given $v_1, \dots, v_i$, the next vertex v_{i+1} is chosen from $V \setminus \{v_1, \dots, v_i\}$ such that it has the largest number, d_{i+1}, of neighbors in $\{v_1, \dots, v_i\}$. We define Builder's strategy in rounds. After round i the so-far presented graph G_i shall consist of a rainbow induced copy of $H[v_1, \dots, v_i]$, for $i = 1, \dots, n$, and perhaps additional vertices adjacent to some vertices of that copy and inducing an independent set. In round 1 Builder just presents a single vertex w_1, i.e., $G_1 = \{w_1\} \cong H[v_1]$. For $i \geq 2$ round i is defined as follows.

1) Let G be the so-far presented graph and $G_{i-1} = G[w_1, \dots, w_{i-1}]$ be the rainbow copy of $H[v_1, \dots, v_{i-1}]$ in G, with w_j corresponding to v_j for $j = 1, \dots, i-1$. Let W_i be the set of vertices in $\{w_1, \dots, w_{i-1}\}$ corresponding to the neighborhood of v_i.
2) Builder presents a new vertex x together with the edges xw, $w \in W_i$ presenting xw_j before $xw_{j'}$ for $j < j'$ one-by-one as long as Painter does not use the colors used in G_{i-1}. As soon as Painter uses a color already present in G_{i-1}, Builder repeats this step by introducing a new vertex and edges from it to W_i. If, on each of the edges from the new vertex to W_i, Painter does not use a color present in G_{i-1}, then call this new vertex w_i and observe that $w_1, \dots, w_i$ induce a rainbow copy of $H[v_1, \dots, v_i]$ with w_i corresponding to v_i. This finishes i-th round.

Let G be the final graph obtained after the n-th round and containing an induced rainbow copy of H on the vertex set $W = \{w_1, \dots, w_n\}$. Note that $X = V(G) \setminus W$ induces an independent set and sends edges to W. Let us partition the edges between X and W into *old* and *new* edges as follows: An edge e that is introduced by Builder in i-th round is called *old* if Painter colors e with a color present in G_{i-1}, and *new* otherwise. Note that a new edge e might have the same color as an edge e' in $G[W]$, but only if e is presented earlier than e. Recall that v_i has d_i neighbors in $\{v_1, \dots, v_{i-1}\}$, i.e., $d_i = |W_i|$. Note that, with $d_1 = 0$, we obtain $d_1 + \cdots + d_n = k$. We need one more parameter: k_i is the number of edges in the subgraph induced by W at the last round involving edges incident to v_i. Formally, $k_i = |E(H[v_1, \dots, v_j])|$, where j is the largest index s.t. $v_i v_{j+1} \in E(H)$.

Claim. *The edges between X and W can be decomposed into $|W \setminus w_n| = n-1$ graphs, where each such graph consists of its center w_i, all old edges incident to w_i (call these endpoints old neighbors of w_i) and all other edges incident to the old neighbors of w_i. Moreover, the size of such a graph with center w_i is at most $d_{i+1}(k_i - \deg_H(v_i) + 1)$.*

We refer to Figure 4 for an illustration of these graphs. To prove the claim, observe that each $x \in X$ sends exactly one old edge to W. Each $w_i \in W$ sends at most $k_i - \deg_H(v_i) + 1$ old edges to X. Indeed, when w_i gets its last incident edge in round $j+1$ the total number of colors in G_j is k_i, and $\deg_H(v_i) - 1$ of those are used on edges incident to w_i and its neighbors in $\{w_1, \dots, w_j\}$. Moreover, if a vertex w_i is adjacent to a vertex $x \in X$ via an old edge, it means that x was introduced at a step j, $j \geq i+1$ of the algorithm. Then x is adjacent exactly to the vertices in W corresponding

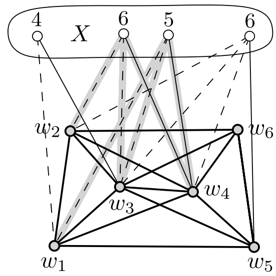

Figure 4: An example graph G after the last round. Edges between X and $W = \{w_1, \dots, w_6\}$ are drawn dashed if they are new and solid if they are old. The vertices in X are labeled with the number of the round in which they have been introduced. The graph with center w_4 is highlighted in gray.

to $\{v_1, \dots, v_i\} \cap N(v_j)$ in H. Since by our choice of the ordering of v_i's, $|\{v_1, \dots, v_i\} \cap N(v_j)| \le |\{v_1, \dots, v_i\} \cap N(v_{i+1})| = d_{i+1}$, we have that x sends at most $d_{i+1} - 1$ new edges to W. This proves the claim.

Now, we bound the number of edges in G by the total number of edges in the double stars described above and the number of edges in H. Here, use the fact H has no isolated vertices, $k_i \le k - 1$ for all $i = 1, \dots, n-1$, and $\sum_{i=1}^{n} d_i = k$.

$$
\begin{aligned}
(4) \qquad AR_o(H) &\le k + \sum_{i=1}^{n-1} (k_i - \deg(v_i) + 1) d_{i+1} \\
&\le k + \sum_{i=1}^{n-1} (k-1) d_{i+1} = k + (k-1)k = k^2.
\end{aligned}
$$

This proves the first part of the theorem.

For the special case of $H = P_k$, we take as v_1 one of the endpoints of the path, which gives the natural vertex ordering $v_1, \dots, v_{k+1}$ along the path. Then one easily sees that $d_i = 1$ for each $i = 2, \dots, k+1$ and $k_i = i - 1$ for each $i = 1, \dots, k$. Plugging into (4) we obtain

$$
AR_o(P_k) \le k + \sum_{i=1}^{k} i - \sum_{i=1}^{k} \deg(v_i) = k + \binom{k+1}{2} - (2k-1) = \binom{k}{2} + 1.
$$

Similarly, when H is the cycle C_k, we have $d_i = 1$ for $i = 2, \dots, k-1$, $d_k = 2$ and $k_i = i - 1$ for each $i = 1, \dots, k-1$, and thus obtain

$$\begin{aligned} AR_o(C_k) &\le k + \sum_{i=1}^{k-1} i + (k-1) - \sum_{i=1}^{k-1} \deg(v_i) - 2 \\ &= 2k - 3 + \binom{k}{2} - 2(k-2) = \binom{k}{2} + 1. \end{aligned}$$

Applying the bound $AR_o(H) \le k^2$ in the first part of the theorem to $H = K_n$, we immediately get the desired upper bound $AR_o(K_n) \le n^4/4$. $\square$

We remark that a more careful analysis for $H = K_n$ using the inequality (4) as we did it for $H = P_k$ and $H = C_k$ in the proof of Theorem 17 gives asymptotically the same bound as the one above. When H is the matching M_k then a straightforward application of Theorem 17 gives only $AR_o(M_k) \le k^2$, while from inequality (4) we get $AR_o(M_k) \le \binom{k+1}{2}$. However, we also get this bound from Theorem 1 and Theorem 6 as follows: $AR_o(M_k) \le AR_s(M_k) = \binom{k+1}{2}$.

Let us define $AR_o^*(H)$ just like $AR_o(H)$ but with the difference that Builder is allowed to present parallel edges. Note that from Theorem 6 and Theorem 7 follows that $AR_s^*(M_k) = AR_s(M_k)$, i.e., allowing multiple edges does not help here. However, in the online setting we get a better upper bound.

Theorem 18. *For all $k \ge 1$, $AR_o^*(M_k) \le \frac{k^2}{3} + \mathcal{O}(k)$.*

Proof. We present edges in k phases, introducing at most $2\lceil \frac{i}{3} \rceil$ edges in phase i, for each $i = 1, \dots, k$ such that for any proper coloring of Painter and every $i = 1, \dots, k$, the following holds. At each stage we identify a pair (F_i, v_i), where

- F_i is a rainbow matching on i edges
- v_i is a vertex of degree $\lceil \frac{i}{3} \rceil$
- v_i not incident to any edge in F_i
- each edge incident to v_i whose color is not already in F_i is not adjacent to any edge in F_i.

In phase 1 present two independent edges e and e', let $F_1 = \{e\}$ and $v_1 \in e'$. For $i \ge 2$ we call colors of edges in F_i old colors and all other colors new colors. Introduce two bundles of $\lceil \frac{i}{3} \rceil$ parallel edges each, one after another. As endpoints of the first bundle we choose v_{i-1} and a new vertex v. First assume that Painter colors in such a way that some edge e at v_{i-1} has a new color. Then we choose the endpoints of the second bundle to be two new vertices u, w and let $F_i = F_{i-1} \cup e$ and $v_i = u$. Otherwise, each edge at v_{i-1} has an old color. Then we choose the endpoints of the second bundle to be v_{i-1} and a new vertex z. Since v_{i-1} now has degree $\lceil\frac{i-1}{3}\rceil + 2\lceil\frac{i}{3}\rceil \geq i$, at least one edge e of the second bundle receives a new color. Set $F_i = F_{i-1} \cup e$ and $v_i = v$.

Since in each phase we introduce $2\lceil\frac{i}{3}\rceil$ edges, we finally obtain that $AR_o^*(M_k) \leq \sum_{i=1}^{k} 2\left\lceil\frac{i}{3}\right\rceil \leq \frac{k^2}{3} + \mathcal{O}(k)$. □

6. Concluding remarks and open questions

In this paper we investigate three anti-Ramsey type functions: $AR_{FF}(H)$, $AR_o(H)$ and $AR_s(H)$. Although these functions might be identical for some graphs, such as stars, in general these are distinct functions. For example, AR_{FF} and AR_o differ from AR_s already in their order of magnitude in case of K_n. For paths or matchings, all three numbers have the same order of magnitude.

In classical size Ramsey theory, in particular in an argument attributed to Chvátal, see [13], it is claimed that if G is a graph with the smallest number of edges such that any 2-coloring contains a monochromatic copy of K_n, then G must be a complete graph. So one might think that in order to force a rainbow clique, i.e., taking $H = K_n$, it is most efficient to color a large clique, i.e., $G = K_N$, instead of some other graph G. That would imply that $AR_s(K_n) = \binom{AR_{loc}(K_n)}{2}$. We prove that this intuition and analogue of size-Ramsey theorem does not hold already for K_4. The following general question remains:

Problem 1. By how much do $AR_s(K_n)$ and $\binom{AR_{loc}(K_n)}{2}$ differ as n goes to infinity?

We believe that the inequality $AR_s(H) \leq \binom{AR_{loc}(H)}{2}$ could be asymptotically tight for $H = K_n$ and for some sparse graphs H. When H is the path P_k, then the best-known upper bound for $AR_s(P_k)$ is given by the best-known upper bound for $AR_{loc}(P_k)$. Next is:

Problem 2. Improve the upper bound for $AR_s(P_k)$.

We proved that $AR_{FF}(H)$ is very closely related to a min-max quantity with respect to matchings in H. We derived an upper bound $AR_{FF}(H) \leq (\tau + 1)k$, which we suspect could be improved by the factor of 2.

Problem 3. Is it true that for every graph H, $AR_{FF}(H) \leq (\tau + 1)k/2$?

For all our lower bounds for AR_o Painter uses the first-fit strategy. This gives rise to the following:

Problem 4. Find a class of graphs for which AR_o and AR_{FF} have different asymptotic behavior.

The theme of this paper was to efficiently force a rainbow copy of a specific graph H in every proper coloring of a constructed graph G. We have measured efficiency by the number of edges required in G and have considered this both in the online and offline setting. Let us mention another concept of efficiency that might be interesting to study: For given graph H what is the *least maximum degree* of a graph G with $G \to H$, i.e., where every proper edge-coloring of G contains a rainbow copy of H? Clearly, the maximum degree of G must be at least $|E(H)|-1$ since otherwise G can be properly colored with less than $|E(H)|$ colors and hence does not contain a rainbow copy of H.

There is also an online variant of this question, which is analogous to the online anti-Ramsey numbers we defined here. One can show that Builder can force a rainbow matching on k edges even if the graph she presents has maximum degree at most $\lceil (k+1)/2 \rceil$ and that no rainbow copy of any k-edge graph H can be forced by Builder if she is restricted to presenting a graph of maximum degree strictly less than $\lceil (k+1)/2 \rceil$.

Problem 5. How large rainbow paths can Builder force, when she presents a graph with bounded maximum degree?

We remark that such questions are closely related to the so-called *restricted setting* considered by Prałat [27] and Grytczuk, Hałuszczak and Kierstead [18].

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References

[1] N. Alon, T. Jiang, Z. Miller, and D. Pritikin (2003). Properly colored subgraphs and rainbow subgraphs in edge-colorings with local constraints. *Random Structures Algorithms* **23**(4), 409–433. MR2016871

[2] N. Alon, H. Lefmann, and V. Rödl (1992). On an anti-Ramsey type result. In: *Sets, Graphs and Numbers (Budapest, 1991)*, Colloq. Math. Soc. János Bolyai, Vol. 60. North-Holland, Amsterdam, pp. 9–22.

[3] D. Avis (1983). A survey of heuristics for the weighted matching problem. *Networks* **13**(4), 475–493. MR0723694

[4] M. Axenovich, T. Jiang, and Z. Tuza (2003). Local anti-Ramsey numbers of graphs. *Combin. Probab. Comput.* **12**(5–6), 495–511. Special issue on Ramsey theory. MR2037066

[5] L. Babai (1985). An anti-Ramsey theorem. *Graphs Combin.* **1**(1), 23–28. MR0796179

[6] J. Beck (1983). On size Ramsey number of paths, trees, and circuits. I. *J. Graph Theory* **7**(1), 115–129. MR0693028

[7] J. Beck (1993). Achievement games and the probabilistic method. In: *Combinatorics, Paul Erdős Is Eighty*, Bolyai Soc. Math. Stud., Vol. 1, János Bolyai Math. Soc., pp. 51–78. MR1249704

[8] J. Böttcher, Y. Kohayakawa, and A. Procacci (2012). Properly coloured copies and rainbow copies of large graphs with small maximum degree. *Random Structures Algorithms* **40**(4), 425–436. MR2925305

[9] H. Broersma and X. Li (1997). Spanning trees with many or few colors in edge-colored graphs. *Discussiones Mathematicae Graph Theory* **17**(2), 259–269. MR1627951

[10] D. Conlon (2009/10). On-line Ramsey numbers. *SIAM J. Discrete Math.* **23**(4), 1954–1963. MR2594965

[11] S. Das, C. Lee, and B. Sudakov (2013). Rainbow Turán problem for even cycles. *European J. Combin.* **34**(5), 905–915. MR3021520

[12] P. Erdős and R. J. Faudree (1992). Size Ramsey functions. In: *Sets, Graphs and Numbers (Budapest, 1991)*, Colloq. Math. Soc. János Bolyai, Vol. 60. North-Holland, pp. 219–238. MR1218192

[13] P. Erdős, R. J. Faudree, C. C. Rousseau, and R. H. Schelp (1978). The size Ramsey number. *Period. Math. Hungar.* **9**(1–2), 145–161. MR0479691

[14] P. Erdős, M. Simonovits, and V. T. Sós (1975). Anti-Ramsey theorems. In: *Infinite and Finite Sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II*, Colloq. Math. Soc. János Bolyai, Vol. 10. North-Holland, pp. 633–643.

[15] E. Friedgut, Y. Kohayakawa, V. Rödl, A. Rucinski, and P. Tetali (2003). Ramsey games against a one-armed bandit. *Combinatorics Probability and Computing* **12**(5), 515–546. MR2037068

[16] S. Fujita, C. Magnant, and K. Ozeki (2010). Rainbow generalizations of Ramsey theory: A survey. *Graphs and Combinatorics* **26**(1), 1–30. MR2606615

[17] H. Gebauer and F. Mousset (2012). On rainbow cycles and paths. Preprint: arXiv:1207.0840. Bachelor Thesis.

[18] J. A. Grytczuk, M. Hałuszczak, and H. A. Kierstead (2004). On-line Ramsey theory. *Electron. J. Combin.* **11**(1), Research Paper 60, 10. MR2097326

[19] J. A. Grytczuk, H. A. Kierstead, and P. Prałat (2008). On-line Ramsey numbers for paths and stars. *Discrete Math. Theor. Comput. Sci.* **10**(3), 63–74. MR2445473

[20] A. Gyárfás, M. Ruszinkó, G. N. Sárközy, and R. H. Schelp (2011). Long rainbow cycles in proper edge-colorings of complete graphs. *Australas. J. Combin.* **50**, 45–53. MR2829278

[21] P. Keevash, D. Mubayi, B. Sudakov, and J. Verstraëte (2007). Rainbow Turán problems. *Combin. Probab. Comput.* **16**(1), 109–126. MR2286514

[22] H. A. Kierstead and G. Konjevod (2009). Coloring number and on-line Ramsey theory for graphs and hypergraphs. *Combinatorica* **29**(1), 49–64. MR2506387

[23] A. Kostochka and M. Yancey (2012). Large rainbow matchings in edge-colored graphs. *Combin. Probab. Comput.* **21**(1–2), 255–263. MR2900062

[24] A. Kurek and A. Ruciński (2005). Two variants of the size Ramsey number. *Discuss. Math. Graph Theory* **25**(1–2), 141–149. MR2152058

[25] X. Li and Z. Xu (2007). On the existence of a rainbow 1-factor in proper coloring of $K_{rn}^{(r)}$. Preprint: arXiv:0711.2847.

[26] M. Maamoun and H. Meyniel (1984). On a problem of G. Hahn about coloured Hamiltonian paths in K_{2n}. *Discrete Math.* **51**(2), 213–214. MR0758880

[27] P. Prałat (2008). A note on small on-line Ramsey numbers for paths and their generalization. *Australas. J. Combin.* **40**, 27–36. MR2381412

[28] V. Rödl and Z. Tuza (1992). Rainbow subgraphs in properly edge-colored graphs. *Random Structures Algorithms* **3**(2), 175–182. MR1151360

[29] H. J. Ryser (1967). Neuere Probleme der Kombinatorik. *Mathematisches Forschungsinstitut Oberwolfach* **July**, 24–29.

[30] C. Thomassen (2013). Decomposing graphs into paths of fixed length. *Combinatorica* **33**(1), 97–123. MR3070088

[31] D. B. West (2001). *Introduction to Graph theory*, Vol. 2. Prentice Hall, Englewood Cliffs. MR1367739

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