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New bijections from n-color compositions

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Combinatorial bijections are given from the set of n-color compositions of ν, for which a part of size n can take on n colors, to the set of compositions of $2\nu - 1$ having only parts of size 1 or 2, the set of compositions of 2ν having only odd parts, and the set of compositions of $2\nu + 1$ having no parts of size 1. A generalized bijection based on similar ideas is then given between the set of compositions of ν into parts congruent to $a(\text{mod } b)$ and the set of compositions of $\nu + b - a$ into parts congruent to $b(\text{mod } a)$ with each part greater than $b - a$.

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1. Introduction

A *composition of* ν is a sequence of positive integers called *parts* that sum to ν. An *n-color composition of* ν is a composition of ν for which a part of size n can take on n colors. It has been shown using generating functions that the number of n-color compositions of ν is $F_{2\nu}$, the 2ν-th Fibonacci number [1]. For example:

$\nu = 1$	$\nu = 2$	$\nu = 3$	$\nu = 4$	
(1_1)	(2_1)	(3_1)	(4_1)	$(2_1, 2_1)$
	(2_2)	(3_2)	(4_2)	$(2_2, 2_1)$
	$(1_1, 1_1)$	(3_3)	(4_3)	$(2_1, 2_2)$
		$(2_1, 1_1)$	(4_4)	$(2_2, 2_2)$
		$(2_2, 1_1)$	$(3_1, 1_1)$	$(2_1, 1_1, 1_1)$
		$(1_1, 2_1)$	$(3_2, 1_1)$	$(2_2, 1_1, 1_1)$
		$(1_1, 2_2)$	$(3_3, 1_1)$	$(1_1, 2_1, 1_1)$
		$(1_1, 1_1, 1_1)$	$(1_1, 3_1)$	$(1_1, 2_2, 1_1)$
			$(1_1, 3_2)$	$(1_1, 1_1, 2_1)$
			$(1_1, 3_3)$	$(1_1, 1_1, 2_2)$
				$(1_1, 1_1, 1_1, 1_1)$
$F_2 = 1$	$F_4 = 3$	$F_6 = 8$	$F_8 = 21$	

The following relationships between restricted compositions and Fibonacci numbers are also well-known (see [3], for example): The number of compositions of ν having only parts of size 1 or 2, referred to here as *1-2 compositions*, is $F_{\nu+1}$. The number of compositions of ν having only odd parts, referred to here as *odd compositions*, is F_ν. The number of compositions of ν having no parts of size 1, referred to here as *1-free compositions*, is $F_{\nu-1}$. Combinatorial bijections are therefore anticipated, for appropriately shifted values of ν, between these sets and the set of n-color compositions; however, to the author's knowledge and according to Heubach and Mansour's recent comprehensive text on compositions [3, p. 88], no such bijection has yet been demonstrated.

In this paper, we represent n-color compositions symbolically using strings of crosses (x) and dashes (−). The symbolic representation provides an unexpected and immediate structural link to 1-2 compositions. We then perform transformations on the character strings in order to obtain odd and 1-free compositions. Finally, we apply an adaptation of those transformations in order to answer the following open problem posed in a recent paper by Diffenderfer [2]: Give a combinatorial bijection between the set of compositions of ν into parts congruent to $a(\text{mod } b)$ and the set of compositions of $\nu+b-a$ into parts congruent to $b(\text{mod } a)$ with each part greater than $b-a$.

Proposition 1. *Let $\lambda = (\lambda_1^{c_1}, \ldots, \lambda_t^{c_t})$ denote an n-color composition of ν with t parts, where $\lambda_i^{c_i}$ is a part of size λ_i with color $1 \le c_i \le \lambda_i$. Let $p = (p_1, \ldots, p_{2t-1})$ denote a list of the positions of the crosses in a string of $\nu+t-1$ characters of which $2t-1$ are crosses and $\nu-t$ are dashes. The set of all such λs is in one-to-one correspondence with the set of all such ps.*

Proof. The symbolic representation of an n-color composition is achieved as follows: A part of size λ_i with color $1 \le c_i \le \lambda_i$ is represented by a string of λ_i characters, all of which are dashes except for the c_i-th character which is a cross. The full representation is then formed by concatenating these character strings (in order) and separating adjacent strings by an additional cross. For example, the n-color compositions of 3 are represented as follows:

n-color comps(3)	cross-and-dash
(3_1)	x − −
(3_2)	− x −
(3_3)	− − x
$(2_1, 1_1)$	x − x x
$(2_2, 1_1)$	− x x x
$(1_1, 2_1)$	x x x −
$(1_1, 2_2)$	x x − x
$(1_1, 1_1, 1_1)$	x x x x x

Note that an n-color composition of ν with t parts corresponds to exactly one string of $\nu+t-1$ characters of which $2t-1$ are crosses and $\nu-t$ are dashes: The number of crosses is $t+(t-1)=2t-1$, one for each part and one between each consecutive part; the number of dashes is $\nu-t$, the sum of all the part sizes minus t for the single crosses marking the color of each part. We will occasionally refer to the cross-and-dash representation as a list of characters, $s=(s_1,\ldots,s_{\nu+t-1})$.

Formally, given any n-color composition $\lambda=(\lambda_1^{c_1},\ldots,\lambda_t^{c_t})$ of ν, build a list $p=(p_1,\ldots,p_{2t-1})$ of the positions of the crosses in a string of $\nu+t-1$ characters by $T(\lambda)=p$, where p_i is given by

$$\begin{cases} p_1 = c_1 & \\ p_{2i} = \sum_{k=1}^{i}(\lambda_k+1) & 1\le i\le t-1\\ p_{2i+1} = \sum_{k=1}^{i}(\lambda_k+1)+c_{i+1} & 1\le i\le t-1. \end{cases}$$

Given any list $p=(p_1,\ldots,p_{2t-1})$ of the positions of the crosses in a string of $\nu+t-1$ characters, build an n-color composition $\lambda=(\lambda_1^{c_1},\ldots,\lambda_t^{c_t})$ of ν by $T^*(p)=\lambda$, where λ_i and c_i are given by

$$\begin{cases} c_1 = p_1 & \\ c_i = p_{2i-1}-p_{2i-2} & 2\le i\le t\\ \lambda_1 = p_2-1 & \\ \lambda_i = p_{2i}-p_{2i-2}-1 & 2\le i\le t-1\\ \lambda_t = \nu+t-1-p_{2t-2}. & \end{cases}$$

Note that, using the above equations, it can be verified directly that $T\circ T^*(p)=p$ and $T^*\circ T(\lambda)=\lambda$. For example, the $2i$-th position in $T\circ T^*(p)$ is

$$\sum_{k=1}^{i}(\lambda_k+1)=(\lambda_1+1)+\sum_{k=2}^{i}(\lambda_k+1)=p_2+\sum_{k=2}^{i}(p_{2k}-p_{2k-2})=p_{2i}$$

and the i-th position in $T^*\circ T(\lambda)$ is

$$p_{2i}-p_{2i-2}-1=\sum_{k=1}^{i}(\lambda_k+1)-\sum_{k=1}^{i-1}(\lambda_k+1)-1=\lambda_i$$

with color

$$p_{2i-1}-p_{2i-2}=\sum_{k=1}^{i-1}(\lambda_k+1)+c_i-\sum_{k=1}^{i-1}(\lambda_k+1)=c_i.$$

Similar calculations can be made in both directions for the remaining positions. □

2. 1-2 compositions

Proposition 2. *Let $\lambda = (\lambda_1^{c_1}, \ldots, \lambda_t^{c_t})$ denote an n-color composition of ν with t parts, where $\lambda_i^{c_i}$ is a part of size λ_i with color $1 \leq c_i \leq \lambda_i$. Let $\omega = (\omega_1, \ldots, \omega_{\nu+t-1})$ denote a 1-2 composition of $2\nu - 1$ with $\nu + t - 1$ parts. The set of all such λs is in one-to-one correspondence with the set of all such ωs.*

Proof. First write the cross-and-dash representation of an n-color composition. Then simply let crosses represent parts of size 1 and let dashes represent parts of size 2. For example:

n-color comps(3)	cross-and-dash	1-2 comps(5)
(3_1)	x − −	$(1,2,2)$
(3_2)	− x −	$(2,1,2)$
(3_3)	− − x	$(2,2,1)$
$(2_1,1_1)$	x − x x	$(1,2,1,1)$
$(2_2,1_1)$	− x x x	$(2,1,1,1)$
$(1_1,2_1)$	x x x −	$(1,1,1,2)$
$(1_1,2_2)$	x x − x	$(1,1,2,1)$
$(1_1,1_1,1_1)$	x x x x x	$(1,1,1,1,1)$

Formally, given any n-color composition of ν and its unique cross-and-dash representation $s = (s_1, \ldots, s_{\nu+t-1})$ (guaranteed by Proposition 1), build a 1-2 composition $\omega = (\omega_1, \ldots, \omega_{\nu+t-1})$ of $2\nu - 1$ by defining

$$\omega_i = \begin{cases} 1 & \text{if } s_i = \mathsf{x} \\ 2 & \text{if } s_i = -. \end{cases}$$

This construction is clearly reversible.

We are guaranteed to create compositions of $2\nu - 1$ in this manner since the number of crosses is $2t - 1$ and the number of dashes is $\nu - t$. Therefore, since crosses represent parts of size 1 and dashes represent parts of size 2, an n-color composition of ν with t parts corresponds to a 1-2 composition of

$$1 \cdot (2t - 1) + 2 \cdot (\nu - t) = 2\nu - 1.$$

□

3. Odd compositions

Proposition 3. *Let $\lambda = (\lambda_1^{c_1}, \ldots, \lambda_t^{c_t})$ denote an n-color composition of ν with t parts, where $\lambda_i^{c_i}$ is a part of size λ_i with color $1 \le c_i \le \lambda_i$. Let $\omega = (\omega_1, \ldots, \omega_{2t})$ denote an odd composition of 2ν with $2t$ parts. The set of all such λs is in one-to-one correspondence with the set of all such ωs.*

Proof. Beginning with the cross-and-dash representation, we now view crosses as separators. To the left and right of every cross, replace a string of j dashes with a string of $2j+1$ dashes, which then represents a part of size $2j+1$. Note that all part sizes are inherently odd. For example:

n-color comps(3)	cross-and-dash	dashes as parts	odd comps(6)
(3_1)	x − −	− x − − − − −	$(1,5)$
(3_2)	− x −	− − − x − − −	$(3,3)$
(3_3)	− − x	− − − − − x −	$(5,1)$
$(2_1, 1_1)$	x − x x	− x − − − x − x −	$(1,3,1,1)$
$(2_2, 1_1)$	− x x x	− − − x − x − x −	$(3,1,1,1)$
$(1_1, 2_1)$	x x x −	− x − x − x − − −	$(1,1,1,3)$
$(1_1, 2_2)$	x x − x	− x − x − − − x −	$(1,1,3,1)$
$(1_1, 1_1, 1_1)$	x x x x x	− x − x − x − x − x −	$(1,1,1,1,1,1)$

Formally, given any n-color composition of ν and its unique cross-and-dash representation (guaranteed by Proposition 1), make the following transformation. To the left and right of every cross, replace a string of j dashes with a string of $2j+1$ dashes ($0 \le j \le \nu - t$), and let ℓ_i be the length of the i-th string of dashes ($i = 1, \ldots, 2t$). Then build an odd composition $\omega = (\omega_1, \ldots, \omega_{2t})$ of 2ν by defining $\omega_i = \ell_i$. This construction is easily reversed by replacing each string of j dashes with a string of $\frac{j-1}{2}$ dashes ($1 \le j \le 2\nu - 2t + 1$).

We are guaranteed to create compositions of 2ν in this manner by the following reasoning: Suppose a given n-color composition has t parts and that its cross-and-dash representation has β_i strings of i consecutive dashes. Then the new number of dashes in the cross-and-dash transformation is

$$\sum_{i=1}^{\nu} (2i+1)\beta_i.$$

Since $\sum_{i=1}^{\nu} i\beta_i = \nu - t$ (the original number of dashes) and $\sum_{i=1}^{\nu} \beta_i = (2t-1)+1 = 2t$ (one more than the number of crosses), we are constructing compositions of

$$\sum_{i=1}^{\nu}(2i+1)\beta_i = 2\sum_{i=1}^{\nu} i\beta_i + \sum_{i=1}^{\nu}\beta_i = 2(\nu - t) + 2t = 2\nu. \qquad \square$$

4. 1-free compositions

Proposition 4. *Let* $\lambda = (\lambda_1^{c_1}, \ldots, \lambda_t^{c_t})$ *denote an* n*-color composition of* ν *with* t *parts, where* $\lambda_i^{c_i}$ *is a part of size* λ_i *with color* $1 \le c_i \le \lambda_i$*. Let* $\omega = (\omega_1, \ldots, \omega_{\nu-t+1})$ *denote a 1-2 composition of* $2\nu+1$ *with* $\nu-t+1$ *parts. The set of all such* λ*s is in one-to-one correspondence with the set of all such* ω*s.*

Proof. Beginning with the cross-and-dash representation, we now view dashes as separators. To the left and right of every dash, replace a string of j crosses with a string of $j+2$ crosses, which then represents a part of size $j+2$. Note that all part sizes are inherently greater than 1. For example:

n-color comps(3)	cross-and-dash	crosses as parts	1-free comps(7)
(3_1)	× − −	× × × − × × − × ×	$(3,2,2)$
(3_2)	− × −	× × − × × × − × ×	$(2,3,2)$
(3_3)	− − ×	× × − × × − × × ×	$(2,2,3)$
$(2_1,1_1)$	× − × ×	× × × − × × × ×	$(3,4)$
$(2_2,1_1)$	− × × ×	× × − × × × × ×	$(2,5)$
$(1_1,2_1)$	× × × −	× × × × × − × ×	$(5,2)$
$(1_1,2_2)$	× × − ×	× × × × − × × ×	$(4,3)$
$(1_1,1_1,1_1)$	× × × × ×	× × × × × × ×	(7)

Formally, given any n-color composition of ν and its unique cross-and-dash representation (guaranteed by Proposition 1), make the following transformation. To the left and right of every dash, replace a string of j crosses with a string of $j+2$ crosses ($0 \le j \le 2t-1$), and let ℓ_i be the length of the i-th string of crosses ($i = 1, \ldots, \nu-t+1$). Then build a 1-free composition $\omega = (\omega_1, \ldots, \omega_{\nu-t+1})$ of ν by defining $\omega_i = \ell_i$. This construction is easily reversed by replacing each string of j crosses with a string of $j-2$ crosses ($2 \le j \le 2t+1$).

We are guaranteed to create compositions of $2\nu+1$ in this manner by the following reasoning: Suppose a given n-color composition has t parts, and its cross-and-dash representation has γ_i strings of i crosses. Then the new number of crosses in the cross-and-dash transformation is

$$\sum_{i=1}^{\nu}(i+2)\gamma_i.$$

Since $\sum_{i=1}^{\nu} i\gamma_i = 2t-1$ (the original number of crosses) and $\sum_{i=1}^{\nu} \gamma_i = (\nu - t) + 1 = \nu - t + 1$ (one more than the number of dashes), we are constructing compositions of

$$\sum_{i=1}^{\nu}(i+2)\gamma_i = \sum_{i=1}^{\nu} i\gamma_i + 2\sum_{i=1}^{\nu}\gamma_i = (2t-1) + 2(\nu - t + 1) = 2\nu + 1. \qquad \square$$

5. Generalized bijection

While this section is not directly related to n-color compositions, it employs the ideas of Propositions 3 and 4 in order to construct a generalized bijective proof of the following theorem by Diffenderfer and Sills. The theorem is proven in [2], while background is given in [4].

Theorem 1. *Suppose $a \leq b$. The number of compositions of ν into parts congruent to $a(\mathrm{mod}\ b)$ equals the number of compositions of $\nu + b - a$ into parts congruent to $b(\mathrm{mod}\ a)$ where each part is greater than $b - a$.*

Bijective proof. The essence of the proof is as follows: Begin with a composition of ν into parts congruent to $a(\bmod b)$. Construct a string of crosses and dashes using the ideas of Proposition 3, i.e. the parts of the composition are represented by strings of dashes, and adjacent parts are separated by crosses. Make a transformation on the strings of dashes by replacing a string of j dashes with a string of $\frac{j-a}{b}$ dashes. Make a second transformation on the strings of crosses by replacing a string of j crosses with a string of $aj + b$ crosses. Extract a composition of $\nu + b - a$, whose parts are congruent to $b(\bmod a)$ and greater than $b - a$, by using the ideas of Proposition 4, i.e. strings of crosses represent the parts of the composition, and dashes separate adjacent parts. The steps are easily reversed by replacing a string of j crosses with a string of $\frac{j-b}{a}$ crosses, then replacing a string of j dashes with a string of $bj + a$ dashes.

We provide a guiding example with $\nu = 10$, $a = 2$, and $b = 3$:

comps(10)	dashes as parts		crosses as parts	comps(11)
$(8,2)$	− − − − − − − − × − −	− − ×	× × × − × × × − × × × × ×	$(3,3,5)$
$(5,5)$	− − − − − × − − − − −	− × −	× × × − × × × × × − × × ×	$(3,5,3)$
$(2,8)$	− − × − − − − − − − −	× − −	× × × × × − × × × − × × ×	$(5,3,3)$
$(2,2,2,2,2)$	− − × − − × − − × − − × − −	× × × ×	× × × × × × × × × × ×	(11)

The left column contains compositions of $\nu = 10$ with parts congruent to $a(\bmod b) = 2(\bmod 3)$. The right column contains compositions of $\nu + b - a = 11$ with parts congruent to $b(\bmod a) = 3(\bmod 2)$, i.e. $1(\bmod 2)$, and greater than $b - a = 1$.

Formally, there are five steps in the bijection, which we will label T_1 through T_5, and five inverses, which we will label T_1^* through T_5^*. The main ideas having been outlined above, we now describe each step in detail. We use the composition $(2, 8)$ from the table above as a guiding example, referenced in brackets after each step.

- $T_1(\lambda) = p$
  Given a composition $\lambda = (\lambda_1, \ldots, \lambda_t)$ of ν with t parts, all of which are congruent to $a(\text{mod } b)$, build a cross-and-dash representation by concatenating strings of dashes of length λ_i, each separated by a cross. There are $t-1$ crosses and ν dashes. Record a list $p = (p_1, \ldots, p_{t-1})$ of the positions of the crosses, where p_i is given by

$$p_i = \sum_{k=1}^{i} (\lambda_k + 1).$$

  [For the composition $(2, 8)$, we have $t-1 = 1$ cross and $\nu = 10$ dashes. We compute $p_1 = \lambda_1 + 1 = 3$ and record $p = (3)$ for the position of the cross.]
- $T_2(p) = \hat{p}$
  Replace a string of j dashes with a string of $\frac{j-a}{b}$ dashes. Since $j = \lambda_i \equiv a(\text{mod } b)$, we have $\frac{j-a}{b} \in \mathbb{Z}_+$. Note that some crosses may now be adjacent. There are now $t-1$ crosses and $\frac{\nu-at}{b}$ dashes, by the following reasoning: Let β_i be the number of strings of dashes of length i. Since $\sum_{i=1}^{\nu} i\beta_i = \nu$ (the original number of dashes) and $\sum_{i=1}^{\nu} \beta_i = t$ (one more than the number of crosses), the new number of dashes is

$$\sum_{i=1}^{\nu} \beta_i \left(\frac{i-a}{b}\right) = \frac{1}{b}\sum_{i=1}^{\nu} i\beta_i - \frac{a}{b}\sum_{i=1}^{\nu} \beta_i = \frac{\nu - at}{b}.$$

  Due to the first term in the equation above and the fact that $\frac{j-a}{b} \in \mathbb{Z}_+$, we have $\frac{\nu-at}{b} \in \mathbb{Z}_+$. Record a list $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_{t-1})$ of the new positions of the crosses, where $\hat{p}_i$ is given by

$$\hat{p}_i = \frac{p_i - i - ai}{b} + i.$$

  [We now have $t-1 = 1$ cross and $\frac{\nu-at}{b} = 2$ dashes. We compute $\hat{p}_1 = \frac{p_1 - 1 - a\cdot 1}{b} + 1 = 1$ and record $\hat{p} = (1)$ for the new position of the cross.]

- $T_3(\hat{p}) = \hat{q}$
  Take the complement of $\hat{p}$ to build a list $\hat{q} = (\hat{q}_1, \ldots, \hat{q}_{\frac{\nu-at}{b}})$ of the positions of the dashes. In other words, begin with a list of all positive integers in the interval $[1, d]$, where $d = t-1+\frac{\nu-at}{b}$ is the current total number of crosses and dashes, and remove all integers that occur in $\hat{p}$. [Since $d = 3$, we record $\hat{q} = (2, 3)$ for the positions of the dashes.]
- $T_4(\hat{q}) = q$
  Replace a string of j crosses with a string of $aj + b$ crosses. The new lengths of all strings of crosses are now inherently congruent to $b(\text{mod } a)$. Moreover, they are at least length b (or equivalently, greater than length $b - a$). There are now $\nu + b - a$ crosses and $\frac{\nu-at}{b}$ dashes, by the following reasoning: Let γ_i be the number of strings of crosses of length i. Since $\sum_{i=0}^{\nu} i\gamma_i = t-1$ (the original number of crosses) and $\sum_{i=0}^{\nu} \gamma_i = \frac{\nu-at}{b} + 1$ (one more than the number of dashes), the new number of crosses is

$$\sum_{i=0}^{\nu} \gamma_i(ai+b) = a\sum_{i=0}^{\nu} i\gamma_i + b\sum_{i=0}^{\nu} \gamma_i = \nu + b - a.$$

  Record a list $q = (q_1, \ldots, q_{\frac{\nu-at}{b}})$ of the new positions of the dashes, where q_i is given by

$$q_i = (\hat{q}_i - i)a + bi + i.$$

  [We now have $\nu+b-a = 11$ crosses and $\frac{\nu-at}{b} = 2$ dashes. We compute $q_1 = (\hat{q}_1 - 1)a + b\cdot 1 + 1 = 6$ and $q_2 = (\hat{q}_2 - 2)a + b\cdot 2 + 2 = 10$. We record $q = (6, 10)$ for the new positions of the dashes.]
- $T_5(q) = \omega$
  Build a composition of $\nu + b - a$ with $\frac{\nu-at}{b} + 1$ parts by concatenating the lengths of the strings of crosses. This composition has parts congruent to $b(\text{mod } a)$, all of which are at least b. The composition is $\omega = (\omega_1, \ldots, \omega_{\frac{\nu-at}{b}+1})$, where ω_i is given by

$$\begin{cases} \omega_1 = q_1 - 1 \\ \omega_i = q_i - q_{i-1} - 1 & 2 \le i \le \frac{\nu-at}{b} \\ \omega_{\frac{\nu-at}{b}+1} = c - q_{\frac{\nu-at}{b}} \end{cases}$$

  and $c = \nu + b - a + \frac{\nu-at}{b}$ is the current total number of crosses and dashes.

[Since $\frac{\nu-at}{b}+1=3$, we build a composition with 3 parts. We compute $\omega_1=q_1-1=5$ and $\omega_2=q_2-q_1-1=3$ and $\omega_3=c-q_2=3$. We record $\omega=(5,3,3)$ as the new composition of $\nu+b-a=11$.]

- $T_5^*(\omega)=q$
  Given a composition $\omega=(\omega_1,\dots,\omega_{\frac{\nu-at}{b}+1})$ of $\nu+b-a$ with $\frac{\nu-at}{b}+1$ parts, all of which are congruent to $b(\text{mod } a)$ and greater than $b-a$, build a cross-and-dash representation by concatenating strings of crosses of length ω_i, each separated by a dash. There are $\nu+b-a$ crosses and $\frac{\nu-at}{b}$ dashes. Record a list $q=(q_1,\dots,q_{\frac{\nu-at}{b}})$ of the positions of the dashes, where q_i is given by

$$q_i=\sum_{k=1}^{i}(\omega_k+1).$$

  [We have $\nu+b-a=11$ crosses and $\frac{\nu-at}{b}=2$ dashes. We compute $q_1=\omega_1+1=6$ and $q_2=\omega_1+1+\omega_2+1=10$. We record $q=(6,10)$ as the positions of the dashes.]
- $T_4^*(q)=\hat{q}$
  Replace a string of j crosses with a string of $\frac{j-b}{a}$ crosses. Since $j=\omega_i\equiv b(\text{mod } a)$, we have $\frac{j-b}{a}\in\mathbb{Z}_+$. Note that some dashes may now be adjacent. There are now $t-1$ crosses and $\frac{\nu-at}{b}$ dashes, by the following reasoning: Let γ_i be the number of strings of crosses of length i. Since $\sum_{i=1}^{\nu} i\gamma_i=\nu+b-a$ (the original number of crosses) and $\sum_{i=1}^{\nu}\gamma_i=\frac{\nu-at}{b}+1$ (one more than the number of dashes), the new number of crosses is

$$\sum_{i=1}^{\nu}\gamma_i\left(\frac{i-b}{a}\right)=\frac{1}{a}\sum_{i=1}^{\nu}i\gamma_i-\frac{b}{a}\sum_{i=1}^{\nu}\gamma_i=t-1.$$

  Record a list $\hat{q}=(\hat{q}_1,\dots,\hat{q}_{\frac{\nu-at}{b}})$ of the new positions of the dashes, where $\hat{q}_i$ is given by

$$\hat{q}_i=\frac{q_i-i-bi}{a}+i.$$

  [We now have $t-1=1$ cross and $\frac{\nu-at}{b}=2$ dashes. We compute $\hat{q}_1=\frac{q_1-1-b\cdot 1}{a}+1=2$ and $\hat{q}_2=\frac{q_2-2-b\cdot 2}{a}+2=3$. We record $\hat{q}=(2,3)$ as the new positions of the dashes.]
- $T_3^*(\hat{q})=\hat{p}$
  Take the complement of $\hat{q}$ to build a list $\hat{p}=(\hat{p}_1,\dots,\hat{p}_{t-1})$ of the positions of the crosses. In other words, begin with a list of all positive

integers in the interval $[1, d]$, where $d = t-1+\frac{\nu-at}{b}$ is the current total number of crosses and dashes, and remove all integers that occur in $\hat{q}$. [Since $d = 3$, we record $\hat{p} = (1)$ as the position of the cross.]

- $T_2^*(\hat{p}) = p$
  Replace a string of j dashes with a string of $bj + a$ dashes. The new lengths of all strings of dashes are now inherently congruent to $a(\text{mod } b)$. (They are also inherently at least length a, although this does not affect the outcome since it is implied that a part congruent to $a(\text{mod } b)$ cannot be smaller than a.) There are now $t-1$ crosses and ν dashes, by the following reasoning: Let β_i be the number of strings of dashes of length i. Since $\sum_{i=0}^{\nu} i\beta_i = \frac{\nu-at}{b}$ (the original number of dashes) and $\sum_{i=0}^{\nu} \beta_i = t$ (one more than the number of crosses), the new number of dashes is

$$\sum_{i=0}^{\nu} \beta_i(bi + a) = b\sum_{i=0}^{\nu} i\beta_i + a\sum_{i=0}^{\nu} \beta_i = \nu.$$

  Record a list $p = (p_1, \ldots, p_{t-1})$ of the new positions of the crosses, where p_i is given by

$$p_i = (\hat{p}_i - i)b + ai + i.$$

  [We now have $t - 1 = 1$ cross and $\nu = 10$ dashes. We compute $p_1 = (\hat{p}_1 - 1)b + a \cdot 1 + 1 = 3$. We record $p = (3)$ as the new position of the cross.]

- $T_1^*(p) = \lambda$
  Build a composition of ν with t parts by concatenating the lengths of the strings of dashes. This composition has parts congruent to $a(\text{mod } b)$, all of which are at least a. The composition is $\lambda = (\lambda_1, \ldots, \lambda_t)$, where λ_i is given by

$$\begin{cases} \lambda_1 = p_1 - 1 \\ \lambda_i = p_i - p_{i-1} - 1 \quad 2 \le i \le t-1 \\ \lambda_t = c^* - p_{t-1} \end{cases}$$

  and $c^* = \nu + t - 1$ is the current total number of crosses and dashes. [Since $t = 2$, we build a composition with 2 parts. We compute $\lambda_1 = p_1 - 1 = 2$ and $\lambda_2 = c^* - p_1 = 8$. We record $\lambda = (2, 8)$ as the new composition of $\nu = 10$.]

Each transformation can be checked to verify that $T_1 \circ T_1^*(p) = p$, $T_1^* \circ T_1(\lambda) = \lambda$, and so forth for T_2 through T_5. $\square$

6. Remarks

While Propositions 2, 3, and 4 describe only the paths from n-color compositions to 1-2, odd, and 1-free compositions, the same transformations can be used to describe paths between the three sets of restricted compositions. The following example shows a path from 1-free compositions of 6 to odd compositions of 5:

comps(6)	crosses as parts		dashes as parts	comps(5)
(6)	× × × × × ×	× × × ×	− × − × − × − × −	$(1,1,1,1,1)$
$(4,2)$	× × × × − × ×	× × −	− × − × − − −	$(1,1,3)$
$(3,3)$	× × × − × × ×	× − ×	− × − − − × −	$(1,3,1)$
$(2,4)$	× × − × × × ×	− × ×	− − − × − × −	$(3,1,1)$
$(2,2,2)$	× × − × × − × ×	− −	− − − − −	(5)

The parts of the 1-free compositions are represented with strings of crosses, each separated by a dash, as shown in the second column in the table above. A string of j crosses is then replaced with a string of $j-2$ crosses, resulting in the third column. A string of j dashes is then replaced with a string of $2j+1$ dashes, resulting in the fourth column. (Note that during this step, a string of 0 dashes will result in a single dash.) Then the parts of the odd compositions are represented by strings of dashes, each separated by a cross. As in the proof of Theorem 1, the character strings in the center column do not correspond with n-color compositions; they are merely stepping stones in the character transformation.

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