Journal of Combinatorics
Volume 4, Number 3, 357–372, 2013

Requiring pairwise nonadjacent chords in cycles

Terry A. McKee

Let $\mathcal{G}_k$ be the class of graphs for which every cycle of length k or more has at least $k-3$ pairwise nonadjacent chords. This makes $\mathcal{G}_4$ the class of chordal graphs and $\mathcal{G}_5$ the class of distance-hereditary graphs. I show that $k \geq 8$ implies that $\mathcal{G}_k$ is the class of graphs that have circumference less than k. I also characterize $\mathcal{G}_6$ and $\mathcal{G}_7$; for instance, a graph is in $\mathcal{G}_7$ if and only if every hamiltonian subgraph of order 7 or more is 3-connected and bipartite.

Motivated by $\mathcal{G}_4 \cap \mathcal{G}_5$ being the class of ptolemaic graphs, I show that a graph is in $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$ if and only if every order-k hamiltonian subgraph has at least $\lfloor k/2 \rfloor$ universal vertices.

AMS 2000 subject classifications: 05C75.
Keywords and phrases: Chord, cycle, chordal graph, distance-hereditary graph, ptolemaic graph.

1. Introduction

A *chord* of a cycle C of a graph G is an edge whose endpoints are nonconsecutive vertices of C, and n chords are *nonadjacent chords* when they have $2n$ distinct endpoints (equivalently, when they form a matching in G). Let $V(C)$ denote the set of vertices of C, let $|C|$ denote the length $|V(C)|$ of C, and let $G[V(C)]$ denote the subgraph of G induced by $V(C)$. A k-cycle is a cycle of length k, and $[v_1, \dots, v_k]$ will denote the k-cycle whose vertices $v_1, \dots, v_k$ come in that order around the cycle. An *i-chord* of C is an edge whose endpoints are a distance $i \geq 2$ apart along C; thus $i \leq |C|/2$. An *odd chord* (respectively, an *even chord*) is an i-chord where i is odd (even). For an i-chord e of C, say that e and C *form a cycle* C' to mean that C' is one of the two cycles that have $e \in E(C') \subset E(C) \cup \{e\}$; thus either $|C'| = |C| - i + 1$ or $|C'| = i + 1$ (and $|C| - i + 1 \geq i + 1$). Two chords ab and cd are *crossing chords* of a cycle C if their endpoints a, c, b, d come in that order around C; thus crossing chords are always nonadjacent chords.

Reference [4] has a very similar title to the present paper, as well as a somewhat similar goal: it describes how the existence of chords in larger cycles is related to writing cycles as sums of specific sizes of smaller cycles (thereby characterizing some of the same graph classes as in the present paper). Similarly, [5] considers the number of chords that cycles have, relative to the length of the cycles. The present paper goes in a different direction, emphasizing pairwise nonadjacent chords instead of features such as crossing chords that have been previously studied. Investigating such nontraditional aspects of graphs can lead to new graph theoretic insights.

For each $k \geq 4$, define $\mathcal{G}_k$ to be the class of all graphs in which every cycle of length k or more has at least $k-3$ pairwise nonadjacent chords. The class $\mathcal{G}_4$ is precisely the class of *chordal graphs*, which are traditionally defined by every cycle of length 4 or more having a chord. There are many other characterizations in [1, 6], such as every minimal vertex separator inducing a complete subgraph.

Theorem 1.1 and Corollary 1.1 will show that $\mathcal{G}_5$ is precisely the class of *distance-hereditary graphs*, which are traditionally defined by the distance between two vertices in a connected subgraph always equaling the distance between those vertices in the entire graph. Another characterization, from [2] (also see [1]), is that every cycle of length 5 or more has at least two crossing chords.

Theorem 1.1 will give a characteristic property of the graphs in $\mathcal{G}_k$ for all $k \geq 4$, and then sections 2, 3, and 4 will characterize, respectively, $\mathcal{G}_6$, $\mathcal{G}_7$, and all $\mathcal{G}_k$ with $k \geq 8$. Investigating such a sequence of graph classes—especially one whose first two classes are so well studied—emphasizes common features of the classes as well suggesting insight into the role of pairwise nonadjacent chords of cycles. The considerable disparity of the characterizations of the classes $\mathcal{G}_k$ when $k \leq 7$ stands in contrast to Theorem 4.1, which shows that being in $\mathcal{G}_k$ with $k \geq 8$ is equivalent to simply having circumference less than k. This suggests that the sequence beginning with $\mathcal{G}_4$, $\mathcal{G}_5$, and $\mathcal{G}_6$ might be the wrong target for investigation. But the characterization of $\mathcal{G}_6$ will be used in section 5 to look at the alternative sequence beginning with $\mathcal{G}_4$ (the class of chordal graphs), $\mathcal{G}_4 \cap \mathcal{G}_5$ (the class of ptolemaic graphs [1, 3]), and $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$. These increasingly stronger classes have more sophisticated characterizations that are able to exploit the power of chordal graph theory.

Theorem 1.1. *For each $k \geq 4$, if every cycle of length k or more in a graph has at least $k-3$ pairwise nonadjacent chords, then each of those chords must cross another of those chords.*

Proof. Suppose $k \geq 4$ and every cycle of length k or more in a graph G has at least $k-3$ pairwise nonadjacent chords. Further assume C is a minimum-length cycle of length $|C| \geq k$ in G that does not have $k-3$ pairwise nonadjacent chords such that each crosses another (arguing by contradiction). Thus

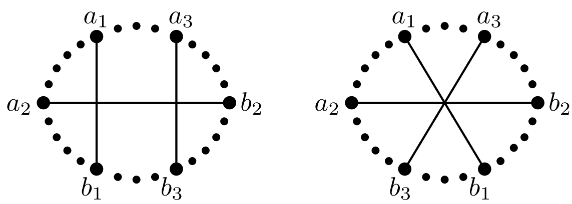

Figure 1: The two ways that a k-cycle with $k \geq 6$ could have three pairwise nonadjacent chords with each one crossing another one.

$k \geq 6$, since nonadjacent chords of a 4-cycle or a 5-cycle do cross. Therefore, C has $k-3 \geq 3$ chords $e_1, \ldots, e_{k-3}$ with $2(k-3)$ distinct endpoints where one of these chords (without loss of generality, say e_{k-3}) does not cross any of the other chords $(e_1, \ldots, e_{k-4})$. Let C_1 and C_2 be the two cycles formed by e_{k-3} and C; thus each $|C_i| < |C|$ and each of $e_1, \ldots, e_{k-4}$ is a chord of C_1 or of C_2. Renumbering if necessary, suppose $|C_1| \geq |C_2|$ and C_1 has the chords $e_1, \ldots, e_h$ with $\lceil \frac{k-4}{2} \rceil \leq h \leq k-4$ and C_2 has any remaining edges among $e_{h+1}, \ldots, e_{k-4}$ as chords.

Suppose for the moment that $h = k-4$, so that all of $e_1, \ldots, e_{k-4}$ are chords of C_1. Their $2(k-4)$ endpoints are then vertices of C_1, as are also the two endpoints of e_{k-3}. Thus, $2k-6 \leq |C_1| < |C|$ and so (since $k \geq 6$) $k \leq k + (k-6) \leq |C_1| < |C|$. Therefore, by the assumed minimality of $|C|$, cycle C_1 does have $k-3$ pairwise nonadjacent chords such that each crosses another. But these $k-3$ chords of C_1 would also be chords of C (contradicting that C does not have $k-3$ such chords).

Therefore, $\lceil \frac{k-4}{2} \rceil \leq h < k-4$ and the $2h \geq k-4$ endpoints of $e_1, \ldots, e_h$ are vertices of C_1, as are also the two endpoints of e_{k-3}. Thus, $k-2 \leq |C_1| < |C|$. Moreover, C_2 has $k-4-h \geq 1$ chords, one of which is e_{h+1}. Let C' be the cycle formed by e_{h+1} and C with $V(C_1) \subset V(C')$. The $k-2$ (or more) vertices of C_1, together with the two endpoints of e_{h+1}, are vertices of C', and so $k \leq |C'| < |C|$. Therefore, by the assumed minimality of $|C|$, cycle C' does have $k-3$ pairwise nonadjacent chords such that each crosses another. But these $k-3$ chords of C' would also be chords of C (contradicting that C does not have $k-3$ such chords). □

Figure 1 illustrates the $k = 6$ instance of Theorem 1.1, with each of the three pairwise nonadjacent chords a_1b_1, a_2b_2, and a_3b_3 of the cycle C crossing another one of the three (C consists of the six 'dotted' paths between labeled vertices, where those paths have arbitrary lengths at least 1, and there might be additional chords of C that are not shown).

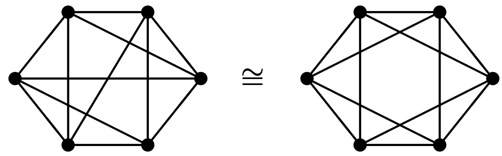

Figure 2: Two views of the complete tripartite graph $K_{2,2,2} \notin \mathcal{G}_6$.

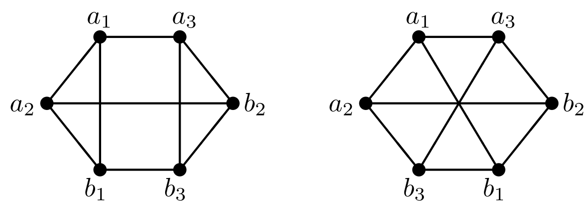

Figure 3: On the left, the triangular prism; on the right, $K_{3,3}$.

Crossing chords are always nonadjacent, but nonadjacent chords need not cross. By Theorem 1.1, however, if every cycle of length 5 or more has at least two nonadjacent chords, then these chords must cross. Using this, Corollary 1.1 is a very simple, previously unnoticed characterization of distance-hereditary graphs that only requires nonadjacent chords in cycles of length 5 or more (as opposed to requiring crossing chords, as in [2]).

Corollary 1.1. *A graph is distance-hereditary if and only if every cycle of length* 5 *or more has two nonadjacent chords.*

2. The graph class $\mathcal{G}_6$

Theorem 2.1 (and Theorem 3.1) will characterize $\mathcal{G}_k$ when $k = 6$ (respectively, $k = 7$) in terms of the induced hamiltonian subgraphs of order k or more. Figure 2 illustrates how one hamiltonian cycle can have three pairwise-nonadjacent chords with each one crossing another one (the 'outside hexagon' on the left has the 'horizontal' chord crossing the two 'vertical' chords) while another 6-cycle (the 'outside hexagon' on the right) of the same graph does not have three such chords; thus the graph shown is not in $\mathcal{G}_6$.

Figure 3 shows two graphs in $\mathcal{G}_6$ that will be important in this section. The graph on the left is the *triangular prism* (isomorphic to the complement of C_6), and the graph on the right is the complete bipartite graph $K_{3,3}$.

Theorem 2.1. *A graph is in* $\mathcal{G}_6$ *if and only if every induced hamiltonian subgraph of order* 6 *or more either* is *a triangular prism or* contains *a* $K_{3,3}$ *subgraph.*

Proof. First suppose $G \in \mathcal{G}_6$ and assume H is a hamiltonian subgraph of minimum order $k \geq 6$ that is not a triangular prism and does not contain a $K_{3,3}$ subgraph (arguing by contradiction). Suppose C is any k-cycle of H (so $H = H[V(C)]$). Let a_1b_1, a_2b_2, a_3b_3 be the three pairwise nonadjacent chords of C guaranteed by $G \in \mathcal{G}_6$, say with a_1b_1 crossing a_2b_2 and with a_2b_2 crossing a_3b_3 as in Theorem 1.1; in fact, say C is as illustrated in Figure 1, with C partitioned into six subpaths $C[a_1, a_2], C[a_1, a_3], C[a_3, b_2]$, and $C[b_2, b_3], C[b_1, b_3], C[a_2, b_1]$ in the left graph or $C[b_1, b_2], C[b_1, b_3], C[a_2, b_3]$ in the right graph (with the indicated endpoints, remembering that C might have more than the three chords shown).

Suppose for the moment that $k = |C| = 6$. By assumption, H is not a triangular prism and does not contain a $K_{3,3}$ subgraph (so no 6-cycle of H has three pairwise crossing chords). Thus H must be as shown on the left in Figure 3 together with at least one additional edge (one additional chord of C). Since each additional chord would have one endpoint in the triangle $a_1a_2b_1$ and the other in the triangle $a_3b_2b_3$, suppose without loss of generality that C has the chord a_3b_1. The 6-cycle $C^* = [a_1, b_1, a_3, b_3, b_2, a_2]$ cannot also have the chord a_1b_3 (it would cross both the chords a_2b_2 and a_3b_1 in C); also C^* cannot have both chords a_1b_2 and a_2a_3 (they would cross each other and the chord b_1b_3 in the 6-cycle $[a_1, a_3, b_3, b_2, a_2, b_1]$), and C^* cannot have both chords a_2b_3 and b_1b_2 (they would cross each other and the chord a_1a_3 in the 6-cycle $[a_1, a_2, b_2, a_3, b_3, b_1]$). These restrictions imply that C^* would be a 6-cycle of G without three pairwise nonadjacent chords (contradicting $G \in \mathcal{G}_6$).

Therefore, $k = |C| \geq 7$. The eight special cases below will show, without loss of generality, that there is always a cycle C' that has $V(C') \subset V(C)$ and $E(C') \subset E(C) \cup \{a_1b_1, a_2b_2, a_3b_3\}$ with $6 \leq |C'| < |C|$ such that $G[V(C')]$ is not a triangular prism. Since every edge and every chord of C' is an edge or chord of C as well, $G[V(C')]$ would also not contain a $K_{3,3}$ subgraph (contradicting the minimality of k that was assumed in the first sentence of this proof).

First suppose that $k = |C| = 7$ with $\{a_1,b_1,a_2,b_2,a_3,b_3,x\} = V(C)$, and consider where x might occur in the graphs H illustrated in Figure 1:

- In the graph on the left, if x is in, say, $C[a_1, a_3]$, then let C' be the 6-cycle $[a_1, b_1, b_3, a_3, b_2, a_2]$; $H[V(C')]$ is not a triangular prism since a_1 is not adjacent to a_3 (otherwise, the hamiltonian subgraph $H - b_1$ of H would contradict the assumed minimality of k).

- In the graph on the left, if x is in, say, $C[a_1, a_2]$, then let C' be the 6-cycle $[a_1, a_3, b_3, b_2, a_2, b_1]$; $H[V(C')]$ is not a triangular prism since a_1 is not adjacent to a_2 (otherwise, $H - b_2$ would contradict the assumed minimality of k).
- In the graph on the right, if x is in, say, $C[a_1, a_3]$, then let C' be the 6-cycle $[a_1, b_1, b_2, a_3, b_3, a_2]$; $H[V(C')]$ is not a triangular prism (whether or not a_1 is adjacent to a_3).

Now suppose that $k = |C| \geq 8$ with $\{a_1,b_1,a_2,b_2,a_3,b_3,x,y\} \subseteq V(C)$, and consider where x and y might occur in the graphs H illustrated in Figure 1:

- In the graph on the left, if x, y are on the same side of the chord a_2b_2 with, say, x and y both in the path $\pi = C[a_1, a_2] \cup C[a_1, a_3] \cup C[a_3, b_2]$, then let C' be the union of the edge a_2b_2 and the path π; $H[V(C')]$ is not a triangular prism, even if $|C'| = 6$, since no three consecutive vertices p, q, r of π can have a 2-chord pr (otherwise, $H - q$ would contradict the assumed minimality of k).
- In the graph on the left, if x, y are on different sides of the chord a_2b_2 with, say, x in $C[a_1, a_2] \cup C[a_1, a_3]$ and y in $C[a_2, b_1] \cup C[b_1, b_3]$, then let C' be the union of the edge a_3b_3 and the path $C[a_1, a_2] \cup C[a_1, a_3] \cup C[a_2, b_1] \cup C[b_1, b_3]$; $H[V(C')]$ is not a triangular prism since $|C'| \geq 7$.
- In the graph on the left, if x, y are on different sides of the chord a_2b_2 with, say, x in $C[a_1, a_2]$ and y in $C[b_2, b_3]$, then let C' consist of the edges a_2b_2 and a_3b_3 and the paths $C[a_1, a_2] \cup C[a_1, a_3]$ and $C[b_2, b_3]$; $H[V(C')]$ is not a triangular prism since $|C'| \geq 7$.
- In the graph on the right, if x, y are on the same side of a chord, say a_2b_2 with x and y both in in $\pi = C[a_1, a_2] \cup C[a_1, a_3] \cup C[a_3, b_2]$, then let C' consist of the edge a_2b_2 and the path π; $H[V(C')]$ is not a triangular prism, even if $|C'| = 6$, since no three consecutive vertices p, q, r of π can have a 2-chord pr (otherwise, $H - q$ would contradict the assumed minimality of k).
- In the graph on the right, if x, y are on different sides of each of the chords a_1b_1, a_2b_2, a_3b_3 with, say, x in $C[a_1, a_3]$ and y in $C[b_1, b_3]$, then let C' consist of the edges a_1b_1, a_2b_2 and a_3b_3 and the paths $C[a_1, a_3]$, $C[b_1, b_2]$ and $C[a_2, b_3]$; $H[V(C')]$ is not a triangular prism since $|C'| \geq 7$.

Conversely, suppose every induced hamiltonian subgraph H of G with order 6 or more either is a triangular prism or contains a $K_{3,3}$ subgraph, and suppose C is a hamiltonian cycle of H. If H is a triangular prism, then the three chords of C are nonadjacent. Otherwise, assume H contains a subgraph $H' \cong K_{3,3}$ and let H° be the subgraph of H' that consists of the chords of C that are in H'. Since each vertex of the 3-connected graph H' is on at least one edge of H°, it takes at least three vertices of $H' \cong K_{3,3}$ to cover all the edges of H°. Therefore, the König Property of bipartite graphs (Theorem 1.1.2 in [1]) implies that H° has at least three pairwise nonadjacent edges, and these are three pairwise nonadjacent chords of C. Therefore, $G \in \mathcal{G}_6$. □

3. The graph class $\mathcal{G}_7$

Lemma 3.1. *If $k \geq 7$ and $G \in \mathcal{G}_k$, then every cycle of G with length k or more must in fact have length $2k-6$ or more.*

Proof. If $G \in \mathcal{G}_k$ contains a cycle C with $|C| \geq k$, then C must have at least $k-3$ pairwise nonadjacent chords, and so at least $2(k-3)$ vertices. □

Lemma 3.2. *Every hamiltonian graph of order 8 in $\mathcal{G}_7$ is 3-connected and bipartite.*

Proof. Suppose a hamiltonian graph $G \in \mathcal{G}_7$ is spanned by an 8-cycle $C = [v_1, \ldots, v_8]$. By Lemma 3.1, G contains no 7-cycles. Let e_1, e_2, e_3, e_4 be four pairwise nonadjacent chords of C. Cycle C cannot have a 2-chord e, since e and C would form a 7-cycle. This leaves only three possibilities, up to relabeling, for e_1, e_2, e_3, e_4: either two 3-chords v_1v_4, v_5v_8 and two 4-chords v_2v_6, v_3v_7 OR four 4-chords $v_1v_5, v_2v_6, v_3v_7, v_4v_8$ OR four 3-chords $v_1v_4, v_2v_7, v_3v_6, v_5v_8$. Only the third possibility can occur without a 7-cycle occurring (namely, $[v_1, v_2, v_3, v_7, v_6, v_5, v_4]$ in the first possibility and $[v_1, v_2, v_3, v_7, v_8, v_4, v_5]$ in the second). If C has even one additional chord that is a 4-chord (say v_1v_5), then G would contain a 7-cycle (namely, $[v_1, v_5, v_4, v_3, v_2, v_7, v_8]$). Thus the only additional chords e that C can have are the four remaining 3-chords: $v_1v_6, v_2v_5, v_3v_8, v_4v_7$. Therefore $G[V(C)]$ is a 3-connected subgraph of the complete bipartite graph $K_{4,4}$ that is formed from C and all eight 3-chords of C. □

Lemma 3.3. *Every hamiltonian graph of order 7 or more in $\mathcal{G}_7$ is bipartite.*

Proof. Suppose a hamiltonian graph $G \in \mathcal{G}_7$ and C is any cycle of G with $|C| \geq 7$. By Lemma 3.1, G contains no 7-cycles. Thus $|C| \geq 8$, say with pairwise nonadjacent chords e_1, e_2, e_3, e_4. If $|C| = 8$, then Lemma 3.2 implies that $G[V(C)]$ is bipartite. Therefore, assume C is a minimum length cycle with $|C| \geq 9$ such that $G[V(C)]$ is not bipartite (arguing by contradiction).

Suppose for the moment that $|C| \geq 10$ is even. Cycle C cannot have an even i-chord e without e and C forming a cycle C' with $|C'| = |C| - i + 1 \geq |C| - \frac{1}{2}|C| + 1 > 5$ where $|C'| \neq 7$ is odd; thus $G[V(C')]$ would not be bipartite and $9 \leq |C'| < |C|$ (contradicting the minimality of $|C|$). Therefore, every chord of the even cycle C is an odd chord, and so all the cycles formed by a chord e and C are even cycles. These would form a basis of even cycles for the cycle space (contradicting that $G[V(C)]$ is not bipartite).

Therefore, $|C| \geq 9$ is odd, say with $C = [v_1, \ldots, v_{|C|}]$. If C has an odd i-chord e, then e and C would form a cycle C' with $|C'| = |C| - i + 1$. Therefore, $|C| > |C'| > |C| - \frac{1}{2}|C| + 1 \geq 6$ where $|C'| \neq 7$ is odd; thus $G[V(C')]$ would be nonbipartite with $9 \leq |C'| < |C|$ (contradicting the minimality of $|C|$). Therefore, every chord of C is an even chord.

Suppose e' is an even i-chord of C (so $2 \leq i \leq \frac{1}{2}(|C| - 1)$). Let C' and C'' be the two cycles formed by e' and C, where $|C'| = |C| - i + 1$ is even with $|C| > |C'| \geq |C| - \frac{1}{2}|C| + 1 \geq 6$ and where $|C''| = i + 1 < |C|$ is odd.

The only way to have $|C'| = 6$, without contradicting $|C''| \neq 7$ or $i \leq \frac{1}{2}(|C| - 1)$, would be to have $|C| = 9$ and $i = 4$. This would make every chord of C be a 4-chord, and so two of e_1, e_2, e_3, e_4 would be 4-chords whose endpoints are not consecutive around C; without loss of generality, say v_1v_5 and v_3v_7. But then those two edges together with v_3v_4, v_4v_5 and v_7v_8, v_8v_9, v_1v_9 would form a 7-cycle (contradicting that C has no 7-cycles).

Therefore, the even number $|C'|$ satisfies $8 \leq |C'| < |C|$. Moreover, $G[V(C')]$ is bipartite (by Lemma 3.2 if $|C'| = 8$, and by the assumed minimality of $|C|$ if $|C'| > 8$), and so C' cannot have any even chords. Since C can only have even chords, the only chords that C' can possibly have are odd chords of C' that are even chords of C. There are only three 3-chords of C' that are even chords of C (for instance, if $e = v_3v_j$, these are v_1v_j and v_2v_{j+1} and v_3v_{j+2}). Thus, at least one of e_1, e_2, e_3, e_4 has to be a chord e'' that is an h-chord of C' with odd $h \geq 5$ such that e'' and C form a cycle C^* where $|C^*| = h + i \neq 7$ is odd. Therefore, $G[V(C^*)]$ would not be bipartite and $9 \leq |C^*| < |C|$ (contradicting the minimality of $|C|$). □

Theorem 3.1. *A graph is in $\mathcal{G}_7$ if and only if every induced hamiltonian subgraph of order 7 or more is 3-connected and bipartite.*

Proof. First suppose $G \in \mathcal{G}_7$ and H is a hamiltonian subgraph of G of minimum order $h \geq 7$ such that H is not a 3-connected bipartite graph (arguing by contradiction). Lemma 3.3 implies $H \in \mathcal{G}_7$ is bipartite. Therefore, h is even and H is not 3-connected and so, by Lemma 3.2, $h \geq 10$. Suppose C is a cycle of H with minimum length $|C|$ satisfying $8 \leq |C| \leq h$.

Suppose for the moment that $|C| = 8$. Let C^* be a hamiltonian h-cycle of the 2-connected hamiltonian graph H. Since H is hamiltonian but not 3-connected, H has a separating set $\{a, b\} \subset V(H)$ that partitions C^* into two a-to-b subpaths π_1 and π_2. Since no chord of C^* can have one endpoint in the interior of each of π_1 and π_2, suppose $V(C) \subseteq V(\pi_1)$. Let a' and b' be, respectively, the neighbors of a and b along π_1. By Lemma 3.2, $H[V(C)]$ is 3-connected and bipartite with at least twelve edges, at least one of which is a chord e of C^* that is not incident with any of a, b, a', b'. The chord e and C^* form a cycle C_e^* that properly contains π_2 where $\{a, b\}$ is a separating set of $H[V(C_e^*)]$ and $|C_e^*| \geq 8$. However, $H[V(C_e^*)]$ would then be a bipartite but not 3-connected graph of order $|C_e^*|$ that satisfies $8 \leq |C_e^*| < |C^*| = h$ (contradicting the assumed minimality of h).

Next suppose that $|C| = 10$, say with $C = [v_1, \ldots, v_{10}]$. Since H is bipartite, every chord of C must be a 3-chord or a 5-chord. If C has a 3-chord e, then e and C would form an 8-cycle (contradicting the minimality of $|C| = 10$). Thus each chord of C is a 5-chord. Since $G \in \mathcal{G}_7$, cycle C' has four pairwise nonadjacent 5-chords, two of which would be 5-chords whose endpoints are not consecutive around C; without loss of generality, say these are v_1v_6 and v_4v_9. But then those two edges together with v_1v_2, v_2v_3, v_3v_4 and v_6v_7, v_7v_8, v_8v_9 would form an 8-cycle (again contradicting the minimality of $|C| = 10$).

Therefore, $|C| \geq 12$. Since $H \in \mathcal{G}_7$ is bipartite, C has an odd i-chord e. Chord e and cycle C would form a cycle C' that has $|C'| = |C| - i + 1 \geq 8$ if $i \in \{3, 5\}$ and that has $|C'| = i + 1 \geq 8$ if $i \geq 7$. Either way, $8 \leq |C'| < |C| = h$ (contradicting the minimality of $|C|$).

Conversely, suppose every induced hamiltonian subgraph of G with order 7 or more of G is 3-connected and bipartite. Suppose C is a cycle of G with $|C| \geq 7$ and $H = G[V(C)]$. Thus H is hamiltonian and bipartite, and so $|C| \geq 8$. Let H° be the subgraph of H that consists of just the chords of C. Since each vertex of the 3-connected graph H is on at least one edge of H°, it takes at least four vertices of H to cover all the edges of H°. Therefore, the König Property of bipartite graphs (Theorem 1.1.2 in [1]) implies that H° has at least four pairwise nonadjacent edges, and these are four pairwise nonadjacent chords of C. Therefore, $G \in \mathcal{G}_7$. □

4. The graph classes $\mathcal{G}_k$ for $k \geq 8$

Theorem 4.1 will show how the disparity of the classes $\mathcal{G}_4$, $\mathcal{G}_5$, $\mathcal{G}_6$ and $\mathcal{G}_7$ ends at $\mathcal{G}_8$. Corollary 4.1 will show that their noncomparability also ends at $\mathcal{G}_8$. The *circumference* of a graph is the length of a longest cycle in the graph.

Theorem 4.1. *If $k \geq 8$, then a graph is in $\mathcal{G}_k$ if and only if its circumference is less than k.*

Proof. Suppose $G \in \mathcal{G}_k$ where $k \geq 8$ and assume C is a cycle of G with minimum length $|C| \geq k$ (arguing by contradiction). Since $G \in \mathcal{G}_k$, the cycle C must have pairwise nonadjacent chords $e_1, \ldots, e_{k-3}$. By Lemma 3.1, $|C| \geq 2k-6$.

Suppose for the moment that $|C| = 2k-6$. If e is an i-chord of C, then e and C form a cycle of length $|C| - i + 1 = 2k-5-i$; thus $k \geq 8$ and the assumed minimality of $|C|$ imply that $i \notin \{2, \ldots, k-5\}$. Therefore, since every i-chord of C must have $2 \leq i \leq \lfloor |C|/2 \rfloor = k-3$, every chord of C must be a $(k-4)$-chord or a $(k-3)$-chord. If $e_1, \ldots, e_{k-3}$ are all $(k-4)$-chords, then $\{e_1, \ldots, e_{k-3}\}$ contains crossing $(k-4)$-chords ab and cd that partition C into subpaths $C[a,c]$, $C[b,c]$, $C[b,d]$, $C[a,d]$ with those endpoints such that $C[a,c]$ has length 1 and both $C[a,d]$ and $C[b,c]$ have length $k-5$. But then the paths $C[a,d]$ and $C[b,c]$ would combine with the edges ab and cd to form a cycle C' with $|C'| = 2k-8$, and so with $|C'| \geq k$ but $|C'| \not\geq 2k-6$ (contradicting Lemma 3.1). Therefore, $e_1, \ldots, e_{k-3}$ cannot all be $(k-4)$-chords. Similarly, $e_1, \ldots, e_{k-3}$ cannot all be $(k-3)$-chords (by the same argument, except now with $C[a,c]$ having length 2 and both $C[a,d]$ and $C[b,c]$ having length $k-5$). Therefore, $\{e_1, \ldots, e_{k-3}\}$ contains at least one $(k-4)$-chord ab and at least $(k-3)$-chord cd. Since $|C| = 2(k-3)$ implies that every vertex of C is on a unique chord in $\{e_1, \ldots, e_{k-3}\}$, there are crossing chords $e_i = ab$ and $e_j = cd$ such that (using the notation above) $C[a,c]$ has length 1 and $C[a,d]$ has length $k-4$ and $C[b,c]$ has length $k-5$. But then the paths $C[a,d]$ and $C[b,c]$ would combine with the edges ab and cd to form a cycle C' with $|C'| = 2k-7$, and so with $|C'| \geq k$ but $|C'| \not\geq 2k-6$ (contradicting Lemma 3.1). Therefore, $|C| \neq 2k-6$, and so $|C| \geq 2k-5$.

Next suppose that $|C| = 2k-5$. Note that $|C| = 2(k-3)+1$ implies that all but one of the vertices of C are on unique chords in $\{e_1, \ldots, e_{k-3}\}$. Arguing as in the preceding paragraph, every i-chord of C must have $i \notin \{2, \ldots, k-4\}$ and must have $i \leq k-3$. Thus, every chord of C must be a $(k-3)$-chord, and so C has crossing $(k-3)$-chords ab and cd where $C[a,c]$ has length 2 and both $C[a,d]$ and $C[b,c]$ have length $k-5$, leading to a cycle of length $2k-8$ (contradicting Lemma 3.1). Therefore, $|C| \neq 2k-5$, and so $|C| \geq 2k-4$.

Now suppose that $|C| = 2k-4$. Note that $|C| = 2(k-3)+2$ implies that all but two of the vertices of C are on unique chords in $\{e_1, \ldots, e_{k-3}\}$. Arguing as in the preceding paragraphs, every i-chord of C must have $i \notin \{2, \ldots, k-3\}$ and must have $i \leq k-2$. Thus, every chord of C must be a $(k-2)$-chord, and so C has crossing $(k-2)$-chords ab and cd where $C[a,c]$ has length 3 and both $C[a,d]$ and $C[b,c]$ have length $k-5$, leading to a cycle of length $2k-8$ (contradicting Lemma 3.1). Therefore, $|C| \neq 2k-4$, and so $|C| \geq 2k-3$.

Finally, suppose $|C| \geq 2k-3$ and let $l = |C|$. Every i-chord e of C must have $i \notin \{2, \ldots, l-k+1\}$ (otherwise e and C would form a cycle C' where $|C'| = l-i+1$ with $k \leq |C'| \leq l-1$, contradicting the assumed minimality of $|C|$). But this contradicts that every i-chord of C must have $2 \leq i \leq \lfloor |C|/2 \rfloor \leq \frac{l-1}{2} \leq l-k+1$ (the last inequality is equivalent to $l \geq 2k-3$). Therefore, $|C| \ngeq 2k-3$.

The preceding paragraphs show that $|C| \ngeq 2k-6$ (contradicting Lemma 3.1 and the assumption that $|C| \geq k$). □

Corollary 4.1. *The inclusion $\mathcal{G}_k \subseteq \mathcal{G}_l$ holds if and only if $8 \leq k \leq l$.*

Proof. If $8 \leq k \leq l$, then then $\mathcal{G}_k \subseteq \mathcal{G}_l$ by Theorem 4.1. If $k > l \geq 4$, then $k \geq 5$ and the l-cycle $C_l \in \mathcal{G}_k$ (by the definition of $\mathcal{G}_k$, since C_l contains no cycle of length k or more) and $C_l \notin \mathcal{G}_l$ (by the definition of $\mathcal{G}_l$, since C_l has no chords); thus $\mathcal{G}_k \not\subseteq \mathcal{G}_l$. The only things left to show are that $\mathcal{G}_k \not\subseteq \mathcal{G}_l$ when $k \in \{4, 5, 6, 7\}$ and $l > k$.

- $\mathcal{G}_4 \not\subseteq \mathcal{G}_l$ when $l > 4$: If G is formed from the l-cycle C_l with a distinguished vertex v by inserting all the $l-3$ possible chords that are incident with v, then $G \in \mathcal{G}_4$, and yet $G \notin \mathcal{G}_l$ by the definition of $\mathcal{G}_l$.
- $\mathcal{G}_5 \not\subseteq \mathcal{G}_l$ when $l > 5$: If $l = 6$ and $G = K_{2,2,2}$, then $G \in \mathcal{G}_5$, and yet $G \notin \mathcal{G}_l$ by the definition of $\mathcal{G}_l$. If $l \geq 7$ and $G = K_{s,s}$ with $s = \lceil l/2 \rceil$, then $G \in \mathcal{G}_5$, and yet $G \notin \mathcal{G}_l$ by Lemma 3.1 applied to a $(2s)$-cycle.
- $\mathcal{G}_6 \not\subseteq \mathcal{G}_l$ when $l > 6$: If $G = K_l$, then $G \in \mathcal{G}_6$ by Theorem 2.1, and yet $G \notin \mathcal{G}_l$ by Theorem 3.1 if $l = 7$ and by Theorem 4.1 if $l \geq 8$.
- $\mathcal{G}_7 \not\subseteq \mathcal{G}_l$ when $l > 7$: If $G = K_{s,s}$ with $s = \lceil l/2 \rceil$, then $G \in \mathcal{G}_7$ by Theorem 3.1, and yet $G \notin \mathcal{G}_l$ by Lemma 3.1 applied to a $(2s)$-cycle. □

5. The graph classes $\mathcal{G}_4 \cap \mathcal{G}_5$ and $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$

Since the chordal distance-hereditary graphs are precisely the *ptolemaic graphs* (see [1, 3]), a graph is in $\mathcal{G}_4 \cap \mathcal{G}_5$ if and only if every k-cycle has at least $\lfloor \frac{3}{2}(k-3) \rfloor$ chords. Reference [5] contains examples showing that $\lfloor \frac{3}{2}(k-3) \rfloor$ is *optimum for* $\mathcal{G}_4 \cap \mathcal{G}_5$ in that there exists a sequence $G_4^*, G_5^*, \ldots$ of graphs in $\mathcal{G}_4 \cap \mathcal{G}_5$ such that each G_k^* has a k-cycle with exactly $\lfloor \frac{3}{2}(k-3) \rfloor$ chords. This compares with a graph being in $\mathcal{G}_4$ if and only if every k-cycle has at least $k-3$ chords, where $k-3$ is optimum for $\mathcal{G}_4$.

Theorem 5.1 will similarly characterize $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$ in terms of a function $f(k)$ that gives the number of required chords for k-cycles—and is optimum

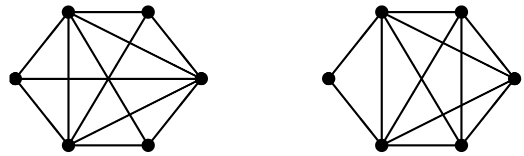

Figure 4: On the left, the graph $K_6 - K_3 \in \mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$; on the right, the graph $K_6 - K_{1,3} \in \mathcal{G}_4 \cap \mathcal{G}_5 - \mathcal{G}_6$.

for $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$—along with a second, more elegant characterization. But first, Corollary 5.1 will state the consequences of Theorem 2.1 for $\mathcal{G}_4 \cap \mathcal{G}_5$, and Lemma 5.1 will detail the structure of the graphs in $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$ based on their being chordal. The graphs shown in Figure 4 will be important in this section.

Corollary 5.1. *A graph in $\mathcal{G}_4 \cap \mathcal{G}_5$ is also in $\mathcal{G}_6$ if and only if every induced hamiltonian subgraph of order 6 or more* contains *a $K_6 - K_3$ subgraph.*

Proof. This follows from Theorem 2.1 and the observation that $K_6 - K_3$ and its order-6 supergraphs are precisely the order-6 graphs in $\mathcal{G}_4 \cap \mathcal{G}_5$. ☐

A vertex is *simplicial* if its neighborhood induces a complete subgraph, and one common characterization of being chordal is that every induced subgraph contains a simplicial vertex. In fact, a chordal graph G that is not complete contains nonadjacent simplicial vertices; see [1, 6]. A vertex is *universal* if it is adjacent to all the other vertices in the graph.

Lemma 5.1. *Suppose $G \in \mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$ is hamiltonian with order $n \geq 6$. Let $G_n = G$ and, for each $i \in \{3, \dots, n-1\}$, define G_i by deleting a simplicial vertex from G_{i+1}.*

Suppose $n \geq 6$ is even and $f(n) = (3n^2 - 10n)/8$. If $3 \leq i \leq \frac{n}{2}$, then $G_i \cong K_i$ and G_i has i universal vertices, and every i-cycle of G_i has $(i^2 - 3i)/2$ chords. If $\frac{n}{2} \leq i \leq n$, then G_i has at least $\frac{n}{2}$ universal vertices, and every i-cycle of G_i has at least $f(n) - \frac{n-2}{2}(n-i)$ chords. In particular, every n-cycle of G has at least $f(n)$ chords.

Suppose $n \geq 7$ is odd and $f(n) = (3n^2 - 12n + 9)/8$. If $3 \leq i \leq \frac{n-1}{2}$, then $G_i \cong K_i$ and G_i has i universal vertices, and every i-cycle of G_i has $(i^2 - 3i)/2$ chords. If $\frac{n-1}{2} \leq i \leq n$, then G_i has at least $\frac{n-1}{2}$ universal vertices, and every i-cycle of G_i has at least $f(n) - \frac{n-1}{2}(n-2) + \frac{n-3}{2}i$ chords. In particular, every n-cycle of G has at least $f(n)$ chords.

Proof. Suppose $G \in \mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$ is hamiltonian with order $n \geq 6$, and suppose the function $f(n)$ and the subgraphs G_i are as in the statement of the theorem, noting that these G_i exist precisely because G is chordal. In fact, since each G_{i-1} results from G_i by deleting a simplicial vertex, all the graphs $G_3, \dots, G_n$ are chordal and hamiltonian.

Suppose $3 < i \leq n$ and C is an i-cycle of G_i and v is the simplicial vertex of G_i for which $G_{i-1} = G_i - v$. Let u and w be the two neighbors of v along C and let C' be the $(i-1)$-cycle with edge set $E(C) \cup \{uw\} - \{uv, vw\}$. Let $v' \notin \{u, v, w\}$ be a second simplicial vertex of G_i with neighbors u' and w' along C, and let C'' be the $(i-1)$-cycle with edge set $E(C) \cup \{u'w'\} - \{u'v', v'w'\}$. Let c' denote the number of chords that C' has, and let u'' denote the number of universal vertices $x \neq v$ that $G[V(C'')]$ has, noting that xv will be an edge or chord of C'' but not of C'. Thus C has at least $c' + (u'' - 2) + 1$ chords, where the -2 term reflects that u or w might be a universal vertex of C'' without uv or vw being a chord of C'', and the $+1$ term reflects that uw is a chord of C that is neither a chord of C' nor incident with v. Therefore, C has at least $c' + u'' - 1$ chords.

Case 1. $n \geq 6$ is even. Argue by induction on $i \in \{3, \dots, n\}$. For the $i = 3$ basis, $G_i \cong K_3$ and G_i has $i = 3$ universal vertices, and every 3-cycle of K_3 has $(3^2 - 3 \cdot 3)/2 = 0$ chords.

If $3 < i \leq \frac{n}{2}$, then the induction hypothesis implies that C has at least $c' + u'' - 1 \geq [(i-1)^2 - 3(i-1)]/2 + (i-1) - 1 = (i^2 - 3i)/2$ chords; thus G_i is complete and has i universal vertices.

If $i = \frac{n}{2}$, then the preceding paragraph implies that C has at least $(i^2 - 3i)/2 = (n^2 - 6n)/8 = f(n) - \frac{n-2}{2}(n - i)$ chords, and G_i has at least $i = \frac{n}{2}$ universal vertices.

If $\frac{n}{2} < i \leq n$, then the induction hypothesis implies that C has at least $c' + u'' - 1 \geq [f(n) - \frac{n-2}{2}(n - (i-1))] + \frac{n}{2} - 1 = f(n) - \frac{n-2}{2}(n - i)$ chords, and G_i has at least $u'' = \frac{n}{2}$ universal vertices.

In particular, if $i = n$, then the preceding paragraph implies that C has at least $f(n) - \frac{n-2}{2}(n - n) = f(n)$ chords.

Case 2. $n \geq 7$ is odd. Argue by induction on $i \in \{3, \dots, n\}$. For the $i = 3$ basis, $G_i \cong K_3$ and G_i has $i = 3$ universal vertices, and every 3-cycle of K_3 has $(3^2 - 3 \cdot 3)/2 = 0$ chords.

If $3 < i \leq \frac{n-1}{2}$, then the induction hypothesis implies that C has at least $c' + u'' - 1 \geq [(i-1)^2 - 3(i-1)]/2 + (i-1) - 1 = (i^2 - 3i)/2$ chords; thus G_i is complete and has i universal vertices.

If $i = \frac{n-1}{2}$, then the preceding paragraph implies that C has at least $(i^2 - 3i)/2 = (n^2 - 8n + 7)/8 = f(n) - \frac{n-1}{2}(n-2) + \frac{n-3}{2}i$ chords, and G_i has at least $i = \frac{n-1}{2}$ universal vertices.

If $\frac{n-1}{2} < i \le n$, then the induction hypothesis implies that C has at least $c' + u'' - 1 \ge [f(n) - \frac{n-1}{2}(n-2) + \frac{n-3}{2}(i-1)] + \frac{n-1}{2} - 1 = f(n) - \frac{n-1}{2}(n-2) + \frac{n-3}{2}i$ chords, and G_i has at least $u'' = \frac{n-1}{2}$ universal vertices.

In particular, if $i = n$, then the preceding paragraph implies that C has at least $u = \frac{n-1}{2}$ universal vertices. Thus G has an order-n subgraph H that consists of u degree-n vertices and $n-u$ independent degree-u vertices, and so H will have $\binom{u}{2} + u(n-u) = (n^2 - 4n + 1)/8$ edges. Since $u < n - u$ and H is spanned by a $K_{u,n-u}$ subgraph, H is not hamiltonian. Since G is hamiltonian, G must have at least one additional edge beyond H, and so G has at least $(n^2 - 4n + 1)/8 + 1 = f(n) + n$ edges. Therefore, the n-cycle C has at least $f(n)$ chords. □

Theorem 5.1. *Each of the following is equivalent to being in $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$:*

(1) *For every k, every k-cycle C has at least $\frac{1}{2}\lfloor\frac{k}{2}\rfloor\lfloor\frac{3}{2}(k-3)\rfloor$ chords, and $G[V(C)] \not\cong K_6 - K_{1,3}$.*

(2) *For every k, every induced hamiltonian subgraph of order k has at least $\lfloor\frac{k}{2}\rfloor$ universal vertices.*

Proof. Define $f(k) = \frac{1}{2}\lfloor\frac{k}{2}\rfloor\lfloor\frac{3}{2}(k-3)\rfloor$ and notice that $f(k) = (3k^2 - 10k)/8$ and $\lfloor\frac{k}{2}\rfloor = \frac{k}{2}$ when k is even, and that $f(k) = (3k^2 - 12k + 9)/8$ and $\lfloor\frac{k}{2}\rfloor = \frac{k-1}{2}$ when k is odd.

The one hamiltonian graph of order $k = n = 3$ in $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$ is K_3, which has $3 > \lfloor\frac{3}{2}\rfloor$ universal vertices, and every hamiltonian cycle has $0 = f(3)$ chords. The two hamiltonian graphs of order $k = n = 4$ in $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$ are $K_{1,1,2}$ and K_4, which have, respectively, 2 or 4 (so at least $\lfloor\frac{4}{2}\rfloor$) universal vertices, and every hamiltonian cycle has 1 or 2 (so at least $f(4) = 1$) chords. The three hamiltonian graphs of order $k = n = 5$ in $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$ are $K_5 - P_3$ and $K_5 - P_2$ and K_5, which have, respectively, 2 or 3 or 5 (so at least $\lfloor\frac{5}{2}\rfloor$) universal vertices, and every hamiltonian cycle has 3 or 4 or 5 (so at least $f(5) = 3$) chords. By Lemma 5.1, the hamiltonian graphs of order $k = n \ge 6$ in $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$ have at least $\lfloor\frac{k}{2}\rfloor$ universal vertices, and every hamiltonian cycle has at least $f(k)$ chords. By Corollary 5.1, $G[V(C)] \not\cong K_6 - K_{1,3}$. Therefore, every graph in $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$ satisfies both conditions (1) and (2).

Conversely, suppose a graph G satisfies condition (1) and has a k-cycle C. If $k \ge 4$, then C has at least $f(k) \ge k - 3$ chords, and so $G \in \mathcal{G}_4$. If $k \ge 5$, then C has at least $f(k) \ge \lfloor\frac{3}{2}(k-3)\rfloor$ chords, and so $G \in \mathcal{G}_5$.

To show $G \in \mathcal{G}_6$, assume C has minimum length $k \ge 6$ such that C does not have three pairwise nonadjacent chords (arguing by contradiction). If $k = 6$, then C has at least $f(6) = 6$ chords and $G[V(C)] \not\cong K_6 - K_{1,3}$, and so Corollary 5.1 implies that $G[V(C)]$ is one of the four supergraphs of $K_6 - K_3$;

but then C would have three pairwise nonadjacent chords (contradicting that C does not have such chords). If $k \geq 7$, let v be a simplicial vertex of $G[V(C)]$ with neighbors u and w along C and let C' be the $(k-1)$-cycle with edge set $E(C) \cup \{uw\} - \{uv, vw\}$. By the assumed minimality of the length of C, the cycle C' does have three pairwise nonadjacent chords, and these are also chords of C (contradicting that C does not have such chords). Thus $G \in \mathcal{G}_6$ and, therefore, $G \in \mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$.

Finally, suppose a graph G satisfies condition (2) and has a k-cycle C. If $k \geq 4$, then $G[V(C)]$ has at least $\lfloor \frac{k}{2} \rfloor \geq 2$ universal vertices v_1 and v_2, and so C has at least the one chord v_1v_2; thus $G \in \mathcal{G}_4$. If $k \geq 5$, then $G[V(C)]$ has at least $\lfloor \frac{k}{2} \rfloor \geq 2$ universal vertices v_1 and v_2, and so C has nonadjacent chords v_1w_1 and v_2w_2 where w_1, w_2 are distinct vertices in $V(C) - \{v_1, v_2\}$; thus $G \in \mathcal{G}_5$. If $k \geq 6$, then $G[V(C)]$ has at least $\lfloor \frac{k}{2} \rfloor \geq 3$ universal vertices v_1, v_2, v_3, and so C has nonadjacent chords v_1w_1, v_2w_2, v_3w_3 where w_1, w_2, w_3 are distinct vertices in $V(C) - \{v_1, v_2, v_3\}$; thus $G \in \mathcal{G}_6$. Therefore, $G \in \mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$. □

The function $\frac{1}{2}\lfloor \frac{k}{2} \rfloor \lfloor \frac{3}{2}(k-3) \rfloor$ can be shown to be optimum for $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$ by constructing a sequence $G_4^*, G_5^*, \ldots$ of graphs as follows: If $k \geq 4$ is even, define G_k^* to be the order-k graph that has $k/2$ vertices of degree k and $k/2$ vertices of degree $k/2$. If $k \geq 5$ is odd, define G_k^* to be the order-k graph obtained by inserting one additional edge into the order-k graph that has $(k-1)/2$ vertices of degree k and $(k+1)/2$ vertices of degree $(k-1)/2$. Each G_k^* is in $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$ and has a hamiltonian cycle with exactly $\frac{1}{2}\lfloor \frac{k}{2} \rfloor \lfloor \frac{3}{2}(k-3) \rfloor$ chords.

In conclusion, Corollary 5.2 will observe that, unsurprisingly, the graph class $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6 \cap \mathcal{G}_7$ is rather ridiculously restricted.

Corollary 5.2. *A graph in $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$ is also in $\mathcal{G}_7$ if and only if it has no cycles of length 7 or more.*

Proof. By Theorem 5.1, every hamiltonian graph of order k in $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$ has at least $\lfloor k/2 \rfloor \geq 1$ universal vertices. By Theorem 3.1, every hamiltonian graph or order 7 or more in $\mathcal{G}_7$ is bipartite, and so cannot have universal vertices. □

References

[1] A. Brandstädt, V. B. Le and J. P. Spinrad (1999). *Graph Classes: A Survey.* Society for Industrial and Applied Mathematics, Philadelphia. MR1686154

[2] E. Howorka (1977). A characterization of distance-hereditary graphs. *Quart. J. Math. Oxford (Ser. 2)* **28** 417–420. MR0485544

[3] E. Howorka (1981). A characterization of ptolemaic graphs. *J. Graph Theory* **5** 323–331. MR0625074

[4] T. A. McKee (2005). Requiring chords in cycles. *Discrete Math.* **297** 182–189. MR2159443

[5] T. A. McKee (2011). When every k-cycle has at least $f(k)$ chords. *J. Graph Theory* **68** 137–147. MR2833956

[6] T. A. McKee and F. R. McMorris (1999). *Topics in Intersection Graph Theory*. Society for Industrial and Applied Mathematics, Philadelphia. MR1672910

Terry A. McKee
Wright State University
Dayton, Ohio 45435
USA
E-mail address: terry.mckee@wright.edu

Received April 18, 2013