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A combinatorial proof of an infinite version of the Hales–Jewett theorem*

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We provide a combinatorial proof of an infinite extension of the Hales–Jewett theorem due to T. Carlson and independently due to H. Furstenberg and Y. Katznelson.

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1. Introduction

The aim of this note is to provide a new combinatorial proof of a well-known infinitary extension of the Hales–Jewett theorem. To state it, we need first to recall the relevant terminology. Let A be an *alphabet* (i.e. any non-empty set). The elements of A will be called *letters*. By $W(A)$ we denote the set of *constant words over* A, that is the set of all finite sequences with elements in A including the empty sequence. For $N \in \mathbb{N}$, by A^N, we denote all finite sequences from A, consisting of N letters. We also fix an element $x \notin A$ which will be regarded as a *variable*. A *variable word over* A is an element in $W(A \cup \{x\}) \setminus W(A)$. The variable words will be denoted by $s(x), t(x), w(x)$, etc. Given a variable word $s(x)$ and $a \in A$, by $s(a)$ we denote the constant word in $W(A)$ resulting from the substitution of the variable x with the letter a. Let $q \in \mathbb{N}$ with $q \geq 1$, then a *q-coloring* of a set X is any map $c : X \to \{1, \ldots, q\}$. A subset Y of X will be called *monochromatic*, if there exists $1 \leq i \leq q$ such that $c(y) = i$, for all $y \in Y$. Finally, for every finite set X, by $|X|$ we denote its cardinality.

We recall the following fundamental result in Ramsey Theory, due to A. Hales and R. Jewett [8].

Theorem 1. *For every positive integers p, q there exists a positive integer $HJ(p,q)$ with the following property. For every finite alphabet A with $|A| = p$, every $N \geq HJ(p,q)$ and every q-coloring of A^N there exists a variable word $w(x)$ of length N such that the set $\{w(a) : a \in A\}$ is monochromatic.*

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The Hales–Jewett theorem gave birth to a whole new branch of research concerning extensions of it in the context of both finite and infinite alphabets (see [3–6, 10, 14, 15]). For an exposition of these results the reader can also refer to [7, 11, 13, 16].

The first theorem that we will prove is due to T. Carlson [3] and independently due to H. Furstenberg and Y. Katznelson [5] and is the following.

Theorem 2. *Let A be a finite alphabet. Then for every finite coloring of $W(A)$ there exists a sequence $(t_n(x))_{n=0}^{\infty}$ of variable words over A such that for every $n \in \mathbb{N}$ and every $m_0 < m_1 < \cdots < m_n$, the words of the form $t_{m_0}(a_0)t_{m_1}(a_1)\cdots t_{m_n}(a_n)$ with $a_i \in A$ for all $0 \leq i \leq n$ are of the same color.*

Our approach for Theorem 2 can be extended in order to provide a proof for a stronger version of it which concerns infinite increasing sequences of finite alphabets and is the following.

Theorem 3. *Let $(A_n)_{n=0}^{\infty}$ be an increasing sequence of finite alphabets and let $A = \cup_{n\in\mathbb{N}} A_n$. Then for every finite coloring of $W(A)$ there exists a sequence $(t_n(x))_{n=0}^{\infty}$ of variable words over A such that for every $n \in \mathbb{N}$ and every $m_0 < m_1 < \cdots < m_n$, the words of the form $t_{m_0}(a_0)t_{m_1}(a_1)\cdots t_{m_n}(a_n)$ with $a_0 \in A_{m_0}, a_1 \in A_{m_1}, \ldots, a_n \in A_{m_n}$ are of the same color.*

Let us point out that there exist easy counterexamples that show that a direct extension of Theorem 2 for an infinite alphabet A is false. Theorem 3 is a consequence of a more general result of T. Carlson (see [3, Theorem 15]). The original proofs of Theorems 2 and 3 are based on topological as well as algebraic notions of the Stone–Čech compactification of the related structures. Our approach is strictly combinatorial and relies on the classical Hales–Jewett theorem. It has its origins in the proof of Hindman's theorem [9] due to J. E. Baumgartner [1] and is close in spirit with the proof of Carlson–Simpson's theorem [4]. In particular, a proof of a weaker version of Theorem 2, given by R. McCutcheon in [13, §2.3], was the motivation for this note.

Clearly, Theorem 2 is a consequence of Theorem 3. Although the proofs of both theorems follow similar arguments, Theorem 3 is more demanding and quite more technical. For this reason and in order to make the presentation more clear, we have decided to start with a detailed exposition of Theorem 2 and then proceed to Theorem 3. The present note is an updated and extended version of part of [12].

2. Proof of Theorem 2

2.1. Preliminaries

In this subsection we introduce some notation and terminology that we will use for the proof of Theorem 2. For the following, we fix a finite alphabet A. Let $V(A)$ be the set of all *variable words* (*over* A). By $V^{<\infty}(A)$ (resp. $V^{\infty}(A)$) we denote the set of all finite (resp. infinite) sequences of variable words. Also let $V^{\leq\infty}(A) = V^{<\infty}(A) \cup V^{\infty}(A)$. Generally, the elements of $V^{\leq\infty}(A)$ will be denoted by $\vec{s}, \vec{t}, \vec{w}$, etc. Also by $\mathbb{N} = \{0, 1, \ldots\}$ we denote the set of all non negative integers.

2.1.1. Constant and variable span of a sequence of variable words over *A* Let $m \in \mathbb{N}$ and $(s_n(x))_{n=0}^m \in V^{<\infty}(A)$.

(a) The *constant span of* $(s_n(x))_{n=0}^m$, denoted by $\langle (s_n(x))_{n=0}^m \rangle_c$, is defined to be the set

$$\bigcup_{n=0}^{m} \big\{ s_{l_0}(a_0) \cdots s_{l_n}(a_n) : 0 \leq l_0 < \cdots < l_n \leq m, a_0, \ldots, a_n \in A \big\}.$$

(b) The *variable span of* $(s_n(x))_{n=0}^m$, denoted by $\langle (s_n(x))_{n=0}^m \rangle_v$, is defined to be the set

$$V(A) \cap \bigcup_{n=0}^{m} \big\{ s_{l_0}(a_0) \cdots s_{l_n}(a_n) : 0 \leq l_0 < \cdots < l_n \leq m, a_0, \ldots, a_n \in A \cup \{x\} \big\}.$$

The above notation is naturally extended to infinite sequences of variable words as follows. Let $(s_n(x))_{n=0}^\infty \in V^{\infty}(A)$. The constant span of $(s_n(x))_{n=0}^\infty$ is the set

$$\langle (s_n(x))_{n=0}^\infty \rangle_c = \{ s_{l_0}(a_0) \cdots s_{l_n}(a_n) : n \in \mathbb{N}, 0 \leq l_0 < \cdots < l_n, a_0, \ldots, a_n \in A \}$$

and the variable span of $(s_n(x))_{n=0}^\infty$, denoted by $\langle (s_n(x))_{n=0}^\infty \rangle_v$, is the set

$$V(A) \cap \big\{ s_{l_0}(a_0) \cdots s_{l_n}(a_n) : n \in \mathbb{N}, 0 \leq l_0 < \cdots < l_n, a_0, \ldots, a_n \in A \cup \{x\} \big\}.$$

In the following we will also write $\langle \vec{s} \rangle_c$ (resp. $\langle \vec{s} \rangle_v$) to denote the constant (resp. variable) span of an $\vec{s} \in V^{\leq\infty}(A)$.

2.1.2. Extracted subsequences of a sequence of variable words We start with the following definition.

Definition 4. Let $\vec{s} = (s_n(x))_{n=0}^{\infty} \in V^{\infty}(A)$.

(a) Let $l \in \mathbb{N}$ and $\vec{t} = (t_n(x))_{n=0}^{l} \in V^{<\infty}(A)$. We say that $\vec{t}$ is a (finite) extracted subsequence of $\vec{s}$ if there exist $0 = m_0 < \cdots < m_{l+1}$ such that

$$t_i(x) \in \langle (s_n(x))_{n=m_i}^{m_{i+1}-1} \rangle_v,$$

for all $0 \leq i \leq l$.

(b) Let $\vec{t} = (t_n(x))_{n=0}^{\infty}$. We say that $\vec{t}$ is an (infinite) extracted subsequence of $\vec{s}$ if for every $l \in \mathbb{N}$ the sequence $(t_n(x))_{n=0}^{l}$ is an finite extracted subsequence of $\vec{s}$.

In the following we will write $\vec{t} \leq \vec{s}$ whenever $\vec{t} \in V^{\leq\infty}(A)$, $\vec{s} \in V^{\infty}(A)$ and $\vec{t}$ is an extracted subsequence of $\vec{s}$. The next fact follows easily from the above definitions.

Fact 5. *Let $\vec{s} \in V^{\infty}(A)$.*

(i) If $\vec{t} \in V^{\leq\infty}(A)$ with $\vec{t} \leq \vec{s}$ then $\langle \vec{t} \rangle_c \subseteq \langle \vec{s} \rangle_c$ and $\langle \vec{t} \rangle_v \subseteq \langle \vec{s} \rangle_v$.
(ii) If $\vec{w} \in V^{\leq\infty}(A)$ and $\vec{t} \in V^{\infty}(A)$ with $\vec{w} \leq \vec{t} \leq \vec{s}$ then $\vec{w} \leq \vec{s}$.

2.1.3. The notion of large families The next definition is crucial for the proof of Theorem 2.

Definition 6. Let $E \subseteq W(A)$ and $\vec{s} \in V^{\infty}(A)$. Then E will be called *large* in $\vec{s}$ if $E \cap \langle \vec{w} \rangle_c \neq \emptyset$, for every infinite extracted subsequence $\vec{w}$ of $\vec{s}$.

We close this subsection with some properties of large families.

Fact 7. *Let $E \subseteq W(A)$ and $\vec{s} = (s_n(x))_{n=0}^{\infty} \in V^{\infty}(A)$ such that E is large in $\vec{s}$. Then for every $\vec{t} \leq \vec{s}$ we have that E is large in $\vec{t}$.*

Moreover, arguing by contradiction, we obtain the following.

Fact 8. *Let $E \subseteq W(A)$ and $\vec{s} \in V^{\infty}(A)$ such that E is large in $\vec{s}$. Let $r \geq 2$ and let $E = \bigcup_{i=1}^{r} E_i$. Then there exist $1 \leq i \leq r$ and $\vec{t} \leq \vec{s}$ such that E_i is large in $\vec{t}$.*

For the following fact, we will need the next definition.

Definition 9. Let $m \in \mathbb{N}$ and $(s_n(x))_{n=0}^{m} \in V^{<\infty}(A)$. We set

$$[(s_n(x))_{n=0}^{m}]_c = \left\{ s_0(a_0) \cdots s_m(a_m) : a_0, \ldots, a_m \in A \right\}$$

and

$$[(s_n(x))_{n=0}^m]_v = V(A) \cap \big\{s_0(a_0)\cdots s_m(a_m) : a_0,\dots,a_m \in A\cup\{x\}\big\}.$$

The next fact is a direct application of the Hales–Jewett theorem. In a sense, it is a strengthening of Definition 6. More precisely, we have the following.

Fact 10. *Let $E \subseteq W(A)$ and $\vec{s} = (s_n(x))_{n=0}^\infty \in V^\infty(A)$ such that E is large in $\vec{s}$. Then there exist $m \in \mathbb{N}$ and $w(x) \in \langle (s_n(x))_{n=0}^m\rangle_v$ such that $\{w(a) : a \in A\} \subseteq E$.*

Proof. Assume to the contrary that the conclusion fails. By induction we construct a sequence $\vec{w} = (w_n(x))_{n=0}^\infty \leq \vec{s}$ such that $\langle\vec{w}\rangle_c \subseteq E^c$ which is a contradiction since E is large in $\vec{s}$. The general inductive step of the construction is as follows. Let $n \geq 1$ and assume that $(w_i(x))_{i=0}^{n-1} \leq \vec{s}$ and $\langle (w_i(x))_{i=0}^{n-1}\rangle_c \subseteq E^c$. Let $n_0 \geq 1$ be the least integer satisfying $w_0(x)\cdots w_{n-1}(x) \in \langle (s_i(x))_{i=0}^{n_0-1}\rangle_v$. We set

$$N = HJ\big(|A|, 2^{(|A|+1)^n}\big).$$

To each $w \in [(s_{n_0+i}(x))_{i=0}^{N-1}]_c$ we assign the set $\big\{uw : u \in \langle (w_i(x))_{i=0}^{n-1}\rangle_c \cup \{\emptyset\}\big\}$. It is easy to see that $|\langle (w_i(x))_{i=0}^{n-1}\rangle_c \cup\{\emptyset\}| \leq (|A|+1)^n$. Therefore, since either $uw \in E$ or $uw \in E^c$, the above correspondence induces a $2^{(|A|+1)^n}$-coloring of the set $[(s_{n_0+i}(x))_{i=0}^{N-1}]_c$. By the choice of N, there exists a variable word $w(x) \in [(s_{n_0+i}(x))_{i=0}^{N-1}]_v$ such that for each $u \in \langle (w_i(x))_{i=0}^{n-1}\rangle_c \cup \{\emptyset\}$, the set $\{uw(a) : a \in A\}$ either is included in E or disjoint from E. By our assumption, there is no $u \in \langle (w_i(x))_{i=0}^{n-1}\rangle_c \cup \{\emptyset\}$ satisfying the first alternative. So setting $w_n(x) = w(x)$ we easily see that $(w_i(x))_{i=0}^{n} \leq \vec{s}$ and $\langle (w_i(x))_{i=0}^{n}\rangle_c \subseteq E^c$. The inductive step of the construction of $\vec{w}$ is complete and as we have already mentioned in the beginning of the proof this leads to a contradiction. □

2.2. The main arguments

We pass now to the core of the proof. We will need the next definition.

Definition 11. Let E and F be non empty subsets of $W(A)$. We define

$$E_F = \{z \in W(A) : wz \in E \text{ for every } w \in F\}.$$

Lemma 12. *Let $E \subseteq W(A)$ and $\vec{s} = (s_n(x))_{n=0}^{\infty} \in V^{\infty}(A)$ such that E is large in $\vec{s}$. Then there exist $m \geq 1$, $w(x) \in \langle (s_n(x))_{n=0}^{m-1} \rangle_v$ and $\vec{t} \in V^{\infty}(A)$ with $\vec{t} \leq (s_n(x))_{n=m}^{\infty}$ such that if we set $F = \{w(a) : a \in A\}$ then $E \cap E_F$ is large in $\vec{t}$.*

Proof. We start with the following claim.

Claim 1. *There exists $n_0 \in \mathbb{N}$ such that for every $z \in \langle (s_i(x))_{i=n_0+1}^{\infty} \rangle_c$ there exists $w(x)$ in $\langle (s_i(x))_{i=0}^{n_0} \rangle_v$ such that $\{w(a)z : a \in A\} \subseteq E$.*

Proof of Claim 1. Assume that the claim is not true. Then for every $n \in \mathbb{N}$ there exists $z \in \langle (s_i(x))_{i=n+1}^{\infty} \rangle_c$ such that for every $w(x) \in \langle (s_i(x))_{i=0}^{n} \rangle_v$, the set $\{w(a)z : a \in A\}$ is not contained in E. Using this assumption we easily find a strictly increasing sequence $(k_n)_{n=0}^{\infty}$ in $\mathbb{N}$ with $k_0 = 0$ and a sequence $(z_n)_{n=0}^{\infty}$ in $W(A)$ such the following are satisfied.

- (i) For every $n \in \mathbb{N}$, we have $z_n \in \langle (s_i(x))_{i=k_n+1}^{k_{n+1}-1} \rangle_c$.
- (ii) For every $n \in \mathbb{N}$ and every variable word $u(x) \in \langle (s_i(x))_{i=k_0}^{k_n} \rangle_v$ we have that $\{u(a)z_n : a \in A\} \nsubseteq E$.

For every $n \in \mathbb{N}$, we set $v_n(x) = s_{k_n}(x)z_n$.

By (i) we get that $(s_{k_0}(x)z_0, \ldots, s_{k_n}(x)z_n) \leq \vec{s}$, for all $n \in \mathbb{N}$ and therefore, $(v_n(x))_{n=0}^{\infty} \leq \vec{s}$. Moreover, notice that every $w(x) \in \langle (v_n(x))_{n=0}^{\infty} \rangle_v$ is of the form $w(x) = u(x)z_n$, for some unique $n \in \mathbb{N}$ and

$$u(x) \in \langle (v_0(x), \ldots, v_{k_{n-1}}(x), s_{k_n}(x) \rangle_v.$$

Hence, since $\langle (v_0(x), \ldots, v_{k_{n-1}}(x), s_{k_n}(x) \rangle_v \subseteq \langle (s_i(x))_{i=k_0}^{k_n} \rangle_v$, by (ii) we get that $\{w(a) : a \in A\} \nsubseteq E$, for every $w(x) \in \langle (v_n(x))_{n=0}^{\infty} \rangle_v$. But since $(v_n(x))_{n=0}^{\infty} \leq \vec{s}$, E is large in $(v_n(x))_{n=0}^{\infty}$ and so by Fact 10 we arrive to a contradiction. □

We set $m = n_0 + 1$. Also, let $L = \langle (s_n(x))_{n=0}^{m-1} \rangle_v$ and for every $w(x) \in L$, let $F(w(x)) = \{w(a) : a \in A\}$.

By Claim 1, we have that $\langle (s_i(x))_{i=m}^{\infty} \rangle_c \subseteq \bigcup_{w(x) \in L} E_{F(w(x))}$ and therefore,

$$E \cap \langle (s_i(x))_{i=m}^{\infty} \rangle_c \subseteq \bigcup_{w(x) \in L} E \cap E_{F(w(x))}.$$

Hence, $\bigcup_{w(x) \in L} E \cap E_{F(w(x))}$ is large in $(s_i(x))_{i=m}^{\infty}$. So, by Fact 8, there exist $w(x) \in L$ and $\vec{t} \leq (s_n(x))_{n=m}^{\infty}$ such that $E \cap E_{F(w(x))}$ is large in $\vec{t}$, as desired. □

Lemma 13. *Let $E \subseteq W(A)$ and $\vec{s} \in V^\infty(A)$ such that E is large in $\vec{s}$. Then there exists a sequence $(w_n(x))_{n=0}^\infty \leq \vec{s}$ such that setting $F_n = \langle (w_i(x))_{i=0}^n \rangle_c$ we have that $E \cap E_{F_n}$ is large in $(w_i(x))_{i=n+1}^\infty$, for all $n \in \mathbb{N}$.*

Proof. Iterating Lemma 12 we obtain a sequence $(w_n(x))_{n=0}^\infty$ of variable words, a sequence $(\vec{s}_n)_{n=0}^\infty$ in $V^\infty(A)$ with $\vec{s}_0 = \vec{s}$ and a sequence $(m_n)_{n=0}^\infty$ in $\mathbb{N}$, with $m_0 = 0$ and $m_n \geq 1$ for all $n \geq 1$, such that setting $\vec{s}_n = (s_i^{(n)}(x))_{i=0}^\infty$ then the following are satisfied.

(i) $w_n(x) \in \langle (s_i^{(n)}(x))_{i=0}^{m_{n+1}-1} \rangle_v$.
(ii) $\vec{s}_{n+1}$ is an extracted subsequence of $(s_i^{(n)}(x))_{i=m_{n+1}}^\infty$.
(iii) The set $E \cap E_{F_n}$ is large in $\vec{s}_{n+1}$.

The above construction is straightforward; we only mention that for the proof of (iii) we use the following identity.

$$(E \cap E_{F_n}) \cap (E \cap E_{F_n})_{\langle w_{n+1}(x) \rangle_c} = E \cap E_{F_{n+1}}.$$

By conditions (i) and (ii) we easily see that $\vec{w} \leq \vec{s}$ and moreover, for every $n \in \mathbb{N}$, $(w_i(x))_{i=n}^\infty \leq \vec{s}_n$. Hence by condition (iii) and Fact 7 we obtain that $E \cap E_{F_n}$ is large in $(w_i(x))_{i=n+1}^\infty$, for all $n \in \mathbb{N}$. □

Corollary 14. *Let $E \subseteq W(A)$ and $\vec{s} \in V^\infty(A)$ such that E is large in $\vec{s}$. Then there exists an extracted subsequence $\vec{t} = (t_n(x))_{n=0}^\infty$ of $\vec{s}$ such that $\langle \vec{t} \rangle_c \subseteq E$.*

Proof. Let $\vec{w} = (w_n(x))_{n=0}^\infty$ be the sequence obtained in Lemma 13. Since E is large in $\vec{s}$ and $\vec{w} \leq \vec{s}$, we obtain that E is large in $\vec{w}$ and therefore, by Fact 10, we have that there exist $r_1 \geq 1$ and $t_0(x) \in \langle (w_i(x))_{i=0}^{r_1-1} \rangle_v$ such that $\{t_0(a) : a \in A\} \subseteq E$. We set $G_0 = \langle t_0(x) \rangle_c$ and $F = \langle (w_i(x))_{i=0}^{r_1-1} \rangle_c$. By Lemma 13, we have that $E \cap E_F$ is large in $(w_i(x))_{i=r_1}^\infty$. Since $G_0 \subseteq F$ we get that $E_F \subseteq E_{G_0}$ and hence, $E \cap E_{G_0}$ is large $(w_i(x))_{i=r_1}^\infty$. Using again Fact 10, applied for $E \cap E_{G_0}$, we find $r_2 > r_1$ and a variable word $t_1(x) \in \langle (w_i(x))_{i=r_1}^{r_2-1} \rangle_v$ such that $\langle t_1(x) \rangle_c \subseteq E \cap E_{G_0}$. We set $G_1 = \langle (t_0(x), t_1(x)) \rangle_c$ and we notice that $G_1 \subseteq E$ and also that $E \cap E_{G_1}$ is large in $(w_i(x))_{i=r_2}^\infty$. Proceeding similarly, we construct a sequence $\vec{t} = (t_n(x))_{n=0}^\infty \leq \vec{w} \leq \vec{s}$ such that $\langle (t_i(x))_{i=0}^n \rangle_c \subseteq E$, for all $n \in \mathbb{N}$. Hence, $\vec{t} \leq \vec{s}$ and $\langle \vec{t} \rangle_c \subseteq E$, as desired. □

Proof of Theorem 2. Let $r \geq 2$ and let $W(A) = \cup_{i=1}^r E_i$. Let $\vec{v} = (x, x, \ldots)$. Then $\langle \vec{v} \rangle_c = W(A) = \cup_{i=1}^r E_i$. Trivially, $\cup_{i=1}^r E_i$ is large in $\vec{v}$. Hence, by Fact 8, there exist $1 \leq i \leq r$ and $\vec{s} \leq \vec{v}$ such that E_i is large in $\vec{s}$. By Corollary 14, there exists $\vec{t} \leq \vec{s}$ such that $\langle \vec{t} \rangle_c \subseteq E_i$ and the proof is complete. □

3. Proof of Theorem 3

In this section we present the proof of Theorem 3. As mentioned in the Introduction, the arguments are similar to those of Theorem 2. The main difficulty that we encountered was the manipulation of the infinite sequence of finite alphabets which, among others, it increases the complexity of the notation. We start by reformulating the basic terminology from Section 2, by taking into consideration the infinite sequence of alphabets.

3.1. Preliminaries

We fix an increasing sequence

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots$$

of finite alphabets and we set

$$A = \bigcup_{n \in \mathbb{N}} A_n.$$

Let $V(A)$ be the set of all *variable words* (*over* A). By $V^{<\infty}(A)$ (resp. $V^{\infty}(A)$) we denote the set of all finite (resp. infinite) sequences of variable words. Also let $V^{\leq\infty}(A) = V^{<\infty}(A) \cup V^{\infty}(A)$.

3.1.1. Constant and variable span of a sequence of variable words with respect to a sequence of finite alphabets Let $m \in \mathbb{N}$, $(s_n(x))_{n=0}^m$ in $V^{<\infty}(A)$ and $(k_n)_{n=0}^m$ be a strictly increasing finite sequence of non negative integers. The *constant span of* $(s_n(x))_{n=0}^m$ *with respect to* $(A_{k_n})_{n=0}^m$ denoted by $\langle (s_n(x))_{n=0}^m \parallel (A_{k_n})_{n=0}^m \rangle_c$ is defined to be the set

$$\bigcup_{n=0}^{m} \{s_{l_0}(a_0) \cdots s_{l_n}(a_n) : 0 \leq l_0 < \cdots < l_n \leq m, a_i \in A_{k_{l_i}}, 0 \leq i \leq n\}.$$

We also define the *variable span of* $(s_n(x))_{n=0}^m$ *with respect to* $(A_{k_n})_{n=0}^m$, denoted by $\langle (s_n(x))_{n=0}^m \parallel (A_{k_n})_{n=0}^m \rangle_v$, to be the set

$$V(A) \cap \bigcup_{n=0}^{m} \{s_{l_0}(a_0) \cdots s_{l_n}(a_n) : 0 \leq l_0 < \cdots < l_n \leq m, \\ a_i \in A_{k_{l_i}} \cup \{x\}, 0 \leq i \leq n\}.$$

The above notation is extended to infinite sequences of variable words as follows. Let $(s_n(x))_{n=0}^\infty \in V^\infty(A)$ and $(k_n)_{n=0}^\infty$ be a strictly increasing sequence of non negative integers. Then the constant span of $(s_n(x))_{n=0}^\infty$ with respect to $(A_{k_n})_{n=0}^\infty$ denoted by $\langle (s_n(x))_{n=0}^\infty \parallel (A_{k_n})_{n=0}^\infty \rangle_c$ is defined to be the set

$$\{s_{l_0}(a_0)\cdots s_{l_n}(a_n) : n \in \mathbb{N}, 0 \leq l_0 < \cdots < l_n, a_i \in A_{k_{l_i}}, 0 \leq i \leq n\}.$$

Similarly, the variable span of $(s_n(x))_{n=0}^\infty$ with respect to $(A_{k_n})_{n=0}^\infty$, denoted by $\langle (s_n(x))_{n=0}^\infty \parallel (A_{k_n})_{n=0}^\infty \rangle_v$, is the set

$$V(A) \cap \{s_{l_0}(a_0)\cdots s_{l_n}(a_n) : n \in \mathbb{N}, 0 \leq l_0 < \cdots < l_n, \\ a_i \in A_{k_{l_i}} \cup \{x\}, 0 \leq i \leq n\}.$$

In the following we also write $\langle \vec{s} \parallel (A_{k_n})_{n=0}^\infty \rangle_c$ (resp. $\langle \vec{s} \parallel (A_{k_n})_{n=0}^\infty \rangle_v$) to denote the the constant (resp. variable) span of $\vec{s} = (s_n(x))_{n=0}^\infty$ with respect to $(A_{k_n})_{n=0}^\infty$.

3.1.2. Extracted *k*-subsequences of a sequence of variable words In this subsection, we extend the notion of extracted subsequences defined in Section 2.

Definition 15. Let $k \in \mathbb{N}$ and $\vec{s} = (s_n(x))_{n=0}^\infty \in V^\infty(A)$.

(a) Let $l \in \mathbb{N}$ and $\vec{t} = (t_n(x))_{n=0}^l \in V^{<\infty}(A)$. We say that $\vec{t}$ is a (finite) extracted k-subsequence of $\vec{s}$ if there exist $0 = m_0 < \cdots < m_{l+1}$ such that

$$t_i(x) \in \langle (s_n(x))_{n=m_i}^{m_{i+1}-1} \parallel (A_{k+n})_{n=m_i}^{m_{i+1}-1} \rangle_v,$$

for all $0 \leq i \leq l$.

(b) Let $\vec{t} = (t_n(x))_{n=0}^\infty \in V^\infty(A)$. We say that $\vec{t}$ is a (infinite) extracted k-subsequence of $\vec{s}$ if for every $l \in \mathbb{N}$, the sequence $(t_n(x))_{n=0}^l$ is a finite extracted k-subsequence of $\vec{s}$.

In the following we will write $\vec{t} \leq_k \vec{s}$, whenever $\vec{t} \in V^{\leq\infty}(A)$, $\vec{s} \in V^\infty(A)$ and $\vec{t}$ is an extracted k-subsequence of $\vec{s}$. Taking into account that the sequence of alphabets $(A_n)_{n=0}^\infty$ is increasing, the next fact follows easily from the above definitions.

Fact 16. *Let $k, l \in \mathbb{N}$ and $\vec{s}, \vec{t}, \vec{w} \in V^\infty(A)$.*

(i) If $\vec{t} \leq_k \vec{s}$ then

$$\langle \vec{t} \parallel (A_{k+n})_{n=0}^\infty \rangle_c \subseteq \langle \vec{s} \parallel (A_{k+n})_{n=0}^\infty \rangle_c$$

and

$$\langle \vec{t} \parallel (A_{k+n})_{n=0}^{\infty} \rangle_v \subseteq \langle \vec{s} \parallel (A_{k+n})_{n=0}^{\infty} \rangle_v.$$

(ii) If $k \leq l$ and $\vec{w} \leq_k \vec{t} \leq_l \vec{s}$ then $\vec{w} \leq_l \vec{s}$.

3.1.3. The notion of k-large families The following is an extension of Definition 6.

Definition 17. Let $k \in \mathbb{N}$, $E \subseteq W(A)$ and $\vec{s} \in V^{\infty}(A)$. Then E will be called *k-large* in $\vec{s}$ if $E \cap \langle \vec{w} \parallel (A_{k+n})_{n=0}^{\infty} \rangle_c \neq \emptyset$, for every infinite extracted k-subsequence $\vec{w}$ of $\vec{s}$.

We close this subsection with some properties of k-large families.

Fact 18. *Let $k \in \mathbb{N}$, $E \subseteq W(A)$ and $\vec{s} = (s_n(x))_{n=0}^{\infty} \in V^{\infty}(A)$ such that E is k-large in $\vec{s}$. Then the following hold true.*

(i) E is k-large in $\vec{t}$, for every infinite extracted k-subsequence $\vec{t}$ of $\vec{s}$.
(ii) For every $m \in \mathbb{N}$, E is $(k+m)$-large in $(s_n(x)_{n=m}^{\infty}$.

Proof. (i) It follows easily using the first part of Fact 16.

(ii) Let $\vec{t} \in V^{\infty}(A)$ such that $\vec{t} \leq_{k+m} (s_n(x)_{n=m}^{\infty}$. It is easy to check that $\vec{t} \leq_k \vec{s}$ and therefore, since E is k-large in $\vec{s}$, we obtain that

$$E \cap \langle \vec{t} \parallel (A_{k+n})_{n=0}^{\infty} \rangle_c \neq \emptyset.$$

Moreover, since the sequence of alphabets $(A_n)_{n=0}^{\infty}$ is increasing we get that

$$\langle \vec{t} \parallel (A_{k+n})_{n=0}^{\infty} \rangle_c \subseteq \langle \vec{t} \parallel (A_{k+m+n})_{n=0}^{\infty} \rangle_c.$$

Hence, $E \cap \langle \vec{t} \parallel (A_{k+m+n})_{n=0}^{\infty} \rangle_c \neq \emptyset$ for every $\vec{t} \leq_{k+m} (s_n(x))_{n=m}^{\infty}$, i.e. E is $(k+m)$-large in $(s_n(x))_{n=m}^{\infty}$. □

Fact 19. *Let $k \in \mathbb{N}$, $E \subseteq W(A)$ and $\vec{s} \in V^{\infty}(A)$ such that E is k-large in $\vec{s}$. Let $r \geq 2$ and let $E = \bigcup_{i=1}^{r} E_i$. Then there exist $1 \leq i \leq r$ and an infinite extracted k-subsequence $\vec{t}$ of $\vec{s}$ such that E_i is k-large in $\vec{t}$.*

Definition 20. Let $m \in \mathbb{N}$ and $(s_n(x))_{n=0}^{m} \in V^{<\infty}(A)$ and let B be a finite subset of A. We set

$$[(s_n(x))_{n=0}^{m} \parallel B]_c = \big\{ s_0(a_0) \cdots s_m(a_m) : a_0, \ldots, a_m \in B \big\}$$

and

$$[(s_n(x))_{n=0}^{m} \parallel B]_v = V(A) \cap \big\{ s_0(a_0) \cdots s_m(a_m) : a_0, \ldots, a_m \in B \cup \{x\} \big\}.$$

Fact 21. *Let* $k \in \mathbb{N}$, $E \subseteq W(A)$ *and* $\vec{s} = (s_n(x))_{n=0}^{\infty} \in V^{\infty}(A)$ *such that* E *is* k*-large in* $\vec{s}$*. Then there exist* $m \in \mathbb{N}$ *and* $w(x) \in \langle (s_n(x))_{n=0}^{m} \parallel (A_{k+n})_{n=0}^{m} \rangle_v$ *such that* $\{w(a) : a \in A_k\} \subseteq E$.

Proof. Assume to the contrary that the conclusion fails. By induction we construct a sequence $\vec{w} = (w_n(x))_{n=0}^{\infty} \leq_k \vec{s}$ such that $\langle \vec{w} \parallel (A_{k+n})_{n=0}^{\infty} \rangle_c \subseteq E^c$ which is a contradiction since E is k-large in $\vec{s}$. The general inductive step of the construction is as follows. Let $n \geq 1$ and assume that $(w_i(x))_{i=0}^{n-1} \leq_k \vec{s}$ and $\langle (w_i(x))_{i=0}^{n-1} \parallel (A_{k+i})_{i=0}^{n-1} \rangle_c \subseteq E^c$. Let $n_0 \geq 1$ be the least integer satisfying

$$w_0(x) \cdots w_{n-1}(x) \in \langle (s_i(x))_{i=0}^{n_0-1} \parallel (A_{k+i})_{i=0}^{n_0-1} \rangle_v$$

and let

$$N = HJ(|A_{k+n}|, 2^{\prod_{i=0}^{n-1}(|A_{k+i}|+1)}).$$

To each $w \in [(s_{n_0+i}(x))_{i=0}^{N-1} \parallel A_{k+n}]_c$ we assign the set of words

$$\big\{uw : u \in \langle (w_i(x))_{i=0}^{n-1} \parallel (A_{k+i})_{i=0}^{n-1} \rangle_c \cup \{\emptyset\}\big\}.$$

Since $uw \in E$ or E^c, it is easy to see that the above correspondence induces a $2^{\prod_{i=0}^{n-1}(|A_{k+i}|+1)}$-coloring of the set $[(s_{n_0+i}(x))_{i=0}^{N-1} \parallel A_{k+n}]_c$. Hence, by the Hales–Jewett theorem and the choice of N, there exists a variable word

$$w(x) \in [(s_{n_0+i}(x))_{i=0}^{N-1} \parallel A_{k+n}]_v$$

such that for each $u \in \langle (w_i(x))_{i=0}^{n-1} \parallel (A_{k+i})_{i=0}^{n-1} \rangle_c \cup \{\emptyset\}$ the set $\{uw(a) : a \in A_{k+n}\}$ either is included in E or is disjoint from E. By our initial assumption, there is no $u \in \langle (w_i(x))_{i=0}^{n-1} \parallel (A_{k+i})_{i=0}^{n-1} \rangle_c \cup \{\emptyset\}$ satisfying the first alternative. Setting $w_n(x) = w(x)$ we easily see that $(w_i(x))_{i=0}^{n} \leq_k \vec{s}$ and $\langle (w_i(x))_{i=0}^{n} \parallel (A_{k+i})_{i=0}^{n} \rangle_c \subseteq E^c$. The inductive step of the construction of $\vec{w}$ is complete and as we have already mentioned in the beginning of the proof this leads to a contradiction. □

3.2. The main arguments

The next lemma corresponds to Lemma 12 and constitutes the core of the proof of Theorem 3.

Recall that for every non empty subsets E, F of $W(A)$ we have set

$$E_F = \{z \in W(A) : wz \in E \text{ for every } w \in F\}.$$

Lemma 22. *Let* $k \in \mathbb{N}$, $E \subseteq W(A)$ *and* $\vec{s} = (s_n(x))_{n=0}^{\infty} \in V^{\infty}(A)$ *such that* E *is* k*-large in* $\vec{s}$*. Then there exist* $m \geq 1$*, a variable word* $w(x)$ *in* $\langle (s_n(x))_{n=0}^{m-1} \parallel (A_{k+n})_{n=0}^{m-1} \rangle_v$ *and* $\vec{t} \in V^{\infty}(A)$ *with* $\vec{t} \leq_{k+m} (s_n(x))_{n=m}^{\infty}$ *such that setting* $F = \{w(a) : a \in A_k\}$ *then* $E \cap E_F$ *is* $(k+m)$*-large in* $\vec{t}$*.*

Proof. We start with the following claim.

Claim 1. *There exists* $n_0 \in \mathbb{N}$ *such that for every*

$$z \in \langle (s_i(x))_{i=n_0+1}^{\infty} \parallel (A_{k+i})_{i=n_0+1}^{\infty} \rangle_c$$

there exists $w(x)$ *in* $\langle (s_i(x))_{i=0}^{n_0} \parallel (A_{k+i})_{i=0}^{n_0} \rangle_v$ *such that*

$$\{w(a)z : a \in A_k\} \subseteq E.$$

Proof of Claim 1. Assume that the claim is not true. Then for every $n \in \mathbb{N}$ there exists $z \in \langle (s_i(x))_{i=n+1}^{\infty} \parallel (A_{k+i})_{i=n+1}^{\infty} \rangle_c$ such that the set $\{w(a)z : a \in A_k\}$ is not contained in E, for every $w(x) \in \langle (s_i(x))_{i=0}^{n_0} \parallel (A_{k+i})_{i=0}^{n_0} \rangle_v$. Using this assumption we easily find a strictly increasing sequence $(k_n)_{n=0}^{\infty}$ in $\mathbb{N}$ with $k_0 = 0$ and a sequence $(z_n)_{n=0}^{\infty}$ in $W(A)$ such the following are satisfied.

(i) For every $n \in \mathbb{N}$, we have $z_n \in \langle (s_i(x))_{i=k_n+1}^{k_{n+1}-1} \parallel (A_{k+i})_{i=k_n+1}^{k_{n+1}-1} \rangle_c$.

(ii) For every $n \in \mathbb{N}$ and every $u(x) \in \langle (s_i(x))_{i=k_0}^{k_n} \parallel (A_{k+i})_{i=k_0}^{k_n} \rangle_v$ we have that $\{u(a)z_n : a \in A_k\} \nsubseteq E$.

We set $v_n(x) = s_{k_n}(x) z_n$, for every $n \in \mathbb{N}$.

By (i), we have that $(s_{k_0}(x)z_0, \ldots, s_{k_n}(x)z_n) \leq_k \vec{s}$, for every $n \in \mathbb{N}$ and therefore, $(v_n(x))_{n=0}^{\infty} \leq_k \vec{s}$. Moreover, since the sequence $(k_n)_{n=0}^{\infty}$ is strictly increasing and the sequence of finite alphabets $(A_n)_{n=0}^{\infty}$ is increasing, by (ii), we obtain that $\{u(a)z_n : a \in A_k\} \nsubseteq E$, for every $n \in \mathbb{N}$ and every

$$u(x) \in \langle (s_{k_0}(x)z_0, \ldots, s_{k_{n-1}}(x)z_{n-1}, s_{k_n}(x) \parallel (A_{k+i})_{i=0}^{n} \rangle_v.$$

Hence, since every $w(x) \in \langle (v_n(x))_{n=0}^{\infty} \parallel (A_{k+n})_{n=0}^{\infty} \rangle_v$ is of the form $w(x) = u(x)z_n$ for some unique $n \in \mathbb{N}$ and some variable word $u(x)$ in $\langle (s_{k_0}(x)z_0, \ldots, s_{k_{n-1}}(x)z_{n-1}, s_{k_n}(x) \parallel (A_{k+i})_{i=0}^{n} \rangle_v$, we conclude that there is no $w(x) \in \langle (v_n(x))_{n=0}^{\infty} \parallel (A_{k+n})_{n=0}^{\infty} \rangle_v$ such that $\{w(a) : a \in A_k\} \subseteq E$. But since $(v_n(x))_{n=0}^{\infty} \leq_k \vec{s}$, we have that E is k-large in $(v_n(x))_{n=0}^{\infty}$ and so by Fact 21 we arrive to a contradiction. □

We set $m = n_0 + 1$. Also, let $L = \langle (s_n(x))_{n=0}^{m-1} \parallel (A_{k+i})_{i=0}^{m-1} \rangle_v$ and for every $w(x) \in L$, let $F(w(x)) = \{w(a) : a \in A_k\}$. By Claim 1, we have that $\langle (s_i(x))_{i=m}^{\infty} \parallel (A_{k+i})_{i=m}^{\infty} \rangle_c \subseteq \bigcup_{w(x) \in L} E_{F(w(x))}$ and therefore,

$$E \cap \langle (s_i(x))_{i=m}^{\infty} \parallel (A_{k+i})_{i=m}^{\infty} \rangle_c \subseteq \bigcup_{w(x)\in L} E \cap E_{F(w(x))}.$$

By part (ii) of Fact 18, we have that E is $(k+m)$-large in $(s_i(x))_{i=m}^{\infty}$. Hence, $\bigcup_{w(x)\in L} E\cap E_{F(w(x))}$ is $(k+m)$-large in $(s_i(x))_{i=m}^{\infty}$. So, by Fact 19 there exist a variable word $w(x) \in L$ and $\vec{t} \leq_{k+m} (s_n(x))_{n=m}^{\infty}$ such that $E \cap E_{F(w(x))}$ is $(k+m)$-large in $\vec{t}$ and the proof is complete. □

Lemma 23. *Let $k \in \mathbb{N}$, $E \subseteq W(A)$ and $\vec{s} = (s_n(x))_{n=0}^{\infty} \in V^{\infty}(A)$ such that E is k-large in $\vec{s}$. Then there exist a sequence $\vec{w} = (w_n(x))_{n=0}^{\infty} \leq_k \vec{s}$ and two strictly increasing sequences $(k_n)_{n=0}^{\infty}$ and $(p_n)_{n=0}^{\infty}$ in $\mathbb{N}$, with $k_0 = k$ and $p_0 = 0$, such that for every $n \in \mathbb{N}$, the following are satisfied.*

(W1) $k + p_n \geq k_n$
(W2) $w_n(x) \in \langle (s_i(x))_{i=p_n}^{p_{n+1}-1} \parallel (A_{k+i})_{i=p_n}^{p_{n+1}-1} \rangle_v$.
(W3) Setting $F_n = \langle (w_i(x))_{i=0}^{n} \parallel (A_{k_i})_{i=0}^{n} \rangle_c$ *then* $E \cap E_{F_n}$ *is* k_{n+1}*-large in* $(w_i(x))_{i=n+1}^{\infty}$.

Proof. We start with the following.

Step 1. Let $k \in \mathbb{N}$, $E \subseteq W(A)$ and $\vec{s} = (s_n(x))_{n=0}^{\infty} \in V^{\infty}(A)$ such that E is k-large in $\vec{s}$. Then there exist (a) a sequence $(w_n(x))_{n=0}^{\infty}$ of variable words, (b) a sequence $(\vec{s}_n)_{n=0}^{\infty}$ in $V^{\infty}(A)$ with $\vec{s}_0 = \vec{s}$, (c) two sequences $(m_n)_{n=0}^{\infty}$ and $(k_n)_{n=0}^{\infty}$ in $\mathbb{N}$, with $m_0 = 0$ and $k_0 = k$ such that setting for every $n \in \mathbb{N}$, $\vec{s}_n = (s_i^{(n)}(x))_{i=0}^{\infty}$ then the following are satisfied.

(i) $m_{n+1} \geq 1$ and $k_{n+1} = k_n + m_{n+1}$.
(ii) $w_n(x) \in \langle (s_i^{(n)}(x))_{i=0}^{m_{n+1}-1} \parallel (A_{k_n+i})_{i=0}^{m_{n+1}-1} \rangle_v$.
(iii) $\vec{s}_{n+1}$ is an extracted k_{n+1}-subsequence of $(s_i^{(n)}(x))_{i=m_{n+1}}^{\infty}$.
(iv) If we set $F_n = \langle (w_i(x))_{i=0}^{n} \parallel (A_{k_i})_{i=0}^{n} \rangle_c$ then $E \cap E_{F_n}$ is k_{n+1}-large in $\vec{s}_{n+1}$.

Proof of Step 1. For $n = 0$ we set $m_0 = 0$, $k_0 = 0$ and $\vec{s}_0 = \vec{s}$. Assume that the construction has been carried out up to some $n \in \mathbb{N}$, i.e. the sequences $(w_i(x))_{i<n}$, $(\vec{s}_i)_{i=0}^{n}$, $(m_i)_{i=0}^{n}$, $(k_i)_{i=0}^{n}$ have been selected. We set $G = E\cap E_{F_{n-1}}$ (if $n = 0$, we set $G = E$). By our inductive assumptions we have that G is k_n-large in $\vec{s}_n$. Therefore, by Lemma 22, there exist $m \geq 1$, a variable word $w(x) \in \langle (s_i^{(n)}(x))_{i=0}^{m-1} \parallel (A_{k_n+i})_{i=0}^{m-1} \rangle_v$ and an extracted $(k_n + m)$-subsequence $\vec{t}$ of $(s_i^{(n)}(x))_{i=m}^{\infty}$, such that $G \cap G_F$ is $(k_n + m)$-large in $\vec{t}$, where $F = \{w(a) : a \in A_{k_n}\}$. We set

$$m_{n+1} = m, k_{n+1} = k_n + m, w_n(x) = w(x) \text{ and } \vec{s}_{n+1} = \vec{t}.$$

Moreover, if $F_n = \langle (w_i(x))_{i=0}^n \parallel (A_{k_i})_{i=0}^n \rangle_c$ then it is easy to check that $G \cap G_F = E \cap E_{F_n}$. The above choices clearly fulfill conditions (i)–(iv) and the proof of the inductive step of the construction is complete. □

Step 2. Let $(\vec{s}_n)_{n=0}^\infty$, $(m_n)_{n=0}^\infty$ and $(k_n)_{n=0}^\infty$ be the sequences obtained in Step 1. Then there exists a strictly increasing sequence $(p_n)_{n=0}^\infty$ in $\mathbb{N}$ with $p_0 = 0$ such that for every $n \in \mathbb{N}$, the following are satisfied.

(v) $k + p_n \geq k_n$.
(vi) The set $\langle (s_i^{(n)}(x))_{i=0}^{m_{n+1}-1} \parallel (A_{k_n+i})_{i=0}^{m_{n+1}-1} \rangle_v$ is a subset of $\langle (s_i(x))_{i=p_n}^{p_{n+1}-1} \parallel (A_{k+i})_{i=p_n}^{p_{n+1}-1} \rangle_v$.
(vii) $\vec{s}_n$ is an extracted $(k + p_n)$-subsequence of $(s_i(x))_{i=p_n}^\infty$.

Proof of Step 2. We set $p_0 = 0$ and we easily see that (v) and (vii) are satisfied for $n = 0$. Let $n \in \mathbb{N}$ and assume that the sequence $(p_i)_{i=0}^n$ has been selected. By (vii) we obtain that for every $m \geq 1$ there exists a sequence $(I_j)_{j=0}^{m-1}$ of successive nonempty intervals of $\mathbb{N}$ with $\min(I_0) = 0$ such that setting $M(m) = \max(I_{m-1}) + 1$, then

$$s_j^{(n)}(x) \in \langle (s_{p_n+i}(x))_{i \in I_j} \parallel (A_{k+p_n+i})_{i \in I_j} \rangle_v \tag{1}$$

for every $j \in \{0, \ldots, m-1\}$ and,

$$(s_i^{(n)}(x))_{i=m}^\infty \leq_{k+p_n+M(m)} (s_i(x))_{i=p_n+M(m)}^\infty. \tag{2}$$

We claim that we may set

$$p_{n+1} = p_n + M(m_{n+1}).$$

Indeed, by our inductive assumptions we have that $k + p_n \geq k_n$ and therefore, since the sequence of the alphabets $(A_n)_{n=0}^\infty$ is increasing, by (1) (for $m = m_{n+1}$), we conclude that

$$\langle (s_i^{(n)}(x))_{i=0}^{m_{n+1}-1} \parallel (A_{k_n+i})_{i=0}^{m_{n+1}-1} \rangle_v \subseteq \langle (s_i(x))_{i=p_n}^{p_{n+1}-1} \parallel (A_{k+i})_{i=p_n}^{p_{n+1}-1} \rangle_v$$

and so (vi) is satisfied. Moreover, notice that $M(m) \geq m$. Hence,

$$k + p_{n+1} = k + p_n + M(m_{n+1}) \geq k + p_n + m_{n+1} \geq k_n + m_{n+1} = k_{n+1}, \tag{3}$$

that is (v) is also satisfied. Finally, by (iii) of Step 1 and (2) above, we have

$$\vec{s}_{n+1} \leq_{k_{n+1}} (s_i^{(n)}(x))_{i=m_{n+1}}^\infty \leq_{k+p_{n+1}} (s_i(x))_{i=p_{n+1}}^\infty.$$

Since $k_{n+1} \leq k + p_{n+1}$, by part (ii) of Fact 16, we obtain that

$$\vec{s}_{n+1} \leq_{k+p_{n+1}} (s_i(x))_{i=p_{n+1}}^{\infty}.$$

Hence, (vii) is also valid and the inductive step of the construction is complete. □

If $n_0 \geq 1$ then starting from $\vec{s}_{n_0}$ and k_{n_0} instead of $\vec{s}_0 = \vec{s}$ and $k_0 = k$ and working as in Step 2 we derive the following.

Step 3. Let $n_0 \geq 1$ and let $(\vec{s}_n)_{n=0}^{\infty}$, $(m_n)_{n=0}^{\infty}$ and $(k_n)_{n=0}^{\infty}$ be the sequences obtained in Step 1. Then there exists a strictly increasing sequence $(q_n)_{n=0}^{\infty}$ in $\mathbb{N}$ with $q_0 = 0$ such that for every $n \in \mathbb{N}$ the following are satisfied.

(v′) $k_{n_0} + q_n \geq k_{n_0+n}$.
(vi′) The set $\langle (s_i^{(n_0+n)}(x))_{i=0}^{m_{n_0+n+1}-1} \parallel (A_{k_{n_0+n}+i})_{i=0}^{m_{n_0+n+1}-1} \rangle_v$ is a subset of the set $\langle (s_i^{(n_0)}(x))_{i=q_n}^{q_{n+1}-1} \parallel (A_{k_{n_0}+i})_{i=q_n}^{q_{n+1}-1} \rangle_v$.
(vii′) $\vec{s}_{n_0+n}$ is an extracted $(k_{n_0} + q_n)$-subsequence of $(s_i^{(n_0)}(x))_{i=q_n}^{\infty}$.

We are now ready to complete the proof of the lemma. Clearly, condition (W1) follows by (v). Also, by (ii) and (vi) we obtain that

$$w_n(x) \in \langle (s_i(x))_{i=p_n}^{p_{n+1}-1} \parallel (A_{k+i})_{i=p_n}^{p_{n+1}-1} \rangle_v,$$

for every $n \in \mathbb{N}$ and so (W2) is also satisfied. It remains to verify (W3). To this end, let n_0 be an arbitrary positive integer. Then, by (ii) and (vi′) we get that

$$w_{n_0+n}(x) \in \langle (s_i^{(n_0)}(x))_{i=q_n}^{q_{n+1}-1} \parallel (A_{k_{n_0}+i})_{i=q_n}^{q_{n+1}-1} \rangle_v,$$

for every $n \in \mathbb{N}$, that is $(w_i(x))_{i=n_0}^{\infty}$ is an extracted k_{n_0}-subsequence of $\vec{s}_{n_0}$. Therefore, for every $n \in \mathbb{N}$, $(w_i(x))_{i=n+1}^{\infty}$ is a k_{n+1}-subsequence of $\vec{s}_{n+1}$ and so, by (iv) of Step 1 and part (i) of Fact 18, we get that for every $n \in \mathbb{N}$, $E \cap E_{F_n}$ is k_{n+1}-large in $(w_i(x))_{i=n+1}^{\infty}$, that is (W3). Finally, setting $\vec{w} = (w_n(x))_{n=0}^{\infty}$, by (W2) we obtain that $\vec{w} \leq_k \vec{s}$ and the proof is complete. □

Corollary 24. *Let $k \in \mathbb{N}$, $E \subseteq W(A)$ and $\vec{s} \in V^{\infty}(A)$ such that E is k-large in $\vec{s}$. Then there exists an infinite extracted k-subsequence $\vec{t}$ of $\vec{s}$ such that*

$$\langle \vec{t} \parallel (A_{k+n})_{n=0}^{\infty} \rangle_c \subseteq E.$$

Proof. Let $\vec{w} = (w_n(x))_{n=0}^{\infty}$, $(k_n)_{n=0}^{\infty}$ and $(p_n)_{n=0}^{\infty}$ be the sequences obtained in Lemma 23. We start with the following claim.

Claim 1. *There exist a strictly increasing sequence $(r_n)_{n=0}^{\infty}$ in $\mathbb{N}$ with $r_0 = 0$ and a sequence $\vec{t} = (t_n(x))_{n=0}^{\infty}$ of variable words such that for every $n \in \mathbb{N}$ the following are satisfied.*

(T1) $t_n(x) \in \langle (w_{r_n+i}(x))_{i=0}^{r_{n+1}-r_n-1} \parallel (A_{k_{r_n}+i})_{i=0}^{r_{n+1}-r_n-1} \rangle_v$.
(T2) $\langle (t_i(x))_{i=0}^{n} \parallel (A_{k_{r_i}})_{i=0}^{n} \rangle_c \subseteq E$.

Proof of Claim 1. Since $\vec{w} \leq_k \vec{s}$ and E is k-large in $\vec{s}$ we get that E is k-large in $\vec{w}$. Therefore, by Fact 21 there exist a positive integer r_1 and a variable word

$$t_0(x) \in \langle (w_i(x))_{i=0}^{r_1-1} \parallel (A_{k+i})_{i=0}^{r_1-1} \rangle_v$$

such that $\{t_0(a) : a \in A_k\} \subseteq E$. Since $k_0 = k$ and $r_0 = 0$ we have that conditions (T1) and (T2) are satisfied for $n = 0$.

We set $G_0 = \{t_0(a) : a \in A_k\}$. Notice that $G_0 \subseteq \langle (w_i(x))_{i=0}^{r_1-1} \parallel (A_{k_i})_{i=0}^{r_1-1} \rangle_c$ and so, by (W3) of Lemma 23, we get that $E \cap E_{G_0}$ is k_{r_1}-large in $(w_i(x))_{i=r_1}^{\infty}$. Hence, again by Fact 21, there exists an integer $r_2 > r_1$ and a variable word

$$t_1(x) \in \langle (w_{r_1+i}(x))_{i=0}^{r_2-r_1-1} \parallel (A_{k_{r_1}+i})_{i=0}^{r_2-r_1-1} \rangle_v$$

such that $\{t_1(a) : a \in A_{k_{r_1}}\} \subseteq E \cap E_{G_0}$. Observe that conditions (T1) and (T2) are also satisfied for $n = 1$. Continuing in the same way, we select the desired sequence $\vec{t}$. □

Claim 2. *For every $n \in \mathbb{N}$, $t_n(x) \in \langle (s_i(x))_{i=p_{r_n}}^{p_{r_{n+1}}-1} \parallel (A_{k+i})_{i=p_{r_n}}^{p_{r_{n+1}}-1} \rangle_v$. Therefore, $\vec{t} \leq_k \vec{s}$.*

Proof of Claim 2. Fix $n \in \mathbb{N}$ and let $j \in \mathbb{N}$ be arbitrary. By (W2) of Lemma 23, we have that

$$w_{r_n+j}(x) \in \langle (s_i(x))_{i=p_{r_n+j}}^{p_{r_n+j+1}-1} \parallel (A_{k+i})_{i=p_{r_n+j}}^{p_{r_n+j+1}-1} \rangle_v.$$

Moreover, by (W1) of Lemma 23 and the monotonicity of the sequence $(k_i)_{i=0}^{\infty}$, we have

$$k + p_{r_n+j} \geq k_{r_n+j} \geq k_{r_n} + j$$

and therefore, since the sequence of alphabets $(A_i)_{i=0}^{\infty}$ is increasing, we get that

$$A_{k_{r_n}+j} \subseteq A_{k+p_{r_n+j}}.$$

Hence,

$$\{w_{r_n+j}(a) : a \in A_{k_{r_n}+j}\} \subseteq \langle (s_i(x))_{i=p_{r_n+j}}^{p_{r_n+j+1}-1} \parallel (A_{k+i})_{i=p_{r_n+j}}^{p_{r_n+j+1}-1} \rangle_c,$$

for every $j \in \mathbb{N}$. Therefore, for every $d \in \mathbb{N}$, we conclude that

$$\langle (w_{r_n+j}(x))_{j=0}^{d} \parallel (A_{k_{r_n}+j})_{j=0}^{d} \rangle_v \subseteq \langle (s_i(x))_{i=p_{r_n}}^{p_{r_n+d+1}-1} \parallel (A_{k+i})_{i=p_{r_n}}^{p_{r_n+d+1}-1} \rangle_v.$$

Setting $d = r_{n+1} - r_n - 1$ and using (T1) of Claim 1, the result follows. □

We are now ready to complete the proof. By Claim 2 we have that $\vec{t} \leq_k \vec{s}$. Moreover, since $(k_n)_{n=0}^{\infty}$ and $(p_n)_{n=0}^{\infty}$ are strictly increasing, we get that $A_{k+i} \subseteq A_{k_{r_i}}$, for all $i \in \mathbb{N}$. Therefore, for every $n \in \mathbb{N}$, we have

$$\langle (t_i(x))_{i=0}^{n} \parallel (A_{k+i})_{i=0}^{n} \rangle_c \subseteq \langle (t_i(x))_{i=0}^{n} \parallel (A_{k_{r_i}})_{i=0}^{n} \rangle_c,$$

and so, by (T2) of Claim 1, we get that $\langle (t_i(x))_{i=0}^{n} \parallel (A_{k+i})_{i=0}^{n} \rangle_c \subseteq E$, for every $n \in \mathbb{N}$. Hence, $\langle \vec{t} \parallel (A_{k+n})_{n=0}^{\infty} \rangle_c \subseteq E$, as desired. □

Proof of Theorem 3. Let $r \geq 2$ and let $W(A) = \cup_{i=1}^{r} E_i$. Let $\vec{v} = (x, x, \ldots)$. Then we have that $\langle \vec{v} \parallel (A_n)_{n=0}^{\infty} \rangle_c \subseteq W(A) = \cup_{i=1}^{r} E_i$. Trivially, $\cup_{i=1}^{r} E_i$ is 0-large in $\vec{v}$. Hence, by Fact 19, there exist $1 \leq i \leq r$ and $\vec{s} \leq_0 \vec{v}$ such that E_i is 0-large in $\vec{s}$. Applying Corollary 24 for $k = 0$, we have that there exists $\vec{t} \leq_0 \vec{s}$ such that $\langle \vec{t} \parallel (A_n)_{n=0}^{\infty} \rangle_c \subseteq E_i$ and the proof is complete. □

Remark 1. In [14, Theorem 2.3] the following version of Theorem 2 was shown.

Theorem 25. *Let A be a finite alphabet. Then for every finite coloring of $W(A)$ there exists a sequence $(t_n(x))_{n=0}^{\infty}$ of variable words over A such that for every $n \geq 1$, $t_n(x)$ is a left variable word and for every $n \in \mathbb{N}$ and every $0 = m_0 < m_1 < \cdots < m_n$, the words of the form $t_{m_0}(a_0)t_{m_1}(a_1)\cdots t_{m_n}(a_n)$ with $a_i \in A$ for all $0 \leq i \leq n$ are of the same color.*

The above theorem is a stronger version of a well-known result of T. Carlson and S. Simpson [4, Theorem 6.3]. We mention also that a left variable version of Theorem 3 does not hold true (see [10, §3]). Although our approach can be applied for [4, Theorem 6.3] (see [12]), it is open for us whether it can also provide an alternative proof of Theorem 25.

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