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A note on spanning trees and totally cyclic orientations of 3-connected graphs

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Merino and Welsh conjectured that for a 2-edge connected graph G with no loops, the number of spanning trees of G is always less than or equal to either the number of acyclic orientations of G, or the number of totally cyclic orientations of G. In this paper, we prove that the Merino-Welsh conjecture holds for a 3-connected simple graph of minimum degree at least 4 and average degree at least 7.02.

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1. Introduction

An orientation of a graph is an arbitrary assignment of a direction to every edge of the graph. An orientation such that every edge is contained in some oriented cycles is said to be totally cyclic orientation. It is well known that an orientation is totally cyclic if and only if the digraph under this orientation is strongly connected (that is, for any two vertices u and v there exists a directed path from u to v). An orientation is said to be acyclic if the digraph under this orientation has no directed cycles.

Let G denote a graph and $V(G)$, $E(G)$ denote its vertex set and edge set respectively. Denote $n = |V(G)|$, $m = |E(G)|$. The Tutte polynomial $T(G, x, y)$, which is a polynomial in two variables, plays an important role in graph theory. We gather some of the naturally occurring interpretations of the Tutte polynomial. Let $c(G)$, $a(G)$, $\Gamma(G)$ and $f(G)$ denote the number of totally cyclic orientations of G, the number of acyclic orientations of G, the number of spanning trees of G and the number of forests of G respectively. Then $T(G, 0, 2) = c(G)$, $T(G, 1, 1) = \Gamma(G)$, $T(G, 2, 0) = a(G)$, $T(G, 2, 1) = f(G)$ (see [2, 5]). It is important to point out that the exact evaluation of any such invariant is NP-hard even for planar bipartite graphs (see [7]).

The motivation for this work is that the number of totally cyclic orientations of a graph is closely related to the number of spanning trees of a

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graph. Merino and Welsh [5] conjectured that for a 2-edge connected graph G with no loops, the number of spanning trees of G is always less than or equal to either the number of acyclic orientations of G, or the number of totally cyclic orientations of G. Recently, Thomassen [6] proved that the Merino-Welsh conjecture holds for all bridgeless and loopless multigraphs with maximum degree at most three. He also proved that $\Gamma(G) < a(G)$ if G has at most $16n/15$ edges, and $\Gamma(G) < c(G)$ if G has at least $4n - 4$ edges.

In this paper, all graphs considered are simple graphs (have no loops and no parallel edges), unless otherwise specified. For the notation and terminology not defined herein, we refer readers to [3]. We prove that the Merino-Welsh conjecture holds for a 3-connected simple graph of minimum degree at least 4 and average degree at least 7.02.

The rest of the paper is organized as follows. In Section 2, we describe a special sequence of 3-augmentations which can be used to generate 3-connected graphs. In Section 3, we give a method to construct some totally cyclic orientations of 3-connected graphs. Then use the method described in [6] to prove the main result.

Theorem 1 (Main Theorem)**.** *Let G be a 3-connected graph with minimum degree $\delta \geq 4$ and $m \geq 3.51n$. Then $\Gamma(G) \leq c(G)$.*

2. Generating 3-connected graphs

Let G be a graph with minimum degree at least 3. If G has an edge e such that $G - e$ is a subdivision of H, where H is a 3-connected graph, then we say that G is obtained from H by 3-augmentation. In particular, if $|V(G)| > |V(H)|$, then this 3-augmentation is called strict 3-augmentation. Let Λ_3 be the set of all graphs that can be generated from complete graph K_4 using a sequence of 3-augmentations. The following well known result was proved by Barnette and Grünbaum.

Theorem 2. [1] *Λ_3 is the set of all 3-connected graphs.*

Suppose xy is an edge of H, and v is a vertex of H other than x, y. Let G be the graph obtained from H by using a new vertex a to subdivide the edge xy, then adding the edge av to the resulting graph. Clearly, $|V(G)| = |V(H)| + 1$ and $|E(G)| = |E(H)| + 2$. We say that the graph G is obtained from H by type-1 strict 3-augmentation with base $\{v, xy\}$. Denote

$$G := H(v, a, xy).$$

Similarly, suppose x_1y_1, x_2y_2 are two edges of H. Let G' be the graph obtained from H by using two new vertices a_1, a_2 to subdivide the edges

x_1y_1, x_2y_2 respectively, then adding the edge a_1a_2 to the resulting graph. Then $|V(G')| = |V(H)| + 2$ and $|E(G')| = |E(H)| + 3$. We say that the graph G' is obtained from H by type-2 strict 3-augmentation with base $\{x_1y_1, x_2y_2\}$. Denote

$$G' := H(a_1, x_1y_1; a_2, x_2y_2).$$

Hence, we can partition 3-augmentations into three types: type-1 strict 3-augmentation, type-2 strict 3-augmentation and edge addition.

By Theorem 2, all 3-connected graphs can be obtained from K_4 by a sequence of 3-augmentations, but how many type-1 (or type-2) strict 3-augmentations contained in this sequence is unsolved. In the following lemma, we answer the question in some sense for a 3-connected graph with minimum degree $\delta \geq k > 3$.

Lemma 3. *Suppose G is a 3-connected graph with minimum degree $\delta(G) \geq k > 3$. There is a sequence of 3-augmentations, by which G is obtained from K_4 such that $n_1 \geq (k-3)n_2$, where n_1, n_2 denote the number of type-1, type-2 strict 3-augmentations contained in this sequence respectively.*

Proof. Suppose G is obtained from K_4 by a sequence of 3-augmentations

$$Q := \{G_1, G_2, \ldots, G_{i-1}, G_i, \ldots, G_{s-1}, G_s\},$$

and Q has the following properties: 1) $G_1 = K_4$, $G_s = G$, and G_i is a proper minor of G, $i = 2, 3, \ldots, s-1$; 2) suppose G_{l+1} is obtained from G_l by type-2 strict 3-augmentation, then G_l could not expand to a minor of G by some type-1 strict 3-augmentations or edge additions. In the following, we prove that Q is just the sequence satisfying lemma condition. Suppose G_{l+1} is obtained from G_l by type-2 strict 3-augmentation. Without loss of generality, let $G_{l+1} := G_l(a_1, x_1y_1; a_2, x_2y_2)$. Now add edges to G_{l+1} as many as possible such that the resulting graph G'_{l+1} is a minor of G. We claim that a_i $(i = 1, 2)$ is not adjacent to any vertices of G_l other than x_i, y_i by 2); that is, $G'_{l+1} = G_{l+1}$. Otherwise, say $xa_1 \in E(G'_{l+1})$, $x \in V(G_l)$ but $x \notin \{x_1, y_1\}$. Let $G' := G_l(x, a_1, x_1y_1)$ and $G'' := G'(a_1, a_2, x_2y_2)$. It is clear that $G'' = G_{l+1}$ is a minor of G. Note that G_l could be expanded into a minor of G by two type-1 strict 3-augmentations, which is contrary to the choice of Q. Hence, $G'_{l+1} = G_{l+1}$. Now expand G_{l+1} into a minor of G as large as possible using type-1 strict 3-augmentations and edge additions. Noting that the degree of a_i $(i = 1, 2)$ is three in G_{l+1}, but $\delta(G) \geq k$, there are at least $k-3$ type-1 strict 3-augmentations before another type-2 strict 3-augmentation having to be chosen in Q. Hence, $n_1 \geq (k-3)n_2$ in Q. The proof is completed. □

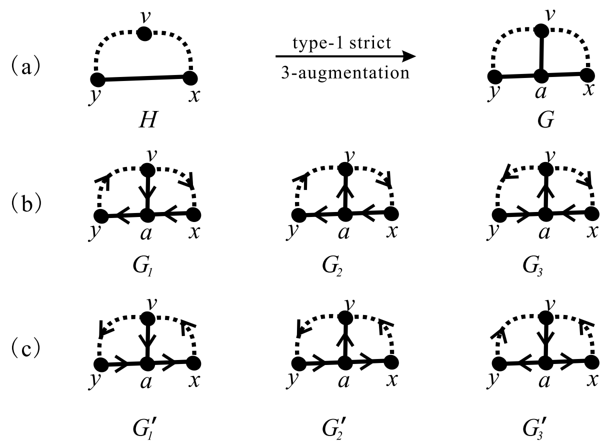

Figure 1: A real line denotes an edge between two vertices, and a dotted line denotes a path between two vertices.

3. Totally cyclic orientations

Lemma 4. *Suppose G is obtained from H by type-1 strict 3-augmentation. Then $c(G) \geq 3c(H)$.*

Proof. Let $G := H(v, a, xy)$ (see Fig. 1(a)). We claim that for each totally cyclic orientation of H, it can be extended to three totally cyclic orientations of G, and for each pair of totally cyclic orientations of H, their corresponding totally cyclic orientations of G are pairwise disjoint. According to the direction of edge xy in H, we partition the totally cyclic orientation set $D(H)$ of H into two subsets $D_1(H)$ and $D_2(H)$, where $D_1(H)$ denotes the subset of $D(H)$ such that the edge xy is directed from x to y and $D_2(H) = D(H) \setminus D_1(H)$.

Given an orientation $\vec{H} \in D_1(H)$, we define three orientations of G which are extended from $\vec{H}$ as follows (see Fig. 1(b)).

$G_1(\vec{H})$: The orientation of G is extended from $E(\vec{H}) - xy$ by directing the edge xa from x to a, the edge ay from a to y, and the edge va from v to a.

$G_2(\vec{H})$: The orientation of G is obtained from $G_1(\vec{H})$ by reversing the orientation of the edge va.

$G_3(\vec{H})$: Since $\vec{H}$ is a totally cyclic digraph, $\vec{H}$ is strongly connected. Hence there exists a directed path P_{yv} from y to v in $\vec{H}$. Then $C := ay - P_{yv} - va$ is a directed cycle in $G_1(\vec{H})$. $G_3(\vec{H})$ is an orientation of G obtained from $G_1(\vec{H})$ by reversing the orientations of all the edges of C.

Given an orientation $\vec{H}' \in D_2(H)$, we define three orientations of G which are extended from $\vec{H}'$ as follows (see Fig. 1(c)).

$G'_1(\vec{H}')$: The orientation of G is extended from $E(\vec{H}') - yx$ by directing the edge ax from a to x, the edge ya from y to a, and the edge va from v to a.

$G'_2(\vec{H}')$: The orientation of G is obtained from $G'_1(\vec{H}')$ by reversing the orientation of the edge va.

$G'_3(\vec{H}')$: Similarly, there exists a directed path P_{vy} from v to y in $\vec{H}'$. Then $C' := av - P_{vy} - ya$ is a directed cycle in $G'_2(\vec{H}')$. $G'_3(\vec{H}')$ is an orientation of G obtained from $G'_2(\vec{H}')$ by reversing the orientations of all the edges of C'.

Let $O_i := \{G_i(\vec{H}), \vec{H} \in D_1(H)\}$, $O'_i := \{G'_i(\vec{H}'), \vec{H}' \in D_2(H)\}$, $i = 1, 2, 3$. For $i, j = 1, 2, 3$, note that in O_i and O'_j the edge ax is directed in reverse; hence, $O_i \cap O'_j = \emptyset$. It is obvious that $O_i \cap O_j = \emptyset$ and $O'_i \cap O'_j = \emptyset$ when $i \neq j$. Hence, the claim holds. □

Lemma 5. *Suppose G is obtained from H by type-2 strict 3-augmentation. Then $c(G) \geq 2.5c(H)$.*

Proof. Let $G := H(a_1, x_1y_1; a_2, x_2y_2)$ (see Fig. 2(a)). According to the direction of the edges x_1y_1, x_2y_2 in H, we partition the totally cyclic orientation set $D(H)$ of H into four subsets $D_{11}(H)$, $D_{12}(H)$, $D_{21}(H)$ and $D_{22}(H)$, where $D_{11}(H)$ denotes the subset of $D(H)$ such that direct the edge x_1y_1 from x_1 to y_1 and the edge x_2y_2 from x_2 to y_2, $D_{12}(H)$ denotes the subset of $D(H)$ such that direct the edge x_1y_1 from x_1 to y_1 and the edge x_2y_2 from y_2 to x_2, $D_{21}(H)$ denotes the subset of $D(H)$ such that direct the edge x_1y_1 from y_1 to x_1 and the edge x_2y_2 from x_2 to y_2, and $D_{22}(H)$ denotes the subset of $D(H)$ such that direct the edge x_1y_1 from y_1 to x_1, and the edge x_2y_2 from y_2 to x_2.

Given an orientation $\vec{H} \in D_{11}(H)$, we define three orientations of G which are extended from $\vec{H}$ as follows (see Fig. 2(b)).

$G^1_{11}(\vec{H})$: The orientation of G is extended from $E(\vec{H}) - x_1y_1 - x_2y_2$ by directing the edge x_1a_1 from x_1 to a_1, the edge a_1y_1 from a_1 to y_1, the edge x_2a_2 from x_2 to a_2, the edge a_2y_2 from a_2 to y_2, and the edge a_1a_2 from a_1 to a_2.

$G^2_{11}(\vec{H})$: The orientation of G is obtained from $G^1_{11}(\vec{H})$ by reversing the orientation of the edge a_1a_2.

$G^3_{11}(\vec{H})$: Since $\vec{H}$ is a totally cyclic digraph, $\vec{H}$ is strongly connected. Hence, there exists a directed path $P_{y_1x_2}$ from y_1 to x_2 in $\vec{H}$. Then $C := a_1y_1 - P_{y_1x_2} - x_2a_2 - a_2a_1$ is a directed cycle in $G^2_{11}(\vec{H})$. $G^3_{11}(\vec{H})$ is an

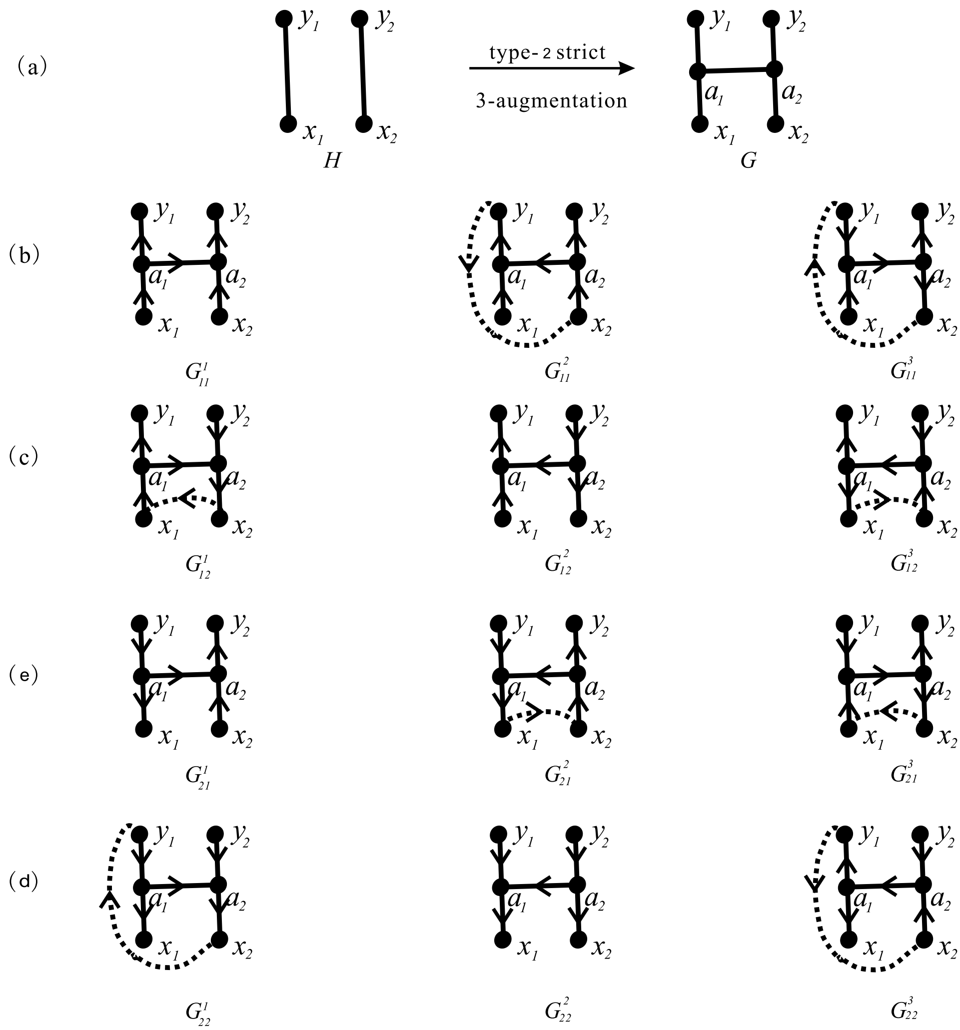

Figure 2: A real line denotes an edge between two vertices, and a dotted line denotes a path between two vertices.

orientation of G obtained from $G^2_{11}(\vec{H})$ by reversing the orientations of all the edges of C.

Given an orientation $\vec{H} \in D_{12}(H)$, we define three orientations of G, which are extended from $\vec{H}$ as follows (see Fig. 2(c)).

$G^1_{12}(\vec{H})$: The orientation of G is extended from $E(\vec{H})-x_1y_1-y_2x_2$ by directing the edge x_1a_1 from x_1 to a_1, the edge a_1y_1 from a_1 to y_1, the edge a_2x_2 from a_2 to x_2, the edge y_2a_2 from y_2 to a_2, and the edge a_1a_2 from a_1 to a_2.

$G_{12}^2(\vec{H})$: The orientation of G is obtained from $G_{12}^1(\vec{H})$ by reversing the orientation of the edge a_1a_2.

$G_{12}^3(\vec{H})$: Since $\vec{H}$ is a totally cyclic digraph, $\vec{H}$ is strongly connected. Hence, there exists a directed path $P_{x_2x_1}$ from x_2 to x_1 in $\vec{H}$. Then $C := a_2x_2 - P_{x_2x_1} - x_1a_1 - a_1a_2$ is a directed cycle in $G_{12}^1(\vec{H})$. $G_{12}^3(\vec{H})$ is an orientation of G obtained from $G_{12}^1(\vec{H})$ by reversing the orientations of all the edges of C.

Given an orientation $\vec{H} \in D_{21}(H)$, we define three orientations of G, which are extended from $\vec{H}$ as follows (see Fig. 2(d)).

$G_{21}^1(\vec{H})$: The orientation of G is extended from $E(\vec{H}) - x_1y_1 - x_2y_2$ by directing the edge x_1a_1 from a_1 to x_1, the edge y_1a_1 from y_1 to a_1, the edge x_2a_2 from x_2 to a_2, the edge a_2y_2 from a_2 to y_2, and the edge a_1a_2 from a_1 to a_2.

$G_{21}^2(\vec{H})$: The orientation of G is obtained from $G_{21}^1(\vec{H})$ by reversing the orientation of the edge a_1a_2.

$G_{21}^3(\vec{H})$: Since $\vec{H}$ is a totally cyclic digraph, $\vec{H}$ is strongly connected. Hence, there exists a directed path $P_{x_1x_2}$ from x_1 to x_2 in $\vec{H}$. Then $C := x_2a_2 - a_2a_1 - a_1x_1 - P_{x_1x_2}$ is a directed cycle in $G_{21}^2(\vec{H})$. $G_{21}^3(\vec{H})$ is an orientation of G obtained from $G_{21}^2(\vec{H})$ by reversing the orientations of all the edges of C.

Given an orientation $\vec{H} \in D_{22}(H)$, we define three orientations of G, which are extended from $\vec{H}$ as follows (see Fig. 2(e)).

$G_{22}^1(\vec{H})$: The orientation of G is extended from $E(\vec{H}) - x_1y_1 - x_2y_2$ by directing the edge x_1a_1 from a_1 to x_1, the edge y_1a_1 from y_1 to a_1, the edge x_2a_2 from a_2 to x_2, the edge y_2a_2 from y_2 to a_2, and the edge a_1a_2 from a_1 to a_2.

$G_{22}^2(\vec{H})$: The orientation of G is obtained from $G_{22}^1(\vec{H})$ by reversing the orientation of the edge a_1a_2.

$G_{22}^3(\vec{H})$: Since $\vec{H}$ is a totally cyclic digraph, $\vec{H}$ is strongly connected. Hence, there exists a directed path $P_{x_2y_1}$ from x_2 to y_1 in $\vec{H}$. Then $C := y_1a_1 - a_1a_2 - a_2x_2 - P_{x_2y_1}$ is a directed cycle in $G_{22}^1(\vec{H})$. $G_{22}^3(\vec{H})$ is an orientation of G obtained from $G_{22}^1(\vec{H})$ by reversing the orientations of all the edges of C.

Note that G_{11}^1, G_{11}^2, G_{11}^3, G_{12}^1, G_{12}^2, G_{12}^3, G_{21}^1, G_{21}^2, G_{22}^1 and G_{22}^2 are distinct orientations of G, but G_{21}^3 and G_{11}^3 may be the same orientation, and G_{22}^3 and G_{12}^3 may be the same orientation. Hence, it is obvious that $c(G) \geq 2.5c(H)$. $\square$

Theorem 6. *Let G be a 3-connected graph with minimum degree $\delta \geq k > 3$. Then*

$$c(G) \geq 3^{\frac{(k-3)n-3k+11}{k-1}} \cdot 2.5^{\frac{n-4}{k-1}} \cdot 2^{m-\frac{(2k-3)n}{k-1}+\frac{5k-9}{k-1}}.$$

Proof. Let $Q = \{G_1, G_2, \ldots, G_s\}$ be a sequence of 3-augmentations which satisfies the conditions of Lemma 3. If G_i is obtained from G_{i-1} by type-1 strict 3-augmentation, then $c(G_i) \geq 3c(G_{i-1})$ by Lemma 4. If G_i is obtained from G_{i-1} by type-2 strict 3-augmentation, then $c(G_i) \geq 2.5c(G_{i-1})$ by Lemma 5. If G_i is obtained from G_{i-1} by edge addition, it is obvious that $c(G_i) \geq 2c(G_{i-1})$. By Lemma 3, $n_1 \geq (k-3)n_2$, where n_i $(i = 1, 2)$, denotes the number of type-i strict 3-augmentations contained in Q. Note that

$$n_1 + 2n_2 = n - 4;$$

hence,

$$n_2 \leq \frac{n-4}{k-1}.$$

It is not difficult to find that $c(K_4) = 24$. Hence,

$$\begin{aligned} c(G) &\geq 24 \cdot 3^{n_1} \cdot 2.5^{n_2} \cdot 2^{m-6-2n_1-3n_2} \\ &= 24 \cdot 3^{n-4-2n_2} \cdot 2.5^{n_2} \cdot 2^{m-6-2(n-4)+n_2} \\ &\geq 3^{\frac{(k-3)n-3k+11}{k-1}} \cdot 2.5^{\frac{n-4}{k-1}} \cdot 2^{m-\frac{(2k-3)n}{k-1}+\frac{5k-9}{k-1}}. \end{aligned}$$ □

Lemma 7. [4] *If G is a graph with vertex-degree sequence $d_1, d_2, \ldots, d_n$. Then the number of spanning trees*

$$\Gamma(G) \leq d_1 d_2 \ldots d_n/(n-1).$$

We prove the main theorem using Theorem 6 and Lemma 7 in the following.

Proof of Main Theorem. Let

$$f(m) = \frac{m + (\frac{1}{3}\log_2(3) + \frac{1}{3}\log_2(2.5) - \frac{8}{3})n - \frac{4}{3}\log_2(2.5) - \frac{1}{3}\log_2(3) + \frac{14}{3}}{n-1}\ln 2 - \ln\frac{m}{n}.$$

Then

$$f'(m) = \frac{\ln 2}{n-1} - \frac{1}{m} > 0$$

when $m \geq \frac{n-1}{\ln 2}$. Thus, $f(m)$ is an increasing function when $m \geq \frac{n-1}{\ln 2}$.

Note that $f(3.51n) > 0$. Hence, $f(m) > 0$ when $m \geq 3.51n$; that is, if $m \geq 3.51n$, then

$$3^{\frac{n-1}{3}} \cdot 2.5^{\frac{n-4}{3}} \cdot 2^{m-\frac{5n}{3}+\frac{11}{3}} \geq \left(\frac{2m}{n}\right)^{n-1}.$$

Then by Lemma 7 and Theorem 6,

$$\begin{aligned}\Gamma(G) &\leq \frac{d_1 d_2 \ldots d_n}{n-1} \\ &\leq \frac{1}{n-1}\left(\frac{d_1 + d_2 + \ldots + d_n}{n}\right)^n \\ &= \left(\frac{2m}{n}\right)^{n-1} \cdot \frac{2m}{n(n-1)} \\ &\leq \left(\frac{2m}{n}\right)^{n-1} \\ &\leq 3^{\frac{n-1}{3}} \cdot 2.5^{\frac{n-4}{3}} \cdot 2^{m-\frac{5n}{3}+\frac{11}{3}} \\ &\leq c(G).\end{aligned}$$

□

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