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Proper connection with many colors

Aydin Gerek, Shinya Fujita* and Colton Magnant†

We say an edge-colored graph is properly connected if, between every pair of vertices, there exists a properly colored path. For a graph G, define the proper connection number $pc(G)$ to be the minimum number of colors k such that there exists a k-coloring of $E(G)$ which is properly connected. In this work, we study conditions on G which force upper bounds on $pc(G)$.

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1. Introduction

In this work, we consider only edge-colorings of graphs. Since Vizing's fundamental result [9], proper edge colorings of graphs, colorings such that no two adjacent edges have the same color, have become an essential topic for every beginning graph theorist. Proper edge colorings have many applications in signal transmission [7], bandwidth allocation [3] and many other areas [5, 6, 8]. See [1] for a survey of the case where two colors (where the term 'alternating' can be used in place of 'proper') are used.

Since many of these applications depend only on properly colored substructures of the graph, not necessarily properly coloring the entire graph, it is natural to restrict our attention to subgraphs. If a graph is properly colored, then every subgraph is properly colored. We relax this condition by requiring only some of the subgraphs to be properly colored. In particular, we say that a colored graph is *properly connected* if, between every pair of vertices, there exists a properly colored path. As defined in [2], the proper connection number of a graph $pc(G)$ is the minimum number of colors k such that there exists a k coloring of G which is properly connected.

As a specific application of proper connectivity to network security, suppose a network administrator would like to create a more secure network.

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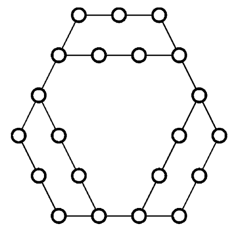

Figure 1: A 2-connected graph with $pc(G) = 3$ (from [2]).

In order to access one computer from another in a network, one must follow a predetermined path, otherwise the system will lock down. Furthermore, on each consecutive pair of connections along these paths, different types of security measures are used. Thus, in order to hack this network, one would have to break different types of security at each step. With colors on edges representing types of security, the proper connection number of a graph is the minimum number of security types needed for such a system.

In [2], the authors consider many conditions on G which force $pc(G)$ to be small. In particular, most results contained in [2] concern graphs G for which $pc(G) \leq 3$. Since a connected graph can have arbitrarily high proper connection number, consider a star, it was natural to focus on 2-connected graphs. Thus, the example in Figure 1 demonstrates a 2-connected graph with proper connection number 3, thereby making results of [2] best possible. Using structures similar to this example, it is easy to demonstrate an infinite class of 2-connected graphs with proper connection number 3. In this work, we prove more general versions of these results, namely, upper bounds on $pc(G)$ which are larger than 3.

Our first result uses a forbidden (induced) subgraph condition to get an upper bound on the proper connection number. Since we clearly have $pc(K_{1,s}) = s$, this result is also a classification of the forbidden subgraphs which provide upper bounds on the proper connection number. For this result, given a graph H, a graph G is said to be H-free if it contains no induced copy of H.

Theorem 1. *For $s \geq 2$, any connected $K_{1,s}$-free graph G has $pc(G) \leq s-1$.*

We present the proof of Theorem 1 in Section 2. This result is sharp by considering any subdivision of a star $K_{1,s-1}$. The central vertex of this subdivided star must have $s-1$ different colors on its incident edges. Theorem 1 immediately implies the following corollary, which is also sharp by simply considering a star. Here $\alpha(G)$ is used to denote the independence number of G.

Corollary 2. *Any connected graph G has $pc(G) \leq \alpha(G)$.*

In [2], the following result concerning the minimum degree was proven.

Theorem 3 ([2])**.** *If G is connected of order $n \geq 68$ and minimum degree $\delta(G) \geq n/4$, then $pc(G) \leq 2$.*

We extend this result to weaker minimum degree conditions in the following result.

Theorem 4. *Suppose that a connected graph G has order $n \geq t^2$ and minimum degree $\delta(G) \geq n/t$ for some $t \geq 5$. Then $pc(G) \leq t - 2$.*

We present the proof of Theorem 4 in Section 3. This result is sharp infinitely often by the following example. Let $G = \cup_{i=1}^{t} K_{n/t}$ where $t|n$ and choose a single vertex from each clique. Between these chosen vertices, we add edges to induce a star (using $t - 1$ edges). Notice that $\delta(G) = n/t - 1$ and this graph requires $t - 1$ colors in any properly connected coloring.

We also prove a similar result for bipartite graphs.

Theorem 5. *Suppose that a connected and bipartite graph G has order $n \geq 2t^2$ and minimum degree $\delta(G) \geq n/2t$ for some $t \geq 4$. Then $pc(G) \leq t - 2$.*

This result is also sharp by a construction similar to the above. Consider t copies of $K_{n/(2t),n/(2t)}$ where $2t|n$. In each copy, select a vertex and connect the selected vertices into a star on $t - 1$ edges. This graph clearly requires $t - 1$ colors to be properly connected. Although the proof of Theorem 5 is similar, in nature, to the proof of Theorem 4, we present the proof in Section 4.

For notation, given a graph G and a subset $A \subseteq V(G)$, let $G[A]$ denote the subgraph of G induced on A. Given a colored graph G and a vertex $v \in G$, let $d_G^c(v)$ denote the number of distinct colors used on edges incident to v. If the graph G is understood, we will simply write $d^c(v)$. The edge chromatic number of a graph G is denoted $\chi'(G)$. A vertex $v \in V(G)$ is called *simplicial* if $G[N(v)]$ is complete. All other notation can be found in [4].

2. Forbidden induced stars

In order to prove Theorem 1, we will actually prove the following slightly stronger result.

Theorem 6. *For $s \geq 2$, any connected $K_{1,s}$-free graph G has a coloring with $s - 1$ colors so that G is properly connected and, for every vertex v, we have $d^c(v) \leq \alpha(N(v))$.*

Proof. This result is proven by induction on $n+s$. If G is complete, then $pc(G)=1$ so we may assume G is not complete and therefore $s \geq 3$. Suppose the result holds when $|G|<n$ and for all graphs which are $K_{1,s-1}$-free. Consider a connected, $K_{1,s}$-free graph G of order n which contains an induced $K_{1,s-1}$.

First suppose $\kappa(G)=1$, let v be a cut vertex and let $H_1, H_2, \ldots, H_t$ be the set of components of $G \setminus v$. Let $a_i = \alpha(N(v) \cap H_i)$ for all i. Apply induction on $H_i' = H_i \cup v$ for each i to obtain an $s-1$ coloring of H_i' which is properly connected and with $d^c_{H_i'}(u) \leq \alpha(N_{H_i'}(u))$ for all $u \in H_i'$. In order to combine these colorings, we permute the colors used in H_1' so that the edges from v to H_1 have colors $1, 2, \ldots, a_1$. Similarly, we permute the colors used in H_i' so that the edges incident to v have colors from the set $\{(\sum_{j=1}^{i-1} a_j)+1, (\sum_{j=1}^{i-1} a_j)+2, \ldots, \sum_{j=1}^{i} a_j (= \sum_{j=1}^{i-1} a_j + a_i)\}$ (recall that the edges incident to v have at most a_i colors). This uses a total of $\sum_{i=1}^{t} a_i = \alpha(N(v)) \leq s-1$ colors on edges incident to v and clearly produces a properly connected coloring of G, thereby proving our desired result in this case. Thus, G is properly connected, so we may assume $\kappa(G) \geq 2$.

Let v be a vertex with $\alpha(N(v)) = s-1$ and suppose first that $G[N(v)]$ contains no edges; namely, v is not in a triangle and so $\alpha(N(v)) = d(v)$. Let $H = G \setminus v$ and apply induction on $n+s$ in H (note that H is connected since G was 2-connected). This produces a coloring of H which uses at most $s-1$ colors such that, for each $u \in H$, we have $d^c_H(u) \leq \alpha(N_H(u))$. Notice that, for every vertex $w \in N(v)$, we have $\alpha(N_H(w)) = \alpha(N_G(w)) - 1 \leq s-2$. This means, on the edges incident to each vertex w, there is at least one color which is not used. Color the edge vw with one such unused color for all $w \in N(v)$. Certainly this provides a coloring of G in which $d^c(u) \leq \alpha(N(u))$ for all $u \in G$. Furthermore, H is properly connected so let u be any vertex of H and we will produce a proper path to v. There is a properly colored path P from u to w for any vertex $w \in N(v)$ and, since the edge wv has a color which was previously unused at w, the path $uPwv$ is a properly colored (u,v)-path. Hence, we may assume $G[N(v)]$ contains an edge.

Let $e = uw$ be an edge in $G[N(v)]$ and again let $H = G \setminus v$ and apply induction on H to produce a coloring satisfying the desired properties with at most $s-1$ colors. Without loss of generality, suppose e receives color 1. For every vertex $x \in N(v)$, let c_x be the largest numbered color which is already present on an edge incident to x (thus, $c_x \neq 1$ as long as x has an edge of another color in H). This is the color that we would like to use on the edge vx.

We will first consider the case where u (or similarly w) is simplicial in G, and hence v is adjacent to all of $N[u] = N(u) \cup \{u\}$. Since $\alpha(N(u)) = 1$,

u is incident to the edge e of color 1 and by induction $d^c_H(u) \leq \alpha(N(u))$, this means that all edges incident to u have color 1. In this case, we know $c_u = 1$ and redefine $c_x = 1$ for all $x \in N_H(u)$ (leaving c_y as defined earlier for all other vertices $y \in N(v)$). Now color all edges xv with the color c_x for all $x \in N(v)$. Since every vertex of H has a proper path P to u on which the edge incident to u has color 1, every vertex must also be connected to v by a proper path $(P \setminus \{u\}) \cup \{v\}$, so G is properly connected. Furthermore, it is easy to check that we have $d^c_G(y) = d^c_H(y) \leq \alpha(N_G(y))$ for all $y \in H$ and $d^c(v) \leq s - 1 = \alpha(N(v))$.

Finally, we may assume that u and w are not simplicial in G. If the only color used in H at u (or similarly w) is color 1, then set $c_u = 2$ (leaving c_x as originally defined for all other vertices x). Otherwise, we leave all values c_x as originally defined. Then we color all edges vx with the color c_x for all $x \in N(v)$. Since we assumed $\alpha(N(u)) \geq 2$ and $\alpha(N(w)) \geq 2$ and for all other vertices in $N(v)$, the color c_x is already used on an edge at x, this coloring of G satisfies $d^c(y) \leq \alpha(N(y))$ for all $y \in G$. It remains only to show that the coloring is properly connected. By induction, H is properly connected so let $y \in H \setminus N(v)$ and we will produce a proper path to v. Since H is properly connected, there exists a proper path from y to u. Note that we may assume this path does not contain w since otherwise we could consider the subpath from y to w and apply the same argument. If the last edge (incident to u) on this path does not have color c_u, we may take the edge uv to complete a proper path to v so suppose this edge has color c_u. Then, since $c_u \neq 1$ which is the color of e, we may take the edge e to w and then the edge wv to complete the proper path to v. This shows that G is properly connected and completes the proof. □

3. Minimum degree

In order to prove Theorem 4, we will use the following result from [2]. For this result, we define what it means to be 2-strongly properly connected. Given a proper path P with an inherent orientation, we say *start*(P) is the color of the first edge of P and *end*(P) is the color of the last edge of P. A colored graph G is 2-*strongly properly connected* (or 2-*strong*) if, between any pair of vertices, there exist at least 2 different (not necessarily disjoint) properly colored paths P and Q such that *start*$(P) \neq$ *start*(Q) and *end*$(P) \neq$ *end*(Q). This notion can be easily extended to k-strong.

Theorem 7 ([2]). *If G is 2-connected, then $pc(G) \leq 3$ and there exists a 3-coloring of G that makes it 2-strong.*

In order to prove our main minimum degree result, we first prove this lemma which gets within 3 of the desired result by using the edge chromatic number of the set of bridges.

Lemma 1. *Let $t \geq 2$, suppose G is connected of order $n \geq t^2$ and let B be the subgraph of G containing only the edges that are bridges in G. If $\delta(G) \geq n/t$, then $|E(B)| \leq t-2$ and $pc(G) \leq \chi'(B) + 3 \leq t+1$.*

Proof. Let G be connected of order $n \geq t^2$ with $\delta(G) \geq n/t$. We first show $|E(B)| \leq t-2$.

Let $\mathscr{H}_1 = \{H \mid H$ is a component of $G \setminus E(B)$ such that there exists a vertex $v \in H$ with $v \cap V(B) = \emptyset\}$. Note that for any component H of $G \setminus E(B)$, we have $|H| \geq \delta(G) + 1 > \frac{n}{t} \geq t$. We first show that every component of $G \setminus E(B)$ is in $\mathscr{H}_1$. Thus, suppose for a contradiction that there exists a component M of $G \setminus E(B)$ such that every vertex of M is an end vertex of an edge in $E(B)$.

If we contract each component of $G \setminus E(B)$ to a single vertex, what remains of G is a tree T on the edges of B. As with any tree, the number of leaves in T is at least $\Delta(T)$. In this case, a leaf of T corresponds to a component in which only a single vertex is incident to an edge of B. Thus, if there exists a component of $G \setminus E(B)$ which contains vertices which are incident to s edges of B, then there must be at least s components in $\mathscr{H}_1$.

Since the components corresponding to leaves of T must be in $\mathscr{H}_1$, it is easy to see that $1 \leq |M| \leq t-1$. Similarly, we also get

$$\begin{aligned} t-1 &\geq |E(M, G-M)| \\ &= \sum_{v \in M} d_{G-M}(v) \\ &\geq \sum_{v \in M} [d_G(v) - (|M|-1)] \\ &\geq \frac{|M|n}{t} - |M|(|M|-1). \end{aligned}$$

This means $n \leq \frac{t(t-1+|M|(|M|-1))}{|M|}$. By convexity, this quantity is maximized when either $m = 1$ or $m = t-1$, both cases yielding a value of $n < t^2$. This contradicts the assumption that $n \geq t^2$, meaning that all components of $G \setminus E(B)$ are in $\mathscr{H}_1$. Since we have $|H| > n/t$ for each $H \in \mathscr{H}_1$, we know T must have fewer than t vertices, which means it has fewer than $t-1$ edges. This gives us $|E(B)| \leq t-2$.

Now consider the standard block-decomposition of $G \setminus E(B)$ into edge-disjoint maximal 2-connected blocks. By Theorem 7, each 2-connected block can be colored with 3 colors so that the block is 2-strong. Color the edges of each block as such using the same three colors, suppose colors $\{1, 2, 3\}$, on all blocks.

We now show that it suffices to properly color the edges of B with $\chi'(B)$ colors $\{4, 5, \ldots, \chi'(B) + 3\}$. Suppose we have such a coloring and let u and v be two vertices of G. Note that, since each block is 2-strong, we can easily see that each component of $G \setminus E(B)$ is also 2-strong. Thus, if u and v are in the same component, then, regardless of the number of cut vertices between u and v, there is clearly a proper path connecting u and v, so suppose u and v are in different components of $G \setminus E(B)$. Again, since each component is 2-strong, it follows almost immediately that this coloring is properly connected. □

In order to complete the proof of Theorem 4, we state one more result.

Theorem 8 ([2])**.** *If G is 2-connected and $diam(G) = 2$, then $pc(G) = 2$.*

We may now prove our main minimum degree result.

Proof of Theorem 4. Let G be a graph with $\delta(G) \geq n/t$ and let B be the subgraph of G containing only the edges that are bridges in G. If $\chi'(B) \leq t - 5$, then by Lemma 1, we have the desired result so we may assume $\chi'(B) \geq t - 4$. Since B induces a forest in G with at most $t - 2$ edges by Lemma 1), this means that B contains a star S and $B \setminus E(S)$ contains at most two edges. Let s be the vertex at the center of this star.

Claim 1. *There exists at most one edge e in $B \setminus E(S)$.*

Proof. If $|E(B)| < t - 2$, since $|E(S)| \geq t - 4$, there is clearly at most one edge in $B \setminus S$. Thus, we may assume $|E(B)| = t - 2$. Let $C_1, C_2, \ldots, C_\ell$ where $\ell = t - 1$ be the set of components of $G \setminus E(B)$. Call a component C_i *bad* if $|C_i| \geq 2\delta(G) - t + 3$.

Fact 1. *If $|E(B)| = t - 2$, then there is no bad component.*

This fact is proven simply by observing that every component of $G \setminus E(B)$, even one which is not bad, has order at least $n/t + 1$ by Lemma 1 and the fact that $\ell = t - 1$.

If $diam(C_i) \geq 3$, since $|E(B)| = t - 2$, we get $|C_i| \geq 2\delta(G) + 2 - (t - 2) \geq 2n/t - t + 4$ so C_i is bad, contradicting Fact 1. Also, if a component C_i is not 2-connected, then $|C_i| \geq 2\delta + 1 > 2n/t - t + 3$ since $t \geq 5$, so again C_i is bad, a contradiction to Fact 1. Thus, every component of $G \setminus E(B)$ is 2-connected and has diameter at most 2. By Theorem 8, there exists a 2-coloring of each component so that C_i is properly connected for all i. Color each component as such with colors 1 and 2.

If $\chi'(B) = t-4$, we may properly color $E(B)$ with colors $3, 4, \ldots, t-2$ and easily produce a properly connected coloring of G so we know $E(S) \geq t-3$ and there is at most one edge e in $B \setminus S$. $\square_{Claim\ 1}$

We now provide a properly connected coloring of the entire graph using at most $t-2$ colors. By Theorem 7, each block can be colored with 3 colors so that the blocks are 2-strong. Since each component C_i is bridgeless, such a coloring of the blocks also makes each component 2-strong. Color each component C_i with colors $\{1, 2, 3\}$ so that each is 2-strong.

Let $m = |E(S)|$ and note that $m \leq t-2$. Color $E(S)$ properly with m colors $t-m-1, t-m, \ldots, t-2$. Since each component is 2-strong, the subgraph consisting of S and all components containing a vertex of S is easily shown to be properly connected. We now observe an easy fact which is a corollary of Theorem 7.

Fact 2. *Let C be a component of $G \setminus E(B)$ such that $|V(B) \cap V(C)| = 1$ where e is the edge of B incident to a vertex of C. Then, for any coloring of e (say, $c(e) = 1$), $e \cup C$ can be properly connected with at most 3 colors and with $c(e) = 1$.*

Thus, if $m = |B|$, the result follows immediately from Fact 2 so we may suppose there exists an edge $e \in B \setminus E(S)$. If we let C_e be the component which is disconnected from S by the removal of e, then $C_e \cup e$ is properly connected regardless of the color assigned to e by Fact 2.

We will assume e has an end u in the component C_s which also contains s. The case where u is in another component follows similarly. Since C_s is 2-strong, there exist two distinct properly colored paths P_1 and P_2 from s to u in C_s such that $start(P_1) \neq start(P_2)$ and $end(P_1) \neq end(P_2)$. Color e with the remaining color in $\{1, 2, 3\} \setminus \{end(P_1), end(P_2)\}$. This allows for proper connection between any pair of vertices in G, thereby completing the proof. $\square$

If we suppose $pc(G) \geq 4$, then by Theorem 7, we have $\kappa(G) = 1$. In fact, we can say a bit more. By the proof of Theorem 4, we obtain the following corollary.

Corollary 9. *If $pc(G) = t \geq 5$, then the number of bridges in G is at least t.*

4. Bipartite graphs

For the proof of Theorem 5, we first recall a result from [2] and then prove a lemma similar to Lemma 1.

Theorem 10 ([2]). *If G is bipartite and 2-connected, then $pc(G) = 2$ and there exists a 2-coloring of G which is 2-strong.*

Lemma 2. *Let $t \geq 4$, G be a connected and bipartite graph of order $n \geq 2t^2$ and let B be the subgraph of G containing only the edges that are bridges in G. If $\delta(G) > \frac{n}{2t}$ then $|E(B)| \leq t-2$ and $pc(G) \leq \chi'(B) + 2 \leq t$.*

Proof. Let G be as given in the statement. If G is 2-connected, then $pc(G) = 2$ by Theorem 10 so suppose $\kappa'(G) = 1$. Let B be the subgraph of G containing only the edges that are bridges of G. As in the proof of Lemma 1, we first prove that $|E(B)| \leq t-2$.

Suppose $|E(B)| \geq t-1$. Let $\mathscr{H}$ be the set of components of $G \setminus E(B)$ that contain at least one vertex in each partite set which is not incident to an edge of B. Each component in $\mathscr{H}$ has order at least $2\delta(G) > 2\frac{n}{2t} = \frac{n}{t}$. This means that $|\mathscr{H}| < t$ so, since there are at least t components of $G \setminus E(B)$, there must exist a component C in which every vertex in at least one partite set is incident to an edge of B. Let k be the number of edges of B with one end in C. Using the same contraction argument as in Lemma 1, we know $|\mathscr{H}| \geq k$. If $k \geq t$, the proof is complete. Now, suppose it is not.

There are at most $t-1$ vertices in C incident to edges of B but every vertex of at least one partite set (call it C_1) of C must be incident to at least one edge of B. This means that $|C_1| \leq t-1$. Since $\delta(G) > \frac{n}{2t} \geq t$, there are not enough vertices in C_1 for each vertex of the opposite set C_2 to have all of their edges inside C. This means that every vertex of C_2 must also be incident to an edge of B. If we let $c_1 = |C_1| \leq t-1$ and $c_2 = |C_2| \leq t-1$, we get

$$|\mathscr{H}| \geq (\delta(G) - c_2)c_1 + (\delta(G) - c_1)c_2 \geq t$$

since $n \geq 2t^2$ and $C \neq \emptyset$. This shows that $|E(B)| \leq t-2$.

Now color the blocks of G with 2 colors so that they are each 2-strong. Such a coloring exists by Theorem 10. This naturally implies that the components of $G \setminus E(B)$ are 2-strong. Now properly color B using the set of colors $\{3, 4, \ldots, \chi'(B) + 2\}$. This produces a coloring of $E(G)$ using $2 + \chi'(B) \leq 2 + |E(B)| \leq 2 + t - 2 = t$ colors. Finally, using the same argument as in Lemma 1, we see that this coloring is properly connected. □

Proof of Theorem 5. Suppose G is connected and bipartite of order $n \geq 2t^2$. By Lemma 2, if $\chi'(B) \leq t-4$, we have $pc(G) \leq t-2$ as desired so we may assume $\chi'(B) \geq t-3$. Since $|E(B)| \leq t-2$, this means that B is a large star S with at most one edge e outside the star. We now observe an easy fact that follows from Theorem 10.

Fact 3. *Let C be a component of $G \setminus E(B)$ such that $|V(B) \cap V(C)| = 1$ where e is the edge of B incident to a vertex of C. Then, for any coloring of e (say, $c(e) = 1$), $e \cup C$ can be properly connected with at most 2 colors and with $c(e) = 1$.*

Fact 3 eliminates the case where the edges of B form a star so we may assume B contains a star S and $B \setminus E(S)$ contains a single edge e with one end in a component which does not contain the center of the star. At this point, using exactly the same argument as at the end of the proof of Theorem 4, the desired result holds. □

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Aydin Gerek
Department of Mathematics
Lehigh University
Bethlehem, PA 18015
USA
E-mail address: ayg207@lehigh.edu

Shinya Fujita
Department of Integrated Design Engineering
Maebashi Institute of Technology
460-1 Kamisadori, Maebashi 371-0816
Japan
E-mail address: shinya.fujita.ph.d@gmail.com

Colton Magnant
Department of Mathematical Sciences
Georgia Southern University
Statesboro, GA 30460
USA
E-mail address: cmagnant@georgiasouthern.edu

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