Partitioning graphs into paths or cycles of prescribed lengths

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In this paper, we consider the path (and cycle) partition problem for graphs with additional length restrictions. More specifically, we prove sufficient degree sum conditions for the vertices of a graph to be partitioned into paths, with fixed end vertices, such that these paths have approximately prescribed lengths. We also prove similar results for partitions into cycles of approximately prescribed lengths each containing a specified vertex.

1. Partition problem into paths or cycles

Many researchers have considered the "path (or cycle) partition problem", that is, determining whether the vertices of a graph can be partitioned into paths or cycles (with some properties, mentioned after). In particular, we are interested in a $\sigma_2(G)$ condition which implies the existence of such a partition, where

$$\sigma_2(G) := \min\{d(x) + d(y) : xy \notin E(G)\}$$

if G is not a complete graph; otherwise let $\sigma_2(G) := +\infty$. For many related problems and results, we refer the reader to the path and cycle partition problem survey [6].

In this paper, we first concentrate on a partition into paths with some special properties. It is easily proved that if $\sigma_2(G) \geq n - t$ for a graph G of order n, there exist t disjoint paths $P_1, P_2, \ldots, P_t$ with $V(G) = \bigcup_{i=1}^t V(P_i)$. By considering the classical result on a hamilton cycle by Ore [13], we can obtain the following fact.

Theorem 1 (Ore [13]). *Let t be an integer with $t \geq 2$ and let G be a graph of order n. If $\sigma_2(G) \geq n$, then for any t vertices $x_1, x_2, \ldots, x_t$, there exists t pairwise disjoint paths $P_1, P_2, \ldots, P_t$ such that $V(G) = \bigcup_{i=1}^t V(P_i)$ and x_i is an end vertex of P_i.*

On the other hand, Johansson [9] gave a sharp $\delta(G)$ condition for the existence of a partition into paths with given order. Chen et al. extended Johansson's result, and we obtain the following $\sigma_2(G)$ condition as a corollary of it.

Theorem 2 (Chen et al. [2]). *Let t be an integer with $t \geq 2$ and also let $a_1, a_2, \ldots, a_t$ be integers with $a_i \geq 2$ for any $1 \leq i \leq t$. Let G be a graph of order $n = \sum_{i=1}^{t} a_i$ and suppose $\sigma_2(G) \geq n - t + \lambda - 2$, where λ is the number of even integers in $\{a_1, a_2, \ldots, a_t\}$. Then there exist t pairwise disjoint paths $P_1, P_2, \ldots, P_t$ such that $V(G) = \bigcup_{i=1}^{t} V(P_i)$ and $|P_i| = a_i$.*

Enomoto and Ota considered the conditions from Theorems 1 and 2 at the same time, that is, a partition into paths with specified end vertices and given order. They also posed the following conjecture.

Conjecture 1 (Enomoto and Ota [7]). *Let t be an integer with $t \geq 2$ and let $a_1, a_2, \ldots, a_t$ be positive integers. Let G be a graph of order $n = \sum_{i=1}^{t} a_i$. If $\sigma_2(G) \geq n + t - 1$, for any t vertices $x_1, x_2, \ldots, x_t$, then there exist t vertex disjoint paths $P_1, P_2, \ldots, P_t$ such that $V(G) = \bigcup_{i=1}^{t} V(P_i)$, x_i is an end vertex of P_i and $|P_i| = a_i$.*

The sharpness of the aforementioned conjecture (if it is true) is given by $G_1 = K_t + (K_1 \cup K_{n-t-1})$. Notice $\sigma_2(G_1) = t + (n - t - 2 + t) = n + t - 2$. If the vertices of the K_t are the prescribed vertices, there is no way to use the vertex of the K_1 in a desired partition if $a_i \geq 3$ for all $1 \leq i \leq t$.

In [7], Enomoto and Ota showed positive results in the case $t = 3$ or $a_i = 3$ for all $1 \leq i \leq t$. Later Kawarabayashi [10] showed that if $\sigma_2(G) \geq \sum_{i=1}^{t} \max\{\lfloor \frac{4}{3}a_i \rfloor, a_i + 1\} - 1$, there exist t vertex disjoint paths desired in Conjecture 1. Magnant and Martin [11] proved an asymptotic version of this Conjecture 1 very similar to the results contained in this work but, in general, the conjecture is still unsolved.

The above results and conjecture are concentrated on a partition into paths with one specified end vertex. Along these lines, we have a question; what happens if we specify both end vertices? Egawa et al. [4] showed a result on a partition into cycles containing specified edges, and as a corollary, we obtain the following result.

Theorem 3 (Egawa et al. [4], Enomoto [6]). *Let t be an integer with $t \geq 2$ and let G be a graph of order $n \geq 4t - 1$. If $\sigma_2(G) \geq n + 2t - 2$, then for any $2t$ vertices $x_1, x_2, \ldots, x_t, y_1, \ldots, y_t$, there exist t vertex disjoint paths $P_1, P_2, \ldots, P_t$ such that $V(G) = \bigcup_{i=1}^{t} V(P_i)$ and x_i and y_i are end vertices of P_i.*

Like Conjecture 1, now we pose a conjecture regarding a partition into paths with both end vertices specified and given order.

Conjecture 2. *Let t be an integer with $t \geq 2$ and let $a_1, a_2, \ldots, a_t$ be positive integers. Let G be a graph of order $n = \sum_{i=1}^{t} a_i$. If $\sigma_2(G) \geq n + 2t - 1$, then for any $2t$ vertices $x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_t$, there exist t vertex disjoint paths $P_1, P_2, \ldots, P_t$ such that $V(G) = \bigcup_{i=1}^{t} V(P_i)$, x_i and y_i are end vertices of P_i and $|P_i| = a_i$.*

The graph $G_2 = K_{2t} + (K_1 \cup K_{n-2t-1})$ shows the sharpness of the $\sigma_2(G)$ condition of Conjecture 2. Notice $\sigma_2(G_2) = 2t + (n - 2t - 2 + 2t) = n + 2t - 2$. When we specify the $2t$ vertices from K_{2t}, there is no way to use the vertex of the K_1 in a desired partition if $a_i \geq 4$.

Note that Conjecture 2 is stronger than Conjecture 1. This is because, supposing that Conjecture 2 is true, let G be a graph satisfying the assumption of Conjecture 1 with specified vertices $x_1, x_2, \ldots, x_t$. We construct a new graph G' by joining t new vertices $y_1, y_2, \ldots, y_t$ to G and set $a_i' := a_i + 1$ and $n' := n + t$ (then n' is the order of G'). Since $\sigma_2(G') \geq n + t - 1 + 2t = n' + 2t - 1$, we can find t vertex disjoint paths $P_1', P_2', \ldots, P_t'$ such that $V(G') = \bigcup_{i=1}^{t} V(P_i')$, x_i and y_i are end vertices of P_i' and $|P_i'| = a_i' = a_i + 1$. Then t disjoint paths $P_i := P_i' - \{y_i\}$ are desired paths in Conjecture 1.

Therefore, we know that Conjecture 2 is at least as hard as Conjecture 1. The difficulty of Conjectures 1 and 2 seems to arise from considering a partition into paths with specified end vertex and given "exact" order. In fact, when we allow some flexibility or "tolerance" in the desired order, the situation is different. The main purpose of this paper is to find a partition into paths with both end vertices specified and with given order up to some "tolerance".

Theorem 4. *Let t be an integer with $t \geq 2$. For any set of t positive real numbers $\gamma_1, \gamma_2, \ldots, \gamma_t$ with $\sum_{i=1}^{t} \gamma_i = 1$ and for every $\epsilon > 0$ with $\epsilon < \min\{\frac{1}{18^2 t^2}, \frac{\gamma_i}{2}\}$, there exists an integer n_0 such that for every $(2t + 1)$-connected graph G of order $n \geq n_0$ with $\sigma_2(G) \geq n + 2t - 2$ and for every $2t$ vertices $x_1, x_2, \ldots, x_t, y_1, \ldots, y_t$, there exist t vertex disjoint paths $P_1, P_2, \ldots, P_t$ such that $V(G) = \bigcup_{i=1}^{t} V(P_i)$, x_i and y_i are end vertices of P_i and $(\gamma_i - \epsilon)n \leq |P_i| \leq (\gamma_i + \epsilon)n$ for all $1 \leq i \leq t$.*

We prove Theorem 4 in Section 4. Since $\sigma_2(G) \geq n + 2t - 1$ implies $\kappa(G) \geq 2t + 1$, this result immediately implies the following, which has a slightly larger degree sum condition in place of the connectivity assumption. Theorem 4 is stated and proven as above so it can be used in the proof of Theorem 6.

Corollary 5. *Let t be an integer with $t \geq 2$. For any set of t positive real numbers $\gamma_1, \gamma_2, \ldots, \gamma_t$ with $\sum_{i=1}^{t} \gamma_i = 1$ and for every $\epsilon > 0$ with $\epsilon < \min\{\frac{1}{18^2 t^2}, \frac{\gamma_i}{2}\}$, there exists an integer n_0 such that for every graph G of order $n \geq n_0$ with $\sigma_2(G) \geq n + 2t - 1$ and for every $2t$ vertices $x_1, x_2, \ldots, x_t$, $y_1, \ldots, y_t$, there exist t vertex disjoint paths $P_1, P_2, \ldots, P_t$ such that $V(G) = \bigcup_{i=1}^{t} V(P_i)$, x_i and y_i are end vertices of P_i and $(\gamma_i - \epsilon)n \leq |P_i| \leq (\gamma_i + \epsilon)n$ for all $1 \leq i \leq t$.*

Now we consider a partition into cycles. Posa [15] showed that with large degrees, one can guarantee a 2-factor with a bounded number of cycles. Egawa et al. [4] gave a $\sigma_2(G)$ condition to guarantee the existence of a partition into cycles each of which contains one specified vertices. On the other hand, El-Zahar [5] considered a partition into cycles with given order and posed a famous conjecture.

Similarly to a partition into paths, we pose the following conjecture. Moreover, in this paper, we also show a result on such a partition with some "tolerance".

Conjecture 3. *Let t be an integer with $t \geq 2$ and let $a_1, a_2, \ldots, a_t$ be positive integers. Let G be a graph of order $n = \sum_{i=1}^{t} a_i$. If $\sigma_2(G) \geq n + 2t - 2$, then for any t vertices $x_1, x_2, \ldots, x_t$, there exist t vertex disjoint cycles $C_1, C_2, \ldots, C_t$ such that $V(G) = \bigcup_{i=1}^{t} V(C_i)$, $x_i \in V(C_i)$ and $|C_i| = a_i$ for all $1 \leq i \leq t$.*

Theorem 6. *Let t be an integer with $t \geq 2$. For any set of t positive real numbers $\gamma_1, \gamma_2, \ldots, \gamma_t$ with $\sum_{i=1}^{t} \gamma_i = 1$ and for every $\epsilon > 0$ with $\epsilon < \min\{\frac{1}{18^2 t^2}, \frac{\gamma_i}{2}\}$, there exists an integer n_0 such that for every graph G of order $n \geq n_0$ with $\sigma_2(G) \geq n + 2t - 2$ and for every t vertices $x_1, x_2, \ldots, x_t$, there exist t vertex disjoint cycles $C_1, C_2, \ldots, C_t$ such that $V(G) = \bigcup_{i=1}^{t} V(C_i)$, $x_i \in V(C_i)$ and $(\gamma_i - \epsilon)n \leq |C_i| \leq (\gamma_i + \epsilon)n$ for all $1 \leq i \leq t$.*

The sharpness of Conjecture 3 is given by $G_3 = K_{2t-1} + (K_t \cup K_{n-3t+1})$. Notice $\sigma_2(G_2) = (t - 1 + 2t - 1) + (n - 3t + 2t - 1) = n + 2t - 3$. If the vertices of the K_t are the prescribed vertices, there is no way to construct t vertex disjoint cycles each of which containing exactly one prescribed vertex regardless of the choice of a_i. The proof of Theorem 6 is also left to Section 4.

Denote the distance, along a path P, between vertices u and v by $dist_P(u, v)$. All standard notation can be found in [1].

2. Preliminary results

Along with the classical results of Ore [13], Menger [12] and Dirac [3], the proofs of Theorems 4 and 6 use the following results of Williamson [17],

Ore [14] and Thomas and Wollan [16]. A graph is said to be *panconnected* (or *hamilton-connected*) if, between any pair of vertices $x, y \in V(G)$, there exists a path of every length l for $d \leq l \leq |G| - 1$ (a path of length $|G| - 1$, respectively), where d is the length of a shortest path connecting x and y. Williamson [17], and Ore [14] proved the following sufficient conditions for a graph to be panconnected, and hamilton-connected, respectively.

Theorem 7 (Williamson [17]). *Given a graph G of order n, if $\delta(G) \geq \frac{n+2}{2}$ then G is panconnected.*

Theorem 8 (Ore [14]). *Given a graph G of order n, if $\sigma_2(G) \geq n + 1$ then G is hamilton-connected.*

A graph is said to be *t-linked* if, for every choice of $2t$ vertices $x_1, \ldots, x_t$ and $y_1, \ldots, y_t$, there exists t vertex disjoint paths P_i such that x_i and y_i are end vertices of P_i for all i. We use the following result.

Theorem 9 (Thomas and Wollan [16]). *If a graph G is $10t$-connected, then G is t-linked.*

Our proof includes three main steps. The first creates a spanning collection of vertex disjoint paths (or cycles) starting at the chosen vertices. The second step moves vertices from paths which are long enough to paths which are too short if certain conditions are satisfied. When these conditions are not satisfied, we prove the graph has strong structure which allows us, in the third step, to build the desired path (or cycle) system directly.

3. Lemmas

Now we consider only a collection of paths. Theorem 4 will be used to prove Theorem 6 so there is no need to consider lemmas explicitly for Theorem 6.

For the sake of notation, we define some terminology. Throughout this and the next section, suppose we are given an integer $t \geq 3$, t positive real numbers $\gamma_1, \ldots, \gamma_t$ with $\sum \gamma_i = 1$ and a real number $0 < \epsilon \leq \frac{1}{18^2 t^2}$ such that $\epsilon \leq \frac{\gamma_i}{2}$ for all $1 \leq i \leq t$. Let G be a $(2t + 1)$-connected graph of order n sufficiently large (compared to t, γ_i's and ϵ) with a set $S = \{x_1, \ldots, x_t, y_1, \ldots, y_t\} \subset V(G)$ of $2t$ prescribed vertices. A collection of vertex disjoint paths $\mathscr{P} = \{P_1, P_2, \ldots, P_t\}$ is called *a path collection for S* if P_i connects x_i and y_i and $|P_i| \geq \epsilon n$ for any $1 \leq i \leq t$. A path collection $\mathscr{P}$ for S is *spanning* if $\bigcup_{i=1}^{t} V(P_i) = V(G)$ and is *desired* if $(\gamma_i - \epsilon)n < |P_i| < (\gamma_i + \epsilon)n$ for all $1 \leq i \leq t$.

Since there are many small constants used in this work, to simplify computations, we will frequently assume the constants, the value of n and

other variables used are carefully chosen to satisfy divisibility. For example, in place of $\lceil \epsilon n \rceil$, we will occasionally treat the quantity ϵn as an integer.

Our first lemma allows us to absorb vertices into path collections. In this process, we must allow some paths to get shorter. We carefully allow only very little decrease in length and only from paths which are long enough to sustain such a loss. The goal of the following lemma is to allow only paths of length at least $2\epsilon n$ to get shorter and only by a limited amount.

Lemma 1 (Absorbing)**.** *Let $\mathscr{P} = \{P_1, P_2, \ldots, P_t\}$ be a path collection for S. If $\sigma_2(G) \geq n + 2t - 2$, $\kappa(G) \geq 2t + 1$ and $n > \frac{20t}{\epsilon}$, then there exists a spanning path collection for S, $\mathscr{P}' = \{P_1', P_2', \ldots, P_t'\}$ such that $|P_i'| \geq |P_i|$ if $|P_i| < 2\epsilon n$ and $\sum_{j=1}^t \max\{|P_j| - |P_j'|, 0\} \leq \frac{20}{\epsilon^2}$.*

Proof. Let G be a graph as given above, let $P = \bigcup_{i=1}^t V(P_i)$ and let $J = V(G) \backslash P$. We suppose $\mathscr{P}$ is the largest such path collection or, in other words, suppose $|J|$ is minimum under the assumption each path is not shorter than its corresponding original. The following claims provide structure with which we prove the desired result.

Claim 1. *For every vertex $v \in J$, $d_P(v) \leq \frac{|P|+2t}{3}$.*

Proof of Claim 1. Let $v \in J$. The vertex v cannot be adjacent to any pairs of vertices u and u^+ which are consecutive along a path P_i as P_i could then be replaced with $P_i' = \ldots, u, v, u^+, \ldots$ to form a longer path collection and contradict the assumptions. This observation implies that $d_P(v) \leq \frac{|P|+t}{2}$, which means $d(v) \leq \frac{|P|+t}{2} + |J| - 1$. Suppose $d_P(v) > \frac{|P|+2t}{3}$. By the pigeonhole principle, there exists a vertex $u \in P_i$ for some $1 \leq i \leq t$ with $vu^+, vu^- \in E(G)$. By the above observation, we know $uv \notin E(G)$ and we may also assume $N_J(u) \cap N_J(v) = \emptyset$. So $d(v) + d(u) \geq n + 2t - 2$. Since $d_P(v) \leq \frac{|P|+t}{2}$ and $d_J(v) + d_J(u) \leq |J| - 1$, we get

$$d_P(u) \geq n + 2t - 2 - d(v) - d_J(u)$$
$$\geq n + 2t - \frac{|P| + t}{2} - |J| - 1$$
$$= \frac{|P|}{2} + \frac{3t}{2} - 1.$$

This implies that u must be adjacent to two consecutive vertices along some path $P_j \in \mathscr{P}$. Therefore, we may absorb u into P_j as above and replace P_i with $P_i' = \ldots, u^-, v, u^+, \ldots$ to form a larger path collection, contradicting our original assumption. $\quad\square_{Claim\ 1}$

Claim 2. *The graph induced on J is hamilton-connected.*

Proof of Claim 2. Suppose there exist vertices $x, y \in J$ with $xy \notin E(G)$. By the assumed degree condition, $d_G(x) + d_G(y) \geq n + 2t - 2$ but by Claim 1, $d_P(x), d_P(y) \leq \frac{|P|+2t}{3}$. Therefore

$$d_J(x) + d_J(y) \geq n + 2t - 2 - 2\left(\frac{|P| + 2t}{3}\right)$$

$$= |J| + \frac{|P| + 2t}{3} - 2$$

$$\geq |J| + 2.$$

By Theorem 8, J is hamilton-connected. $\square_{Claim\ 2}$

Claim 3. $|\bigcup_{v \in J} N_P(v)| \geq 2t + 1$. *In particular, there exists a path P_i such that $|\bigcup_{v \in J} N_{P_i}(v)| \geq 3$.*

Proof of Claim 3. Since G is $(2t + 1)$-connected, the first statement is obvious. The second statement of Claim 3 directly follows from the fact that $|\bigcup_{v \in J} N_P(v)| \geq 2t + 1$ and the pigeonhole principle. $\square_{Claim\ 3}$

Let $J_P = \cup_{j \in J} N_P(j)$ be the set of all neighbors in P of vertices of J.

Claim 4. *For every vertex $v \in J$, $d_P(v) \geq \frac{\epsilon}{4}n$.*

Proof of Claim 4. Suppose that there exists a vertex $v \in J$ such that $d_P(v) < \frac{\epsilon}{4}n$. Let A be a shortest path segment of P (not including end vertices) between vertices in J_P and let P_i be the path containing A. Note that $|A| < \frac{|P_i|}{2}$, by Claim 3. The goal of this claim is to move the vertices of A into $P_i \setminus A$ and absorb v into P_i as above, which contradicts the minimality of $|J|$.

Let w be any vertex of A. The segment A was chosen to be the smallest between edges from J so $wx \notin E(G)$ for all $x \in J$ and, in particular, $wv \notin E(G)$. Therefore we know

$$d_P(w) \geq n + 2t - 2 - d(v)$$

$$> n - |J| - \frac{\epsilon}{4}n + 2t - 1$$

$$\geq |P| - \frac{|P_i|}{4} + 2t - 1$$

$$\geq (|P| - |P_i|) + \frac{3|P_i|}{4} + 2t - 1$$

$$> (|P| - |P_i|) + \frac{|A|}{2} + \frac{|P_i|}{2} + 2t - 1$$

$$= (|P| - |P_i|) + |A| + \frac{|P_i| - |A|}{2} + 2t - 1.$$

This implies that for each $w \in A$, there exists a pair of consecutive vertices $u, u^+ \in P_i \setminus A$ such that $wu, wu^+ \in E(G)$ so we may move the vertices of A, one at a time in order to reapply the above argument, into $P_i \setminus A$ and absorb at least one vertex of J (or possibly all of J) into P_i. $\qquad \square_{Claim\ 4}$

Claim 5. $|J| < \frac{5}{\epsilon}$.

Proof of Claim 5. Suppose $|J| \geq \frac{5}{\epsilon}$. Let ℓ be the number of path segments between vertex disjoint edges from J to P. Since J is hamilton-connected (by Claim 2), we know these segments must have length at least $|J| + 1$. Let L be the set of vertices in the segments above including the endvertices. Thus $|L| \geq \ell(|J| + 1)$ but furthermore, $d_L(j) \leq \ell + t$ for all $j \in J$ so L has at most $(\ell + t)|J|$ edges to J. Also note that every vertex of P with at least 2 edges to J must be in L. This means that $P \setminus L$ has at most $|P \setminus L|$ edges to J. Conversely, by Claim 4, $d_P(j) \geq \frac{\epsilon n}{4}$ for all $j \in J$ meaning that there are at least $\frac{|J|\epsilon n}{4}$ edges between P and J. This implies

$$(\ell + t)|J| + |P \setminus L| \geq \frac{\epsilon n |J|}{4}$$

or, since $|J| \geq \frac{5}{\epsilon}$,

$$|P \setminus L| \geq \frac{5n}{4} - \frac{5(\ell + t)}{\epsilon}.$$

With $|L| \geq \ell(|J| + 1) > \frac{5\ell}{\epsilon}$ and $|P| < n$, this implies that $\frac{n}{4} < \frac{5t}{\epsilon}$, a contradiction. $\qquad \square_{Claim\ 5}$

By Claims 4 and 5, we may assume $d_P(v) \geq \frac{\epsilon n}{4}$ for any $v \in J$ and $|J| < \frac{5}{\epsilon}$. The rest of the proof is similar to the proof of Claim 4. We again consider a segment A between adjacencies of a vertex in J. Unfortunately, this time we have to be more careful about how we remove the vertices of A from the path.

Let $J' := \{v \in J : d_P(v) \geq \frac{n}{2t+1}\}$ and let $\bar{J} := J - J'$. First, we will absorb a vertex in J' into paths in $\mathscr{Q}$, where $\mathscr{Q} := \{P_i \in \mathscr{P} : |P_i| \geq 2\epsilon n\}$. Note that $\mathscr{Q} \neq \emptyset$. Let $\mathscr{Q} = \{Q_1, Q_2, \dots\}$ and let Q be the set of vertices in paths of $\mathscr{Q}$. There are only t total paths, so we know $|Q| \geq (1 - 2(t-1)\epsilon)n - |J|$

and, since n is sufficiently large and $|J| < \frac{5}{\epsilon}$, we get $|Q| \geq (1 - 3t\epsilon)n$. Since ϵ was chosen such that $\epsilon \leq \frac{1}{6t(2t+1)}$, we know that $|Q| \geq (1 - \frac{1}{2(2t+1)})n$.

Let $v \in J'$. Since $|P \setminus Q| \leq \frac{n}{2(2t+1)}$ and $d_P(v) \geq \frac{n}{2t+1}$, we have $d_Q(v) \geq \frac{n}{2(2t+1)}$. Let A be the smallest segment of a path $Q_i \in \mathcal{Q}$ between (not including end vertices) two vertices adjacent to v. Since $d_Q(v) \geq \frac{n}{2(2t+1)}$, we may choose n sufficiently large such that

$$|A| \leq \frac{(1 - \frac{1}{2(2t+1)})n}{\frac{n}{2(2t+1)} - t} \leq 3(2t+1).$$

Let $w \in A$. Of course $vw \notin E(G)$ so by Claim 1,

$$d_P(w) \geq n + 2t - 2 - \left(\frac{|P| + 2t}{3} + |J| - 1 \right) - (|J| - 1)$$
$$= \frac{|P| + t + |A|}{2} + \frac{|P|}{6} + \frac{13t}{6} - \frac{|A|}{2} - |J|$$
$$\geq \frac{|P| + t + |A|}{2},$$

and we may absorb w into $P \setminus A$. This same process may be repeated for all $w \in A$ and we may then absorb v into Q. By this process, the path Q_i loses at most $|A| \leq 3(2t+1)$ vertices. We repeat this process for each vertex of J'. This means that paths of $\mathcal{Q}$ lose at most a total of $|J'|(6t+3)$ vertices.

Note that we are now left with $d_P(v) < \frac{n}{2t+1}$ for all $v \in \bar{J}$. Let $v \in \bar{J}$ and choose a smallest segment A along a path P_i between two adjacencies of v. Since $d_P(v) \geq \frac{\epsilon n}{4}$, we know the length of this segment is at most $|A| \leq \frac{n}{\frac{\epsilon n}{4} - t} < \frac{5}{\epsilon}$ for n sufficiently large. If $|P_i| \geq 2\epsilon n$, we absorb v as above. This creates a new path P_i' with $|P_i'| \geq |P_i| - \frac{5}{\epsilon}$. Therefore we suppose A is a subpath of a path P_j where $\epsilon n \leq |P_j| < 2\epsilon n$.

Let u be a vertex of A and note that, since A was chosen to be the smallest such segment, $uv \notin E(G)$. This means that

$$d(u) \geq n + 2t - 2 - d(v)$$
$$> n + 2t - 2 - \left(\frac{n}{2t+1} + |J| - 1 \right)$$
$$\geq \left(1 - \frac{1}{2t} \right) n$$
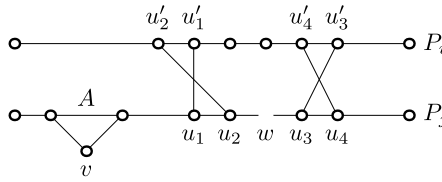$$\geq \frac{5n}{6}$$

Figure 1: Rearrangement of paths.

since n is sufficiently large and $t \geq 3$. Certainly the same is true for every vertex $u \in V(P_j) \setminus N(v)$. Assign an ordering ℓ to the vertices of each path P_i such that $\ell(u) = dist_{P_i}(x_i, u) + 1$. Since v cannot be adjacent to two vertices of P_j with only one vertex in between, there exist four distinct vertices $u_1, u_2, u_3, u_4 \in P_j \setminus (A \cup N(v))$ such that $\ell(u_1) = \ell(u_2) - 1$, $\ell(u_3) = \ell(u_4) - 1$ and $\ell(u_2) + 1 \leq \ell(u_3) \leq \ell(u_2) + 2$.

Let $p = |P \setminus Q|$ and suppose $\frac{5n}{6} - p < \frac{19}{24}(n - p - |J|)$. Since $p \leq 2t\epsilon n$, this inequality implies $n < 10t\epsilon n - 19|J|$, contradicting the fact that n is sufficiently large. Thus, $d_Q(u_i) \geq \frac{5n}{6} - p \geq \frac{19}{24}(n - |J| - p) = \frac{19}{24}|Q|$ for each vertex u_i. By the pigeonhole principle, there exists at least $2 \cdot \frac{19}{24}|Q| - t - |Q| = \frac{7}{12}|Q| - t$ pairs of distinct vertices u_1', u_2' (similarly u_3', u_4') in paths of $\mathcal{Q}$ with $u_i u_i' \in E(G)$ for all i and $\ell(u_1') = \ell(u_2') + 1$ (similarly $\ell(u_3') = \ell(u_4') + 1$).

Since $2(\frac{7}{12}|Q| - t) - |Q| - 3t > 0$, we may again apply the pigeonhole principle. Hence, there exists a set of four vertices u_1', u_2', u_3', u_4' in a single path $P_i \in \mathcal{Q}$ such that $\ell(u_1') = \ell(u_2') + 1$, $\ell(u_3') = \ell(u_4') + 1$ and $\ell(u_4') = \ell(u_1') + 3$. See Figure 1 where the shaded vertex w may or may not be present.

At this point we may swap $P_j = \ldots, u_1, u_2, \ldots, u_3, u_4, \ldots$ for $P_j' = \ldots, u_1, u_1', \ldots, u_4', u_4, \ldots$ and swap $P_i = \ldots, u_2', u_1', \ldots, u_4', u_3', \ldots$ for $P_i' = \ldots, u_2', u_2, \ldots, u_3, u_3', \ldots$. This swap makes $|P_j| + 1 \leq |P_j'| \leq |P_j| + 2$ and $|P_i| - 2 \leq |P_i'| \leq |P_i| - 1$. The swap process may be repeated as necessary to make $|P_j'| \geq |P_j| + |A|$ (thereby removing at most $|A| + 1 < \frac{5}{\epsilon} + 1$ vertices from paths of $\mathcal{Q}$).

We may now absorb v into P_j', losing the vertices of A to other paths as above. Again, we may repeat this process to absorb all of $\bar{J}$. In both processes, we lose at most a total of $|J'|(6t + 3) + |\bar{J}|(\frac{5}{\epsilon} + 1 - 1) \leq |J|\frac{5}{\epsilon}$ vertices from $\mathcal{Q}$. Thus, for all $P_j \in \mathcal{Q}$, we have $|P_j'| \geq 2\epsilon n - \frac{5}{\epsilon} \cdot \frac{5}{\epsilon} \geq \epsilon n$. Moreover, we also know that $\sum_{i=1}^{t} \max\{|P_i| - |P_i'|, 0\} \leq \frac{20}{\epsilon^2}$. This completes the proof of Lemma 1. $\square$

Our next lemma requires some specific definitions. Suppose we are given a path collection $\mathcal{P} = \{P_1, \ldots, P_t\}$. Let $f(P_i) = \gamma_i n - |P_i|$. Without loss of

generality, suppose the paths P_i are ordered such that $f(P_i) \geq f(P_{i+1})$. By the definition of $f(P_i)$ and γ_i's, note that $f(P_1) \geq 0$, because otherwise,

$$0 > \sum_{i=1}^{t} f(P_i)$$
$$= \sum_{i=1}^{t} \gamma_i n - \sum_{i=1}^{t} |P_i|$$
$$\geq n - n = 0,$$

a contradiction. Define:

$$\mu(\mathscr{P}) = \sum_{i=1}^{t} (t + 1 - i) f(P_i).$$

Proposition 1. *If $\mathscr{P}$ is a spanning path collection but not desired, then there exists an integer k such that $f(P_k) - f(P_{k+1}) > \frac{\epsilon}{t^2} n$ and $f(P_{k+1}) < -\frac{\epsilon}{t^2} n$.*

Proof. If $f(P_t) > -\frac{\epsilon}{t} n$, we get $|P_i| < (\gamma_i + \frac{\epsilon}{t}) n < (\gamma_i + \epsilon) n$ for all i, which implies $|P_i| = n - \sum_{j \neq i} |P_j| > (\gamma_i - \epsilon) n$ for all i. This contradicts the assumption $\mathscr{P}$ is not a desired path collection. Thus, $f(P_t) \leq -\frac{\epsilon}{t} n$.

Suppose that $f(P_k) - f(P_{k+1}) \leq \frac{\epsilon}{t^2} n$ for all $1 \leq k \leq t - 1$. Then $f(P_1) \leq f(P_t) + (t-1) \frac{\epsilon}{t^2} n < \frac{-\epsilon}{t^2} n < 0$ because $f(P_t) \leq -\frac{\epsilon}{t} n$, a contradiction. Therefore there exists an integer k such that $f(P_k) - f(P_{k+1}) > \frac{\epsilon}{t^2} n$. If we take such an integer k as large as possible, we have $f(P_{k+1}) < -\frac{\epsilon}{t^2} n$ because $f(P_t) < -\frac{\epsilon}{t} n$ and $f(P_i) - f(P_{i+1}) \leq \frac{\epsilon}{t^2} n$ for any $k + 1 \leq i \leq t - 1$. □

Next we extract a lemma from [8] in a form that is appropriate to our current situation. This lemma takes two paths and, under certain conditions, shows the existence of two other paths which are, in some sense, better. For ease of notation, let $c = \frac{\epsilon^2}{24 \cdot 48 t^2}$.

Lemma 2 (Swapping). *Let $\mathscr{P} = \{P_1, P_2, \ldots, P_t\}$ be a spanning path collection for S but not desired. If $\kappa(G) \geq 2t + 1$, $\sigma_2(G) \geq n + 2t - 2$ and if $e(\mathscr{A}, \mathscr{P} - \mathscr{A}) \geq ct^2 n^2$ for any subcollection $\mathscr{A}$ of $\mathscr{P}$, then there exists a spanning path collection $\mathscr{Q}$ such that $\mu(\mathscr{Q}) < \mu(\mathscr{P})$.*

Proof. Let $\mathscr{P}$ be as stated. By Proposition 1, there exists an integer k with $f(P_k) - f(P_{k+1}) \geq \frac{\epsilon}{t^2} n$ and $f(P_{k+1}) < \frac{-\epsilon}{t^2} n$. Let $\mathscr{A} = \{P_i : 1 \leq i \leq k\}$. Since $e(\mathscr{A}, \mathscr{P} - \mathscr{A}) \geq ct^2 n^2$, there must exist paths $P_i \in \mathscr{A}$ and $P_j \in \mathscr{P} - \mathscr{A}$ with $e(P_i, P_j) \geq cn^2$. Notice that $1 \leq i \leq k$ and $k + 1 \leq j \leq t$.
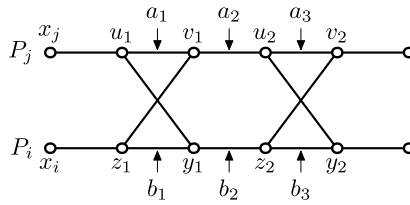
Figure 2: Swapping.

Let $A = V(P_j)$ and $B = V(P_i)$. Let $A' \subseteq A$ denote the set of vertices $v \in A$ with $d_B(v) \geq \frac{cn^2}{2|A|}$. Since $|A| \cdot |B| \leq \frac{n^2}{4}$, note that $d_B(v) \geq \frac{cn^2}{2|A|} \geq 2c|B|$ for any $v \in A'$. Since $e(A, B) \geq cn^2$, we find

$$|A'| \geq \frac{cn^2 - (|A| - |A'|)\left(\frac{cn^2}{2|A|}\right)}{|B|}$$

$$\geq \frac{cn^2}{2|B|} \geq 2c|A|.$$

Assign a labeling $l(v)$ of the vertices of A and B given by the distance along P_j or P_i from x_j or x_i respectively. Define a *crossing pair* to be a pair of edges uy and vz with $u, v \in A$ and $y, z \in B$ such that $l(u) < l(v)$ and $l(z) < l(y)$. Define the *gap* of a crossing pair to be $|l(y) - l(z)| - 1$. We will concern ourselves only with crossing pairs with gap length at most $\frac{4}{c}$.

Consider Figure 2 consisting of two crossing pairs $u_1 y_1$, $v_1 z_1$ and $u_2 y_2$, $v_2 z_2$ where $a_1 = l(v_1) - l(u_1) - 1 \geq 0$, $a_2 = l(u_2) - l(v_1) + 1 > 0$, $a_3 = l(v_2) - l(u_2) - 1 > 0$, $b_1 = l(y_1) - l(z_1) - 1 > 0$, $b_2 = l(z_2) - l(y_1) + 1 \geq 0$ and $b_3 = l(y_2) - l(z_2) - 1 > 0$. The goal of this lemma is to create new paths $P'_j = x_j, \ldots, u_1, y_1, \ldots, z_2, v_2, \ldots, y_j$ and $P'_i = x_i, \ldots, z_1, v_1, \ldots, u_2, y_2, \ldots, y_i$ and by using Lemma 1 to absorb vertices which are removed by this operation. Note that $|P'_j| = |P_j| - (a_1 + a_2 + a_3) + b_2$ and $|P'_i| = |P_i| - (b_1 + b_2 + b_3) + a_2$.

First, we find two crossing pairs as in Figure 2 which satisfy:

$$a_2 - b_2 \geq \frac{64t + 112}{c^2},$$

$$a_1 + a_3 + b_1 + b_3 \leq \frac{32 + 8c}{c^2},$$

$$a_1 + a_2 + a_3 - b_2 < \frac{\epsilon}{2t^2} n,$$

$$a_2 - (b_1 + b_2 + b_3) < \frac{\epsilon}{2t^2} n.$$

Partition the vertices of A' into collections of $\lceil \frac{4}{3c} \rceil$ consecutive (within A') vertices. As mentioned earlier, we will assume constants are chosen so divisibility is satisfied so, as opposed to sticking with $\lceil \frac{4}{3c} \rceil$, we will simply write $\frac{4}{3c}$. Call each such collection a *chunk*.

Claim 6. *Given a chunk C, there are at least $\frac{c|B|}{2}$ crossing pairs within C, all vertex disjoint in B with gap length at most $\frac{4}{c}$.*

Proof of Claim 6. We will find many desired crossing pairs iteratively. Suppose we already found r crossing pairs. In other words, $2r$ vertices of B are used for the r crossing pairs and we cannot use such $2r$ vertices for crossing pairs we will find after this. If $r \geq \frac{c|B|}{2}$, there is nothing to prove. So we may assume that $r < \frac{c|B|}{2}$. Suppose that there exists no more crossing pair with gap length at most $\frac{4}{c}$. Label the vertices of the chunk C as $v_1, v_2, \ldots$ such that $l(v_g) < l(v_h)$ for all $g < h$. We know $d_B(v_1) \geq 2c|B|$ but at most $2r$ vertices of them are already used for the previous crossing pairs. Therefore there are at least $2c|B| - 2r \geq c|B|$ new adjacencies of v_1.

Now suppose we have considered $v_1, v_2, \ldots, v_{h-1}$ and let v_h be the leftmost remaining vertex of C which has not yet been considered. Suppose that $\frac{4}{c}$ consecutive vertices of B have at least two vertices, say y and z with $l(y) > l(z)$, which are not used for the previous crossing pairs and which are shared by v_h and one of $v_1, v_2, \ldots, v_{h-1}$ as neighbors. Then we can find two edges $v_g y$ and $v_h z$ for some $1 \leq g \leq h-1$, which is a new crossing pair. Therefore v_h has at least $2c|B| - 2r - |B|/\frac{4}{c} > \frac{3c|B|}{4}$ new adjacencies. This implies that each vertex v_h of C forces at least $\frac{3c|B|}{4}$ new vertices of B not to have any adjacencies in $C \setminus \{v_1, v_2, \ldots, v_{h-1}\}$.

After considering $\frac{4}{3c}$ vertices of C, at least

$$c|B| + \left(\frac{4}{3c} - 1 \right) \frac{3c|B|}{4} = \left(1 + \frac{c}{4} \right) |B|$$

vertices of B are used as adjacencies of $v_1, v_2, \ldots, v_{\frac{4}{3c}}$, which is a contradiction and completes the proof of the claim. $\square_{Claim\ 6}$

Given two crossing pairs $u_1 y_1, v_1 z_1$ and $u_2 y_2, v_2 z_2$, we say these pairs form a *swapping structure* if $l(u_i) > l(v_j)$ and $l(z_i) > l(y_j)$ for some choice of $i, j \in \{1, 2\}$. For this choice of i and j, define the *distance* of the swapping structure to be $l(z_i) - l(y_j) + 1$ (or the distance in B between the vertices of the crossing pairs).

Claim 7. *Within any collection of $\frac{4}{c}$ chunks, there exists a swapping structure with distance h such that $0 \leq h \leq \frac{16}{c^2}$.*

Proof of Claim 7. This claim employs a proof almost identical to that of Claim 6. Label the chunks as $C_1, C_2, \ldots$ in the order of l. We first consider the left-most chunk C_1. This chunk has at least $\frac{c|B|}{2}$ crossing pairs so there are at least $\frac{c|B|}{2}$ right-vertices of these pairs.

As before, we suppose $h-1$ chunks $C_1, C_2, \ldots, C_{h-1}$ have been considered and consider the left-most chunk C_h of the remaining set of chunks. For every $\frac{16}{c^2}$ consecutive vertices of B, suppose that C_h shares at least $\frac{4}{c}$ right-vertices of crossing pairs with $C_1, C_2, \ldots, C_{h-1}$. Then, there exist two right-vertices y_1 and y_2 which are shared by C_h and some chunk of $C_1, C_2, \ldots, C_{h-1}$ such that $l(y_2) - l(y_1) \geq \frac{4}{c} - 1$. Let z_2 be a left-vertex of the crossing pair in C_h which corresponds to y_2. Since the gap of this crossing pair is at most $\frac{4}{c}$, so $l(z_2) \geq l(y_2) - \frac{4}{c} + 1 \geq l(y_1)$. Thus, the crossing pair of C_h containing y_2 and z_2 and the one of C_g containing y_1 for some $1 \leq g \leq h-1$ create a swapping structure with distance at most $\frac{16}{c^2}$. Therefore this chunk may share a total of at most $(\frac{4}{c})(\frac{|B|}{16/c^2}) = \frac{c|B|}{4}$ right-vertices with preceding chunks. This implies that at least $\frac{c|B|}{4}$ right-vertices are new.

Hence, after consideration of the $\frac{4}{c}$ chunks, there are

$$|B| - \frac{c|B|}{2} - \left(\frac{4}{c} - 1\right)\frac{c|B|}{4} = \frac{-c|B|}{4} < 0$$

vertices available in B for right-vertices of crossing pairs which is again a contradiction completing the proof of the claim. $\square_{Claim\ 7}$

Given a chunk C, define the *span* of C to be the number of vertices of A between the left-most and right-most vertex of the chunk. Since the chunks have $\frac{4}{3c}$ vertices of A' and $|A'| \geq 2c|A|$, the total number of chunks is $|A'|/\frac{4}{3c} \geq \frac{3c^2}{2}|A|$. Suppose that the number of chunks of span at most $\frac{16}{c^2}$ is at most $\frac{23c^2}{16}|A|$. Then $|A| > \frac{16}{c^2}(\frac{3c^2}{2} - \frac{23c^2}{16})|A| = |A|$, a contradiction. Thus, we see that the number of chunks of span at most $\frac{16}{c^2}$ is at least $\frac{23c^2}{16}|A|$. We call such short chunks *good* and since there are many such chunks, we consider only those which are good.

We now mark chunks that are at the desired distance apart within A. Start at the beginning of A (in terms of the original labeling) and mark the first good chunk. We skip the next $\frac{64(t+2)}{c^2}$ vertices. We then mark the next good chunk which starts after the skipped vertices and repeat this process until we have crossed the entire length of A. Note that at most $\frac{64(t+2)}{c^2}/\frac{4}{3c} + 1 = \frac{48(t+2)}{c} + 1$ good chunks are contained in the skipped vertices.

Certainly there are at least $\frac{23c^2}{16}|A|/(\frac{48(t+2)}{c}+2) = \frac{23c^3|A|}{16(48t+96+2c)}$ marked chunks.

Consider any segment of $\frac{\epsilon}{4t^2}n$ consecutive vertices of A and suppose that any such segments has at most $\frac{4}{c}$ marked chunks. Then the total number of marked chunks is at most $\frac{8}{c}(|A|/\frac{\epsilon}{4t^2}n) = \frac{32t^2|A|}{c\epsilon n}$. Therefore $\frac{32t^2|A|}{c\epsilon n} \geq \frac{23c^3|A|}{16(48t+96+2c)}$, or $n \leq \frac{16\cdot 32t^2(48t+96+2c)}{23c^4\epsilon}$, which contradicts the assumption n is sufficiently large. So there must exist a segment containing at least $\frac{4}{c}$ marked chunks.

By Claim 7, there exists a swapping structure within these chunks. This is the desired swapping structure since (using the notation from Figure 2)

$$a_2 - b_2 \geq \frac{64(t+2)}{c^2} - \frac{16}{c^2} = \frac{64t+112}{c^2},$$
$$a_1 + a_3 + b_1 + b_3 \leq 2\frac{16}{c^2} + 2\frac{4}{c} = \frac{32+8c}{c^2},$$
$$a_1 + a_2 + a_3 - b_2 \leq \frac{\epsilon}{4t^2}n + 2\cdot\frac{16}{c^2} < \frac{\epsilon}{2t^2}n, \quad \text{and}$$
$$a_2 - (b_1 + b_2 + b_3) \leq \frac{\epsilon}{4t^2}n < \frac{\epsilon}{2t^2}n$$

for n sufficiently large.

As mentioned before, we obtain two new paths P_i' and P_j' within this structure. Let $\mathscr{P}' = (\mathscr{P} - \{P_i, P_j\}) \cup \{P_i', P_j'\}$. Since $|P_j| = \gamma_j n - f(P_j) \geq \gamma_j n - f(P_{k+1}) > \gamma_j n + \frac{\epsilon}{t^2}n$, note that $|P_j'| = |P_j| - (a_1 + a_2 + a_3) + b_2 > \gamma_j n + \frac{\epsilon}{t^2}n - \frac{\epsilon}{2t^2}n \geq \epsilon n$. We know that $|P_i'| = |P_i| - (b_1 + b_2 + b_3) + a_2 > |P_i| \geq \epsilon n$. So $|P_l'| \geq \epsilon n$ for any $P_l' \in \mathscr{P}'$. Now we calculate the difference $\mu(\mathscr{P}) - \mu(\mathscr{P}')$.

We reorder paths in $\mathscr{P}'$ and rename them Q_l's so that $f(Q_l) \geq f(Q_{l+1})$ for all $1 \leq l \leq t-1$. Define a mapping ϕ from $\{1, 2, \ldots, t\}$ to $\{1, 2, \ldots, t\}$ so that $Q_{\phi(l)} = P_l$ for $l \neq i, j$ and $Q_{\phi(i)} = P_i'$ and $Q_{\phi(j)} = P_j'$. By the definition of P_i' and P_j',

$$\begin{aligned}
f(P_i') - f(P_j') &= \gamma_i n - |P_i'| - \gamma_j n + |P_j'| \\
&= \gamma_i n - \gamma_j n - \big(|P_i| - (b_1 + b_2 + b_3) + a_2\big) \\
&\quad + \big(|P_j| - (a_1 + a_2 + a_3) + b_2\big) \\
&= f(P_i) - f(P_j) - (a_1 + a_2 + a_3 - b_2) \\
&\quad - (a_2 - (b_1 + b_2 + b_3)) \\
&> \left(\frac{\epsilon}{t^2} - 2\frac{\epsilon}{2t^2}\right)n \\
&= 0.
\end{aligned}$$

This implies that $f(P_i') > f(P_j')$ and hence $\phi(i) < \phi(j)$. So,

$$
\phi(l) = \begin{cases}
l & \text{for } 1 \le l \le i-1, \text{ or } \phi(i)+1 \le l \le \phi(j)-1 \\
& \quad \text{or } j+1 \le l \le t, \\
l-1 & \text{for } i+1 \le l \le \phi(i), \\
l+1 & \text{for } \phi(j) \le l \le j-1.
\end{cases}
$$

Denote $g(Q_{\phi(l)}) := \gamma_l n - |Q_{\phi(l)}|$. Then $\mu(\mathscr{Q}) = \sum_{l=1}^{t}(t-\phi(l)+1)g(Q_{\phi(l)})$. Since

$$
\mu(\mathscr{P}) - \mu(\mathscr{Q}) = \sum_{l=1}^{t}\Big((t-l+1)f(P_l) - (t-\phi(l)+1)g(Q_{\phi(l)})\Big),
$$

it suffices to calculate each member of the right side. By the definition of ϕ, for $1 \le l \le i-1$, or $\phi(i)+1 \le l \le \phi(j)-1$ or $j+1 \le l \le t$,

$$
\begin{aligned}
(t-l+1)f(P_l) &- (t-\phi(l)+1)g(Q_{\phi(l)}) \\
&= (t-l+1)f(P_l) - (t-l+1)f(P_l) \\
&= 0.
\end{aligned}
$$
(1)

For $i+1 \le l \le \phi(i)$,

$$
\begin{aligned}
(t-l+1)f(P_l) &- (t-\phi(l)+1)g(Q_{\phi(l)}) \\
&= (t-l+1)f(P_l) - (t-l+2)f(P_l) \\
&= -f(P_l),
\end{aligned}
$$
(2)

and for $\phi(j) \le l \le j-1$,

$$
\begin{aligned}
(t-l+1)f(P_l) &- (t-\phi(l)+1)g(Q_{\phi(l)}) \\
&= (t-l+1)f(P_l) - (t-l)f(P_l) \\
&= f(P_l).
\end{aligned}
$$
(3)

For $l = i$,

$$
\begin{aligned}
(t-i+1)&f(P_i) - (t-\phi(i)+1)g(Q_{\phi(i)}) \\
&= (t-i+1)\big(\gamma_i n - |P_i|\big) - (t-\phi(i)+1)\big(\gamma_i n - |Q_{\phi(i)}|\big) \\
&= (t-i+1)\big(\gamma_i n - |P_i|\big) \\
&\quad - (t-\phi(i)+1)\big(\gamma_i n - |P_i| + (b_1+b_2+b_3) - a_2\big)
\end{aligned}
$$

$$= (\phi(i) - i)(\gamma_i n - |P_i|) - (t - \phi(i) + 1)(b_1 + b_2 + b_3 - a_2)$$

(4)
$$= (\phi(i) - i)f(P_i) - (t - \phi(i) + 1)(b_1 + b_2 + b_3 - a_2),$$

and for $l = j$,

$$(t - j + 1)f(P_j) - (t - \phi(j) + 1)g(Q_{\phi(j)})$$
$$= (t - j + 1)(\gamma_j n - |P_j|) - (t - \phi(j) + 1)(\gamma_j n - |Q_{\phi(j)}|)$$
$$= (t - j + 1)(\gamma_j n - |P_j|)$$
$$\quad - (t - \phi(j) + 1)(\gamma_j n - |P_j| + (a_1 + a_2 + a_3) - b_2)$$
$$= -(j - \phi(j))(\gamma_j n - |P_j|) - (t - \phi(j) + 1)(a_1 + a_2 + a_3 - b_2)$$

(5)
$$= -(j - \phi(j))f(P_j) - (t - \phi(j) + 1)(a_1 + a_2 + a_3 - b_2),$$

Therefore, Equations (1)–(5) imply

$$\mu(\mathscr{P}) - \mu(\mathscr{Q}) = -(t - \phi(i) + 1)(b_1 + b_2 + b_3 - a_2)$$
$$- (t - \phi(j) + 1)(a_1 + a_2 + a_3 - b_2) - \sum_{l=i+1}^{\phi(i)} f(P_l)$$
$$+ \sum_{l=\phi(j)}^{j-1} f(P_l) + (\phi(i) - i)f(P_i) - (j - \phi(j))f(P_j).$$

Since $f(P_l) \leq f(P_i)$ for any $i + 1 \leq l \leq \phi(i)$, we have $\sum_{l=i+1}^{\phi(i)} f(P_l) \leq (\phi(i) - i)f(P_i)$, and since $f(P_l) \geq f(P_j)$ for any $\phi(j) \leq l \leq j - 1$,

$$\sum_{l=\phi(j)}^{j-1} f(P_l) \geq (j - \phi(j))f(P_j).$$

So,

$$\mu(\mathscr{P}) - \mu(\mathscr{Q}) \geq -(t - \phi(i) + 1)(b_1 + b_2 + b_3 - a_2)$$
$$- (t - \phi(j) + 1)(a_1 + a_2 + a_3 - b_2)$$
$$= (\phi(j) - \phi(i))(a_2 - b_2)$$
$$- (t - \phi(i) + 1)(a_1 + a_3) - (t - \phi(j) + 1)(b_1 + b_3)$$
$$\geq (a_2 - b_2) - t(a_1 + a_3 + b_1 + b_3)$$

(6)
$$\geq \frac{64t + 112}{c^2} - t(a_1 + a_3 + b_1 + b_3).$$

Now we apply Lemma 1 and we get a spanning path collection $\mathcal{Q}'$. Since $|Q_l| \geq \epsilon n$ for all $1 \leq l \leq t$ and $|V(G)| - \sum_{l=1}^{t} |P_l| \leq a_1 + a_3 + b_1 + b_3 \leq \frac{32+8c}{c^2} \leq \frac{n}{8}$, Lemma 1 can work for $\mathcal{Q}$. We now show that $\mu(\mathcal{Q}') < \mu(\mathcal{P})$. The upper bound of increase from $\mathcal{Q}$ to $\mathcal{Q}'$ is at most t times the number of vertices which may be added to paths, because t is the maximum weight over the sum of μ. In particular, in the application of Lemma 1, we add at most $a_1 + a_3 + b_1 + b_3$ since we do not have control over where these vertices fall and Lemma 1 adds at most an additional $\frac{20}{\epsilon^2}$ vertices. Hence,

$$(7) \qquad \mu(\mathcal{Q}') - \mu(\mathcal{Q}) \leq t \cdot (a_1 + a_3 + b_1 + b_3) + t \cdot \frac{20}{\epsilon^2}.$$

Thus, using Inequalities (6) and (7), we get

$$
\begin{aligned}
\mu(\mathcal{P}) - \mu(\mathcal{Q}') &\geq \frac{64t + 112}{c^2} - 2t(a_1 + a_3 + b_1 + b_3) - t \cdot \frac{20}{\epsilon^2} \\
&\geq \frac{64t + 112}{c^2} - \frac{64t + 16ct}{c^2} - \frac{20t}{\epsilon^2} \\
&= \frac{112}{c^2} - \frac{16t}{c} - \frac{20t}{\epsilon^2} \\
&\geq \frac{112 \cdot 24^2 \cdot 48^2 t^4}{\epsilon^4} - \frac{16 \cdot 24 \cdot 48 t^3}{\epsilon^2} - \frac{20t}{\epsilon^2} \\
&> 0,
\end{aligned}
$$

because $c = \frac{\epsilon^2}{24 \cdot 48 t^2}$ and $\epsilon \leq \frac{1}{6t(2t+1)}$. Thus, $\mu(\mathcal{Q}') < \mu(\mathcal{P})$. This completes the proof of Lemma 2. $\qquad\square$

Our next lemma constructs the desired path system directly if the conditions of Lemma 2 are not satisfied. The proof is similar to the proof of the corresponding Rebuilding Lemma in [8].

Lemma 3 (Rebuilding)**.** *Given a set of fractions $\gamma_1, \ldots, \gamma_t$, let c_1 and c_2 be positive real numbers with $c_1 \leq \frac{\epsilon}{24}$ and $c_2 < \min\{\frac{c_1}{3}, \gamma_i/16\}$ for all i. Suppose that n is sufficiently large (compared to c_1 and c_2). If $\kappa(G) \geq 2t+1$, $\sigma_2(G) \geq n + 2t - 2$ and $V(G)$ can be partitioned into sets A and B with $|A|, |B| \geq (4c_1 + c_2)n$ and $e(A, B) < c_1 c_2 n^2$ then G contains a spanning desired path collection for S.*

Proof. Let D_A be the set of vertices in A with at least $c_1 n$ edges into B and let D_B be the set of vertices in B with at least $c_1 n$ edges into A. We know $e(A, B) < c_1 c_2 n^2$ so $|D_A|, |D_B| < c_2 n$. Let $A' := A \setminus D_A$, $B' := B \setminus D_B$ and let $u \in A'$ and $v \in B' \setminus N(u)$. Since $d_B(u) < c_1 n$, for every choice of u,

there exists such a v. The edge $uv \notin E(G)$ so $d(u) + d(v) \geq n + 2t - 2$ but $d_B(u), d_A(v) < c_1 n$ so $d_A(u) + d_B(v) > n - 2c_1 n + 2t - 2$. In particular, this implies that for all vertices $u \in A'$, we have $d_A(u) > |A| - 2c_1 n + 2t - 2$. By symmetry $d_B(v) > |B| - 2c_1 n + 2t - 2$ for all $v \in B'$.

Claim 8. *For any $\tilde{A} \subseteq A'$ with $|\tilde{A}| \geq 4c_1 n$, $G[\tilde{A}]$ is panconnected, and for any $\tilde{B} \subseteq B'$ with $|\tilde{B}| \geq 4c_1 n$, $G[\tilde{B}]$ is panconnected. In particular, $G[A']$ and $G[B']$ are panconnected. Also, this means that if $|\tilde{A}| \geq \frac{\epsilon n}{6}$, then $G[\tilde{A}]$ is panconnected and similarly for $\tilde{B}$.*

Proof of Claim 8. We now recall that $d_A(v) > |A| - 2c_1 n + 2t - 2$ for any $v \in A'$. Therefore

$$\delta(G[\tilde{A}]) > |A| - 2c_1 n + 2t - 2 - |A \setminus \tilde{A}|$$
$$= |\tilde{A}| - 2c_1 n + 2t - 2$$
$$\geq \frac{|\tilde{A}|}{2} + 2t - 2.$$

By Theorem 7 we know $G[\tilde{A}]$ is panconnected. In particular, $|A'| = |A| - |D_A| \geq (4c_1 + c_2)n - c_2 n = 4c_1 n$, which implies that $G[A']$ is panconnected. By the symmetry, both $G[\tilde{B}]$ and $G[B']$ are also panconnected. $\square_{Claim\ 8}$

Now, by Menger's Theorem [12], there is a matching $\mathscr{M}$ between $A' \cup D_B$ and $B' \cup D_A$ in G with $|\mathscr{M}| = 2t + 1$. Choose such $\mathscr{M}$ so that $|V(\mathscr{M}) \cap S|$ is as small as possible. Notice $\mathscr{M}$ contains exactly $2t + 1$ vertices of $A' \cup D_B$ and $2t + 1$ vertices of $B' \cup D_A$. Let $\mathscr{M} \cap (A' \cup D_B) := \{a_1, a_2, \ldots, a_{2t+1}\}$ and $\mathscr{M} \cap (B' \cup D_A) := \{b_1, b_2, \ldots, b_{2t+1}\}$ so that $a_i b_i \in \mathscr{M}$. The choice of $\mathscr{M}$ implies the following claim.

Claim 9. *Every edge in $\mathscr{M}$ has at most one vertex in S or G contains a spanning path collection for S.*

Proof of Claim 9. Let $u \in A' \cup D_B \setminus (V(\mathscr{M}) \cup S)$ and let $v \in B' \cup D_A \setminus (V(\mathscr{M}) \cup S)$. Since $|A'| \geq 4c_1 n > 4t + 1$, there exists such a vertex $u \in A'$, and by symmetry there exists such a vertex $v \in B'$. If $uv \in E(G)$, then no edge in $\mathscr{M}$ contains a vertex in S. (Otherwise we can switch out such an edge for the edge uv.) Thus, we may assume that $uv \notin E(G)$.

If there exists $w \in (N(u) \cap N(v)) \setminus V(\mathscr{M})$, then no edge in $\mathscr{M}$ contains two vertices in S. (Again otherwise we can change such an edge with an edge uw or vw.) So we may assume that

$$N(u) \cap N(v) \subseteq \{a_1, a_2, \ldots, a_{2t+1}, b_1, \ldots, b_{2t+1}\}.$$

If $a_j \in N(u) \cap N(v)$, then $b_j \notin S$; otherwise we can change an edge $a_j b_j$ with an edge $a_j v$. Suppose that $a_j, b_j \in N(u) \cap N(v)$ for some j. By the above fact, $a_j, b_j \notin S$. Moreover, if there exists an edge $a_k b_k \in \mathscr{M}$ such that $a_k \in S$ or $b_k \in S$, then we can change two edges $a_j b_j$ and $a_k b_k$ with two edges $a_j v$ and $u b_j$, which contradicts the choice of $\mathscr{M}$. Therefore, no edge in $\mathscr{M}$ contains a vertex in S, so we may assume that $a_j \notin N(u) \cap N(v)$ or $b_j \notin N(u) \cap N(v)$.

If $|N(u) \cap N(v)| \le 2t - 1$, then $d(u) + d(v) = |N(u) \cup N(v)| + |N(u) \cap N(v)| \le n + 2t - 3$, a contradiction. Thus, $|N_G(u) \cap N_G(v)| \ge 2t$. Hence by changing the index if necessary, we may assume that $a_j \in N(u) \cap N(v)$ or $b_j \in N(u) \cap N(v)$ for all $2 \le j \le 2t + 1$. This implies that $a_j \notin S$ or $b_j \notin S$ for all $2 \le j \le 2t + 1$. If $a_1 \notin S$ or $b_1 \notin S$, then we obtain the conclusion of the claim. Hence we may again assume that $a_1 \in S$ and $b_1 \in S$. In this case, all the above inequalities hold, so $N(u) \cup N(v) = V(G) - \{u, v\}$. Since $N(u) \subset A' \cup D_B \cup \{b_1, \dots, b_{2t+1}\}$, we obtain $B' \cup D_A \subset N(v)$. By symmetry, $A' \cup D_B \subset N(u)$. Because this condition holds for any $u \in A' \cup D_B \setminus (V(\mathscr{M}) \cup S)$, any two vertices of $A' \cup D_B$ are connected by an edge except for two vertices in $V(\mathscr{M}) \cup S$. By symmetry, any two vertices of $B' \cup D_A$ are connected by an edge except for two vertices in $V(\mathscr{M}) \cup S$. Construction of the desired paths is trivial in this situation. This completes the proof of the claim. $\square_{Claim\ 9}$

Let $v \in D_B$. Since $|D_A| \le c_2 n$ and $d_A(v) \ge c_1 n$, we have $d_{A'}(v) \ge d_A(v) - |D_A| > (c_1 - c_2)n$. Then the conditions "$c_1 > c_2$" and "$n$ is sufficiently large" implies the existence of a matching $\mathscr{M}_A$ between $V(\mathscr{M}) \cap D_B$ and $A' \setminus (V(\mathscr{M}) \cup S)$. By the symmetry, we have a matching $\mathscr{M}_B$ between $V(\mathscr{M}) \cap D_A$ and $B' \setminus (V(\mathscr{M}) \cup S)$.

For any $a_j \in V(\mathscr{M}) \cap D_B$, let $a_j a_j'$ be an edge in the matching $\mathscr{M}_A$ and for any $b_j \in V(\mathscr{M}) \cap D_A$, let $b_j b_j'$ be an edge in the matching $\mathscr{M}_B$. For convenience, let $a_i' = a_i$ if $a_i \in V(\mathscr{M}) \cap A'$ and $b_i' = b_i$ if $b_i \in V(\mathscr{M}) \cap B'$.

Let $Q_j := a_j'(a_j)(b_j)b_j'$. Notice that Q_j is a path of length one, two or three, and has at most one vertex in S. Let $\mathscr{Q} := \{Q_j : 1 \le j \le 2t + 1\}$. The paths in $\mathscr{Q}$ will be used for "transportation" from A' to B' when we construct the desired paths.

If $x_i \in D_A$ (resp. $y_i \in D_A$), then we take a vertex x_i' (resp. y_i') in $B' \setminus (S \cup V(\mathscr{Q}))$. Since $d_{B'}(v) \ge (c_1 - c_2)n$ for any $v \in D_A$, we can take such vertices x_i' and y_i' so that all of x_i''s and y_i''s are distinct. We similarly define x_i' and y_i' in A' for $x_i, y_i \in D_B$. For convenience, let $x_i' = x_i$ and let $y_i' = y_i$ for $x_i, y_i \in A' \cup B'$.
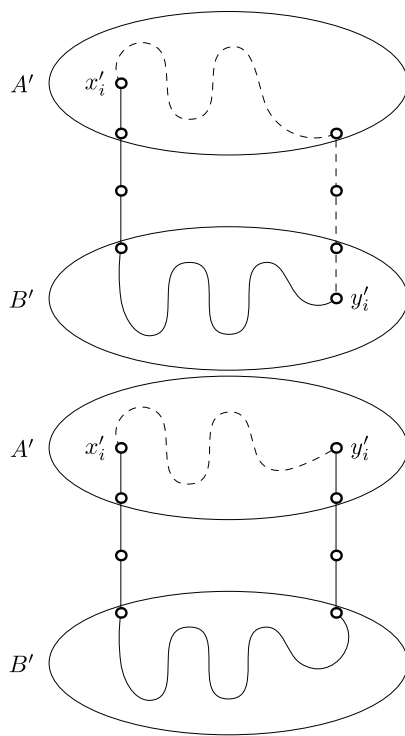
Figure 3: Routing paths.

We now provide a process for constructing the desired path collection. Without loss of generality, we may assume that $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_t$. The paths are constructed as follows. Before constructing formally, we show the outline of the proof.

For each $1 \leq i \leq t-1$ start at the vertex x_i' (suppose x_i' is in A'). Suppose that $y_i' \in B'$. Then we can choose a desired path from two choices depending on the length of γ_i. One of them is starting from x_i', "traversing" A' until attaining the appropriate length, "jumping" to B' and ending at y_i'; the other "traverses" B' after "jumping" to B'. If γ_i is too small for our path to "fit" into A', then we choose the first option; otherwise we choose the second. If $y_i' \in A'$, then there are also two choices, one of them "traverses" A' without "jumping"; another "jumps" to B', "traverses" B' and "jumps" back to A'. In each case, we create such a path using no more than two paths in $\mathscr{Q}$. After constructing $t-1$ paths, we finally construct a path P_t. Since at least three paths in $\mathscr{Q}$ remain, we can "traverse" both A' and B' until attaining the appropriate length. Figure 3 depicts the two cases listed above and their two subcases for constructing these paths.

Formally, we will show the existence of a spanning desired path collection. We first connect the vertices of $D_A \setminus \mathcal{Q}$ into a path. Since each vertex of D_A has at least $c_1 n$ edges to B and $|D_A| < c_2 n < c_1 n/3$, there must exist a system of two distinct neighbors in B of each vertex of D_A. By Claim 8, we may easily connect these vertices by short paths in B' so that all of $D_A \setminus \mathcal{Q}$ is on a single path of length at most $4c_2 n < \gamma_i n/4$ which begins and ends in B'. The same can be said for $D_B \setminus \mathcal{Q}$ so, when constructing the desired paths, we simply use the path through $D_A \setminus \mathcal{Q}$ the first time we use vertices of B' and use the path through $D_B \setminus \mathcal{Q}$ the first time we use vertices of A'.

Suppose we have constructed paths $P_1, \ldots, P_{i-1}$ for $1 \le i \le t-1$ such that each of them contains vertices in at most two paths in $\mathcal{Q}$ and $(\gamma_j - \frac{\epsilon}{2t})n \le |P_j| \le (\gamma_j - \frac{\epsilon}{2t})n + 4$ for all $j \le i-1$. We would like to construct P_i. Let $R_i = V(P_1) \cup \cdots \cup V(P_{i-1})$ and let $\tilde{A} = [A' \setminus (S \cup R_i \cup V(\mathcal{Q}))]$ and let $\tilde{B} = [B' \setminus (S \cup R_i \cup V(\mathcal{Q}))]$. By symmetry, we may assume that $x_i' \in A'$. If $(\gamma_i - \frac{\epsilon}{2t})n \le |\tilde{A}| - \frac{\epsilon}{6}n$, use Claim 8 to complete the path P_i of order $(\gamma_i - \frac{\epsilon}{2t})n$ in A' from x_i' to a_j', where a_j' is chosen so that $y_i' \in V(Q_j)$ if $y_i' \in V(\mathcal{Q})$; otherwise Q_j is not used in the previous paths $P_1, \ldots, P_{i-1}$ and does not contain a vertex in S. Since each path in $\mathcal{Q}$ contains at most one vertex in S and at most $2(i-1)$ paths in $\mathcal{M}$ are used for the previous paths $P_1, \ldots, P_{i-1}$, we can find such Q_j. If $y_i' \in A'$, construct a short path (of length at most 2) from a_j' to y_i' in A' and finish with the path to y_i. Otherwise take the path Q_j from a_j' to b_j' (or y_i' if $y_i' \in V(Q_j) \setminus \{b_j'\}$) and take a path of length at most 2 in $B' \cup D_A$ from b_j' to y_i'. Notice $|\tilde{A} \setminus P_i| \ge |\tilde{A}| - (\gamma_i - \frac{\epsilon}{2t})n > \frac{\epsilon}{6}n$ and since $\frac{\epsilon}{6} \ge 4c_1$, we see that $G[\tilde{A} \setminus P_i]$ is still panconnected.

Suppose that $(\gamma_i - \frac{\epsilon}{2t})n > |\tilde{A}| - \frac{\epsilon}{6}n$. Then $|R_i \cap A'| \ge |A'| - |\tilde{A}| - |S| - |V(\mathcal{Q})| > |A'| - (\gamma_i - \frac{\epsilon}{2t})n - \frac{\epsilon}{6}n - 6t$. Hence

$$
\begin{aligned}
|R_i \cap B'| &\le \sum_{j=1}^{i-1} |P_j| - |R_i \cap A'| \\
&< \sum_{j=1}^{i-1} \left(\left(\gamma_j - \frac{\epsilon}{2t} \right)n + 4 \right) - |A'| + (\gamma_i - \frac{\epsilon}{2t})n + \frac{\epsilon}{6}n + 6t \\
&\le \left(\sum_{j=1}^{i} \gamma_j \right)n - \frac{i\epsilon}{2t}n + 4(i-1) + \frac{\epsilon}{6}n - \left(n - |B'| - \frac{\epsilon}{36}n \right) + 6t \\
&= |B'| - \left(\sum_{j=i+1}^{t} \gamma_j \right)n - \frac{i\epsilon}{2t}n + 4(i-1) + \frac{7\epsilon}{36}n + 6t
\end{aligned}
$$

which implies:

$$|\tilde{B}| - \frac{\epsilon}{6}n = |B'| - |R_i \cap B'| - \frac{\epsilon}{6}n$$

$$> \left(\sum_{j=i+1}^{t} \gamma_j\right)n + \frac{i\epsilon}{2t}n - 4(i-1) - \frac{7\epsilon}{36}n - 6t - \frac{\epsilon}{6}n$$

$$> \left(\sum_{j=i+1}^{t} \gamma_j\right)n + \frac{i\epsilon}{2t}n - 4(i-1) - \frac{3\epsilon}{8}n$$

$$\geq \gamma_t n + \frac{t-i-1}{2t}\epsilon n + \frac{i\epsilon}{2t}n - 4(i-1) - \frac{3\epsilon}{8}n$$

$$= \gamma_t n + \frac{t-1}{2t}\epsilon n - 4(i-1) - \frac{3\epsilon}{8}n$$

$$\geq \gamma_t n - \frac{1}{2t}\epsilon n$$

$$= \left(\gamma_i - \frac{\epsilon}{2t}\right)n.$$

We use the fact that $\tilde{A}$ is panconnected to create a path from x_i' to a_j' of length 2 for some j such that a_j' is not contained in $R_i \cup S$ except for x_i'. We construct this short path from x_i' to a_j' to preserve the panconnectivity of $\tilde{A}$ and connect x_i' to Q_j. Since $(\gamma_i - \frac{\epsilon}{2t})n < |\tilde{B}| - \frac{\epsilon}{6}n$, we can use the panconnectivity of $\tilde{B}$ to complete the path P_i of length $(\gamma_i - \frac{\epsilon}{2t})n$ to y_i' if $y_i' \in B'$. If $y_i' \in A'$, then take a_k' so that Q_k is not used in $R_i \cup Q_j$. In particular, we choose Q_k so that $a_k' = y_i'$ if possible. After that, $G[\tilde{B} \setminus P_i]$ is still panconnected since $|\tilde{B} \setminus P_i| \geq \frac{\epsilon n}{6}$.

Suppose we have constructed $t-1$ paths as above. Let $R_t = V(P_1) \cup \cdots \cup V(P_{t-1})$. Without loss of generality, suppose $x_t' \in A'$ and let Q_1, Q_2, Q_3 be remaining paths of $\mathcal{Q}$. Such paths exist because we use at most two paths of $\mathcal{Q}$ to construct each path P_i and $|\mathcal{Q}| = 2t+1$. Let $\tilde{A} = [A' \setminus (S \cup R_t \cup V(\mathcal{Q}))]$ and let $\tilde{B} = [B' \setminus (S \cup R_t \cup V(\mathcal{Q}))]$ and note that $|\tilde{A}|, |\tilde{B}| \geq \frac{\epsilon n}{6}$ so, by Claim 8, both sets are still panconnected. By symmetry, we may assume that $x_t' \neq a_1', a_2'$. If y_t' are in A', (suppose $y_t' \neq a_2'$) use the panconnectivity of $G[\tilde{A}]$ to connect x_t' to a_1' using at most one intermediate vertex and connect y_t' to a_2' using all of $\tilde{A}$. We then take the paths, through Q, to $\tilde{B}$ and use the panconnectivity of $\tilde{B}$ to pick up all of B' between b_1' and b_2'. This produces a path of order $|P_t| \leq n - \sum_{j=1}^{t-1} |P_i| - |D| \leq \gamma_t n + \frac{2(t-1)\epsilon}{2t}n + \frac{\epsilon}{36}$.

Otherwise suppose $y_t' \in B'$. By symmetry, $a_1' \neq x_t'$ and $b_1' \neq y_t'$. Again use the panconnectivity of $G[A \setminus R_t]$ and $G[B \setminus R_t]$ to connect x_t' to a_1' using all of A', through Q_1 and connect b_1' to y_t' using all of B'.

For any $P_j \in \mathscr{P}$, we have $|P_j| < \gamma_j n + \frac{2(t-1)\epsilon}{2t}n + \frac{\epsilon}{24} \leq (\gamma_j + \epsilon)n$ for n sufficiently large. On the other hand, $|P_j| \geq (\gamma_j - \epsilon)n$, which completes the proof of Lemma 3. $\qquad\square$

Our final lemma gives necessary starting structure for the proofs of our main results.

Lemma 4 (Setup). *Suppose that n is sufficiently large. If $\kappa(G) \geq 2t + 1$ and $\sigma_2(G) \geq n + 2t - 2$, there exists a spanning path collection for S.*

Proof. First we will show the existence of a path collection (possibly not spanning). After finding a path collection, we obtain the desired path collection by applying Lemma 1.

In order to show this, we divide the proof into two parts depending on the connectivity. Suppose first that $\kappa(G) < 2t\epsilon n$, and let K be a minimum cutset of order less than $2t\epsilon n$. Since $\sigma_2(G) \geq n + 2t - 2$ and $\epsilon > 0$ is small, $G \setminus K$ has exactly two components, so we know there exists a component A of $G \setminus K$ with $|A| \geq \frac{(1-2t\epsilon)n}{2}$. Let $B = V(G) - (A \cup K)$.

If $|B \cup K| \geq 9n\sqrt{t\epsilon}$, then we know $e(A, B \cup K) \geq 2t\epsilon n^2$, because otherwise we can apply Lemma 3 by letting $c_1 := 2\sqrt{t\epsilon}$ and $c_2 := \frac{\sqrt{t\epsilon}}{2}$. Since all these edges must be incident to vertices in K, there exists a vertex $v \in K$ such that,

$$d_A(v) > \frac{2t\epsilon n^2}{2t\epsilon n} = n.$$

So we may assume that $|B \cup K| < 9n\sqrt{t\epsilon}$. We know $d_G(v) \leq |B| + |K| - 1$ for any $v \in B$, so B induces a clique, because otherwise, $d_G(v_1) + d_G(v_2) \leq 2|B| + 2|K| - 2 < 18n\sqrt{t\epsilon} \leq n$, a contradiction, where $v_1, v_2 \in B$ such that $v_1 v_2 \notin E(G)$. Consider a vertex $v \in B$. Since $u \in A$ has $d_G(u) \geq n+2t-2-|B|-|K| = |A|+2t-2$, $d_A(u) \geq |A|+2t-2-|K| > |A|-2t\epsilon n+2t-2$. This implies that for any $\tilde{A} \subseteq A$ with $|\tilde{A}| \geq 4t\epsilon n$, $\tilde{A}$ is panconnected, because $d_{\tilde{A}}(u) \geq |A| - 2t\epsilon n + 2t - |A \setminus \tilde{A}| = |\tilde{A}| - 2t\epsilon n + 2t - 2 \geq \frac{|\tilde{A}|+2}{2}$.

Since G is $2t$-connected and $|(B \cup K) \cap S| = |S| - |A \cap S| = 2t - |A \cap S|$, there exists a collection of paths $\mathscr{Q}$ from $(B \cup K) \cap S$ to $A - (A \cap S)$ in $G - A \cap S$. For $x_i \in (B \cup K) \cap S$ ($y_i \in (B \cup K) \cap S$), let Q_{x_i} (Q_{y_i}) be a path in $\mathscr{Q}$ starting from x_i (y_i, respectively). Let x_i' (y_i') be another end vertex of Q_{x_i} (Q_{y_i}).

For convenience, let $x_i' := x_i$ and $y_i' := y_i$ for any $x_i, y_i \in A \cap S$. We will show that there exist t disjoint paths $P_1', P_2', \ldots, P_t'$ such that P_i' connects x_i' and y_i' with $|P_i'| = \epsilon n$. Suppose first that we have already found i disjoint paths $P_1', P_2', \ldots, P_i'$ for $i < t$. Let $\tilde{A} = A - cup_{j=1}^{i} V(P_j')$. Note that

$|\tilde{A}| = |A| - \sum_{j=1}^{i} |P'_j| > (1 - 9\sqrt{t\epsilon})n - i\epsilon n \geq 4t\epsilon n$, because $\epsilon < \frac{1}{18^2 t}$ (which implies $1 > 4t\epsilon + i\epsilon + 9\sqrt{t\epsilon}$). Hence $\tilde{A}$ is panconnected. This implies that we can find a path P'_{i+1} in $\tilde{A}$ connecting x'_{i+1} and y'_{i+1} with $|P_{i+1}| = \epsilon n$. By joining Q_{x_i}, P'_i and Q_{y_i}, we obtain a path P_i which connects x_i and y_i. Moreover, These t paths are pairwise disjoint, so $\{P_i\}$ is a path collection. Thus, we may assume that $\kappa(G) \geq 2t\epsilon n$.

Since $\delta(G) \geq \kappa(G) \geq 2t\epsilon n$, we may create t vertex disjoint paths Q_1, $Q_2, \ldots, Q_t$ in $G - \{y_1, y_2, \ldots, y_t\}$ such that $|Q_i| = 2\epsilon n - 9$ which starts at x_i for any $1 \leq i \leq t$. For $1 \leq i \leq t$, let z_i be an end vertex of Q_i other than x_i, and let $\mathscr{Q} = \bigcup_{i=1}^{t} V(Q_i)$. Notice $|\mathscr{Q}| \leq \kappa(G) - 9t$ so $G - (\mathscr{Q} - \{z_1, z_2, \ldots, z_t\})$ is at least $10t$-connected. By Theorem 9, we know $G - (\mathscr{Q} - \{z_1, z_2, \ldots, z_t\})$ is t-linked. This implies that we may link z_i and y_i, and obtain a path collection $P_1, P_2, \ldots, P_t$, where P_i is a path combining Q_i and the link between z_i and y_i.

By Lemma 1, this completes the proof of Lemma 4.   □

4. Proof of Theorem 4

First, we apply Lemma 4 to obtain a spanning path collection $\mathscr{P} = \{P_1, \ldots, P_t\}$. Take such a spanning path collection $\mathscr{P}$ with $\mu(\mathscr{P})$ minimum and suppose $\mathscr{P}$ is not desired.

If for any subcollection $\mathscr{A}$ of $\mathscr{P}$, $e(\mathscr{A}, \mathscr{P} - \mathscr{A}) \geq \frac{\epsilon^2}{24 \cdot 48} n^2$, then we may apply Lemma 2 to find a collection of paths $\mathscr{P}'$ with $\mu(\mathscr{P}') < \mu(\mathscr{P})$. This contradicts our assumptions on $\mathscr{P}$.

So there exists a subcollection $\mathscr{A}$ of $\mathscr{P}$ with $e(\mathscr{A}, \mathscr{P} - \mathscr{A}) < \frac{\epsilon^2}{24 \cdot 48} n^2$. Letting $c_1 := \frac{\epsilon}{24}$ and $c_2 := \frac{\epsilon}{75}$. Since $\sum_{P_i \in \mathscr{A}} |P_i| \geq \epsilon n \geq (4c_1 + c_2)n$ and $\sum_{P_i \in \mathscr{P} - \mathscr{A}} |P_i| \geq \epsilon n \geq (4c_1 + c_2)n$, we can apply Lemma 3 to find a spanning desired path collection $\mathscr{P}'$, thus completing the proof.   □

5. Proof of Theorem 6

First we consider a matching $\mathscr{M}$ with t edges such that each edge contains a vertex in S. Since $\sigma_2(G) \geq n + 2t - 2$, we know $\delta(G) \geq 2t$, and hence there exists such a matching $\mathscr{M}$. Let $\mathscr{M} := \{x_1 y_1, x_2 y_2, \ldots, x_t y_t\}$ and let $S' := S \cup \{y_i : 1 \leq i \leq t\}$. If G is $(2t+1)$-connected, we can apply Theorem 4 for S'. Then G contains a spanning collection $\mathscr{P}$ of vertex disjoint paths $P_i = x_i, \ldots, y_i$ such that $(\gamma_i - \epsilon)n < |P_i| < (\gamma_i + \epsilon)n$ for all $1 \leq i \leq t$. In this case, $\{C_1, C_2, \ldots, C_t\}$ is a desired spanning collection of vertex disjoint cycles, whehe $C_i := P_i \cup \{x_i y_i\}$. So we may assume that G has a cut set T of order $2t$.

By the degree sum condition, we know that $G - T$ has exactly two components and both of them are cliques. In this case, we can easily find a desired spanning collection of vertex disjoint cycles. □

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