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The Erdős-Lovász Tihany Conjecture and complete minors

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The Erdős-Lovász Tihany Conjecture [*Theory of Graphs (Proc. Colloq., Tihany, 1966)*, Academic Press, 1968] states that for any pair of integers $s, t \geq 2$ and for any graph G with chromatic number equal to $s + t - 1$ and clique number less than $s + t - 1$ there are two disjoint subgraphs of G with chromatic number s and t, respectively. The Erdős-Lovász Tihany Conjecture is still open except for a few small values of s and t. Given the same hypothesis as in the Erdős-Lovász Tihany Conjecture, we study the problem of finding two disjoint subgraphs of G with complete minors of order s and t, respectively. If Hadwiger's Conjecture holds, then this latter problem might be easier to settle than the Erdős-Lovász Tihany Conjecture. In this paper we settle this latter problem for a few small additional values of s and t.

Keywords and phrases: Graph colouring, graph decompositions, complete minors.

1. Introduction

In this paper we study certain relaxed versions of the Erdős-Lovász Tihany Conjecture [12] (Conjecture 1.1). The study documented here is a continuation of that initiated in [18, 27].

First a bit of standard notation and terminology. All graphs considered in this paper are assumed to be simple and finite.[1] Let G denote a graph. An *independent k-set* of G is an independent set of G of size k. The complete graph on k vertices is referred to as a *k-clique*, and the 3-clique is also referred to as a *triangle*. Given a vertex v in G, the *open neighbourhood $N(v)$ of v in G* is the set of vertices in G adjacent to v, and the *closed neighbourhood $N[v]$ of v in G* is the set $N(v) \cup \{v\}$. Given a subset S of the vertices of G, the subgraph of G induced by the vertices of S is denoted $G[S]$. Given two

[1] The reader is referred to [5] for definitions of any graph-theoretic concept used but not explicitly defined in this paper.

graphs G and H, we say that H is a *minor* of G (and that G contains an H *minor*) if there is a collection $\{V_h \mid h \in V(H)\}$ of non-empty disjoint subsets of $V(G)$ such that the induced graph $G[V_h]$ is connected for each $h \in V(H)$, and for any two adjacent vertices h_1 and h_2 in H there is at least one edge in G joining some vertex of V_{h_1} to some vertex of V_{h_2}. The sets V_h are called the *branch sets* of the minor H of G. We may write $H \leq G$ or $G \geq H$, if G contains H as a minor. Given an edge $e = uv$ of a graph G, we denote by G/e the graph obtained from G by contracting the edge e into a new vertex v_e which in G/e is adjacent to all the former neighbours of u and of v in G. A graph G contains H as a minor if and only if some graph F isomorphic to H can be obtained from G by a series of edge contractions and deletions of edges and vertices. The *Hadwiger number* $\eta(G)$ is the largest integer k for which G contains a K_k minor, while the *Hajós number* $h(G)$ is the largest integer ℓ for which G contains a subdivision of K_ℓ.

Conjecture 1.1 (Erdős-Lovász Tihany Conjecture [12]). *For any pair of integers $s, t \geq 2$ and any graph G with $\omega(G) < \chi(G) = s + t - 1$ there are two disjoint subgraphs G_1 and G_2 of G with $\chi(G_1) \geq s$ and $\chi(G_2) \geq t$.*

Conjecture 1.1 holds for (s, t) equal to $(2, 2)$, $(2, 3)$, $(2, 4)$, $(3, 3)$, $(3, 4)$, and $(3, 5)$ (see [6, 25, 31, 32]). Kostochka and Stiebitz [21] proved it to be true for line graphs of multigraphs, while Balogh et al. [4] proved it to be true for quasi-line graphs and for graphs with independence number 2.

Given integers $s, t \geq 2$ with $s \leq t$, an *(s, t)-graph* is a connected $(s + t - 1)$-chromatic graph which does not contain two disjoint subgraphs with chromatic number s and t, respectively. In terms of (s, t)-graphs, the Erdős-Lovász Tihany Conjecture states that every (s, t)-graph contains an $(s+t-1)$-clique. Inspired by Hadwiger's Conjecture [16] — which states that every k-chromatic graph contains a K_k minor — and the Erdős-Lovász Tihany Conjecture, we propose the following conjectures.

Conjecture 1.2. *Every (s, t)-graph contains a K_{s+t-1} minor.*

Conjecture 1.2 is weaker than the Erdős-Lovász Tihany Conjecture, since in Conjecture 1.2 we only require a K_{s+t-1} as a minor, not as a subgraph. Hence Conjecture 1.2 is settled in the affirmative for all values of (s, t) for which the Erdős-Lovász Tihany Conjecture is settled in the affirmative. Conjecture 1.2 is also weaker than Hadwiger's Conjecture. Hence Conjecture 1.2 is settled in the affirmative for all graphs for which Hadwiger's Conjecture is settled in the affirmative.

Any (s, t)-graph with $s = 2$ is referred to as a *double-critical graph*. The Erdős-Lovász Tihany Conjecture with $s = 2$ is equivalent to the *Double-Critical Graph Conjecture* which states that the complete graphs are the only double-critical graphs. (Conjecture 1.2 with $s = 2$ is referred to as the *Double-Critical Hadwiger Conjecture*.)

In [18], we proved that every double-critical k-chromatic graph with $k \in \{6, 7\}$ contains a K_k minor. In [27], it was proved that every double-critical 8-chromatic graph contains a K_8^- minor. (Here K_8^- denotes the complete 8-graph with one edge missing.) Here is yet another related conjecture.

Conjecture 1.3. *For any pair of integers $s, t \geq 2$ and any graph G with $\omega(G) < \chi(G) = s + t - 1$ there are two disjoint graphs G_1 and G_2 of G with $\eta(G_1) \geq s$ and $\eta(G_2) \geq t$.*

Note that if both Hadwiger's Conjecture and the Erdős-Lovász Tihany Conjecture are true for a given class $\mathcal{C}$ of graphs, then Conjecture 1.3 is true for all graphs of $\mathcal{C}$ as well. Hence it follows from theorems by Balogh et al. [4] and Chudnovsky and Ovetsky Fradkin [8] that Conjecture 1.3 is true for all quasi-lines graphs, that is, graphs in which the neighbourhood of every vertex is coverable by two cliques.

Conjecture 1.3 is true for all 6-colourable graphs G. Here is an argument for this claim. Let G denote a connected graph with $\omega(G) < \chi(G) \leq 6$. If $\chi(G) \leq 5$, then the desired conclusion follows immediately from the above-mentioned results. Suppose $\chi(G) = 6$. Then there are two possible values for (s, t), namely, $(2, 5)$ and $(3, 4)$. If $(s, t) = (3, 4)$, then the already settled case $(3, 4)$ of Conjecture 1.1 applies. Suppose $(s, t) = (2, 5)$. If G contains two disjoint subgraphs G_1 and G_2 such that $\chi(G_1) \geq 2$ and $\chi(G_2) \geq 5$, then the desired conclusion follows from the fact that every 5-chromatic graph contains a K_5 minor (see Theorem 2.6). Otherwise G is double-critical, and so, by a theorem presented in [18] (see Theorem 2.8), G contains a K_6 minor. The existence of this K_6 minor in G and the fact that G does not contain a 6-clique implies that G contains a $\overline{K_{2,5}}$ minor. This completes the argument.

The following conjecture is Conjecture 1.3 restricted to (s, t)-graphs.

Conjecture 1.4. *For any pair of integers $s, t \geq 2$ and any (s, t)-graph G with $\omega(G) < s + t - 1$, there are two disjoint graphs G_1 and G_2 of G with $\eta(G_1) \geq s$ and $\eta(G_2) \geq t$.*

2. Preliminaries

The following results will be useful in our search for disjoint complete minors.

Theorem 2.1 ((i) Bush [7]; (ii) Greenwood & Gleason [14])**.**

(i) *Every graph on* 6 *vertices contains an independent* 3*-set or a* 3*-clique.*
(ii) *Every graph on* 9 *vertices contains an independent* 3*-set or a* 4*-clique.*

A vertex x in a graph G is said to be *bisimplicial* if the induced graph $G[N(x)]$ can be covered by two cliques, and an *even hole* is an induced cycle of even length.

Theorem 2.2 (Addario-Berry et al. [1]). *Every non-empty even-hole-free graph G contains a bisimplicial vertex and satisfies $\chi(G) \leq 2\omega(G) - 1$.*

Vertex partitions with degree or colouring constraints

Theorem 2.3 (Stiebitz [33]). *Let G denote an arbitrary graph. If $\delta(G) \geq s+t+1$ for $s, t \in \{0\} \cup \mathbb{N}$, then there are two disjoint subgraphs G_1 and G_2 such that $\delta(G_1) \geq s$ and $\delta(G_2) \geq t$.*

Theorem 2.4 (Stiebitz [34]). *For any pair of integers $s, t \geq 2$ and any graph G with $\omega(G) < \chi(G) = s+t-1$ there are two disjoint subgraphs G_1 and G_2 of G with*

$$\begin{array}{cccc} \textit{either} & \chi(G_1) \geq s & \textit{and} & \mathrm{col}(G_2) \geq t, \\ \textit{or} & \mathrm{col}(G_1) \geq s & \textit{and} & \chi(G_2) \geq t. \end{array}$$

Lemma 2.1 (Stiebitz [32, Corollary 3.2]). *Every s-clique (t-clique) of an (s,t)-graph G is contained in at least $t-1$ ($s-1$) cliques each of order $s+1$ ($t+1$).*

Proof of Lemma 2.1 can also be found in [4].

The following result, which we shall use repeatedly, shows that every (s,t)-graph with clique number at most $s+t-2$ has clique number at most $t-1$.

Lemma 2.2 (Stiebitz [32, Lemma 3.7]). *If an (s,t)-graph contains a t-clique, then it contains an $(s+t-1)$-clique.*

As a convenience to the reader, we include a proof of Lemma 2.2.

Given any subset S of the vertex set of a graph G, let $T(S:G)$ denote the set of vertices in $V(G) \setminus S$ which are adjacent to every vertex of S in G.

Proof of Lemma 2.2. Let G denote an (s,t)-graph. According to Lemma 2.1, every t-clique X of G is contained in at least $s-1$ $(t+1)$-cliques; hence $|T(V(X):G)| \geq s-1$.

Suppose $\omega(G) \geq t$, and let X_0 denote the vertex set of a t-clique of G with vertices labelled $x_1, \ldots, x_t$. Let $y_1, \ldots, y_r$ denote a longest sequence of pairwise distinct vertices of $V(G) \setminus X_0$ satisfying:

(1) for each $i \in [r]$, $X_i := \{y_1, \ldots, y_i, x_{i+1}, \ldots, x_t\}$ induce a t-clique in G, and
(2) for each $i \in [r]$, $y_i \in T(X_{i-1} : G)$.

Now, the set $\{x_1, \ldots, x_r\}$ induces an r-clique in G, and so, in particular, $\chi(G[\{x_1, \ldots, x_r\}]) = r$. The graph $G - \{x_1, \ldots, x_r\}$ contains the t-clique $G[X_r]$, and so, in particular, $\chi(G - \{x_1, \ldots, x_r\}) \geq t$. Thus, since G is an (s,t)-graph, r can be at most $s-1$. Suppose $T(X_r : G)$ contains some vertex y not in $\{x_1, \ldots, x_r\}$. Then y is adjacent to every vertex of $X_r = \{y_1, \ldots, y_r, x_{r+1}, \ldots, x_t\}$, y is distinct from $y_1, \ldots, y_r$, and $y \in V(G) \setminus X_0$. This contradicts the fact that $y_1, \ldots, y_r$ was chosen as a longest sequence of pairwise distinct vertices of $V(G) \setminus X_0$ for which (1) and (2) are satisfied. This shows that $T(X_r : G)$ is a subset of $\{x_1, \ldots, x_r\}$, in particular, $|T(X_r : G)| \leq r$. According to Lemma 2.1, $|T(X_r : G)| \geq s-1$, since $G[X_r]$ is a t-clique. Hence $s-1 \leq |T(X_r : G)| \leq r \leq s-1$, in particular, $r = s-1$ and $T(X_r : G) = \{x_1, \ldots, x_r\}$. This shows that the vertex set $X_r \cup \{x_1, \ldots, x_{s-1}\}$ induces a $(t+s-1)$-clique in G, and so the proof is complete. □

Proposition 2.1 (Kawarabayashi, Pedersen & Toft [18]). *Suppose G is a non-complete double-critical k-chromatic graph. Then*

(i) *G has minimum degree at least $k+1$,*
(ii) *G has clique number at most $k-2$, and*
(iii) *the endvertices of any edge in G have at least $k-2$ common neighbours.*

Complete minors in k-chromatic graphs

König [19, 20] observed that a graph is 2-colourable if and only if it does not contain an odd cycle. This immediately implies the following observation.

Observation 2.1. *Every 3-chromatic graph contains an odd cycle.*

Theorem 2.5 (Hadwiger [16]; Dirac [10, 11]). *Every graph of minimum degree at least 3 contains a subdivision of K_4, in particular, every 4-chromatic graph contains a subdivision of K_4.*

The following theorem, which states that Hadwiger's Conjecture is true for 5-chromatic graphs, follows from Wagner's Theorem [37] and the Four Colour Theorem [2, 3, 30].

Theorem 2.6. *Every 5-chromatic graph contains a K_5 minor.*

Theorem 2.7 (Robertson, Seymour & Thomas [29]). *Every 6-chromatic graph contains a K_6 minor.*

The Four Colour Theorem is also applied in the proof of Theorem 2.7, but it is not applied in the proof of the following theorem.

Theorem 2.8 (Kawarabayashi, Pedersen & Toft [18]). *If G is a double-critical k-chromatic graph with $k \in \{6, 7\}$, then G contains a K_k minor.*

Complete minors in graphs with many edges

Theorem 2.9 (Mader [24]). *For any positive integer p less than 8, any graph G with $m(G) \geq (p-2)n(G) - \binom{p-1}{2} + 1$ and $n(G) \geq p$ contains a K_p minor.*

The results for $p = 5$ and $p = 6$ of Theorem 2.9 were also obtained, independently, by Győri [15].

Theorem 2.10 (Jørgensen [17]). *Every graph G with $m(G) \geq 6n(G) - 19$ and $n(G) \geq 8$ contains a K_8 minor.*

Theorem 2.11 (Jørgensen [17, Remark following Theorem 4]). *Every graph on at most 11 vertices and with minimum degree at least 6 contains a K_6 minor.*

3. Good values for (s, t)

It is straightforward to see that the conclusion of the Erdős-Lovász Tihany Conjecture holds for $s = \omega(G)$ or $t = \omega(G)$ (see also Lemma 2.1). Motivated by this observation and the difficulty in settling the Erdős-Lovász Tihany Conjecture for arbitrary values of s and t, Bjarne Toft posed the following problem.

Problem 3.1. *Given an arbitrary graph G with $\omega(G) < \chi(G)$, prove that there are integers $s, t \in \mathbb{N} \setminus \{1, \omega(G)\}$ with $\chi(G) = s + t - 1$ such that G contains two disjoint subgraphs G_1 and G_2 with $\chi(G_1) \geq s$ and $\chi(G_2) \geq t$.*

Problem 3.1 only asks for the existence of integers $s, t \in \mathbb{N} \setminus \{1, \omega(G)\}$ for which the conclusion of the Erdős-Lovász Tihany Conjecture holds. Nevertheless, we expect Problem 3.1 to be very difficult, since a positive solution of Problem 3.1 restricted to 6-chromatic graphs would imply a positive solution to the Double-Critical Graph Conjecture for 6-chromatic graphs.

Observation 3.1. *A positive solution to Problem 3.1 restricted to 6-chromatic graphs implies that K_6 is the only double-critical 6-chromatic graph, that is, the Double-Critical Graph Conjecture is true for 6-chromatic graphs.*

Proof. Suppose that we have a positive solution to Problem 3.1 restricted to 6-chromatic graphs. We shall use this assumption to prove that K_6 is the only double-critical 6-chromatic graph. Let G denote a double-critical 6-chromatic graph, and assume that G is non-complete. Then, by Proposition 2.1 (ii-iii), $\omega(G) \in \{3, 4\}$. Now, since in the case $\chi(G) = 6$ there are only two possible values of (s, t) (with $s \leq t$), namely $(2, 5)$ and $(3, 4)$, the positive solution to Problem 3.1 restricted to 6-chromatic graphs implies that G contains two disjoint subgraphs G_1 and G_2 with $\chi(G_1) \geq 2$ and $\chi(G_2) \geq 5$. This contradiction implies that G must be complete, and so, as desired, $G \simeq K_6$. □

Given Observation 3.1, we might ask whether there is some integer k such that Problem 3.1 has a positive solution for all graphs with chromatic number at least k.

Observation 3.2. *Suppose G is a graph with $\chi(G) > \omega(G)$ which contains no isolated vertices. If G contains a maximal clique K of order different from $\omega(G)$ and $\chi(G) - \omega(G) + 1$, then $s := n(K)$ and $t := \chi(G) - n(K) + 1$ are integers in $\mathbb{N} \setminus \{1, \omega(G)\}$ such that $\chi(G) = s + t - 1$ and $\chi(K) = s$ and $\chi(G - V(K)) \geq t$.*

Thus, when considering Problem 3.1 we may assume that every maximal clique of G has order 1, $\omega(G)$, or $\chi(G) - \omega(G) + 1$.

Proof of Observation 3.2. Suppose G is a graph with $\chi(G) > \omega(G)$, $\delta(G) \geq 2$, and a maximal clique K of order different from $\omega(G)$ and $\chi(G) - \omega(G) + 1$. Define $s := n(K)$ and $t := \chi(G) - n(K) + 1$.

Then $\chi(K) = n(K) = s$ and $\chi(G - V(K)) \geq \chi(G) - \chi(K) = t - 1$.

By assumption, $s \notin \{1, \omega(G), \chi(G) - \omega(G) + 1\}$. If $t = \omega(G)$ then $s = n(K) = \chi(G) - \omega(G) + 1$, a contradiction. If $t = 1$ then $\chi(G) = n(K)$ which contradicts the assumption $\chi(G) > \omega(G)$. Hence, $t \notin \{1, \omega(G)\}$.

If $\chi(G - V(K)) \geq t$ we immediately obtain the desired conclusion, and so we may assume $\chi(G - V(K)) = t - 1$. Let φ_1 and φ_2 denote an s-colouring and a $(t-1)$-colouring of K and $G - V(K)$, respectively, using colours from $[s]$ and $[s+t-1] \setminus [s]$. Then φ_1 and φ_2 can be combined into a vertex colouring φ of G. Now, φ uses exactly $\chi(G)$ colours to colour G, and so each colour class of G (under φ) contains a vertex adjacent to at least one vertex in each of the other colour classes (see, for instance, [32, Lemma 3.1]), in particular, some vertex x of colour $s + 1$ is adjacent to every vertex in $V(K)$. This, however, contradicts the assumption that K was a maximal clique, and so the proof is complete. □

Observation 3.2 can be found — in a slightly different formulation — in [4].

To the best of our knowledge, the Erdős-Lovász Tihany Conjecture remains open for triangle-free graphs. Here we solve Problem 3.1 for triangle-free graphs. For any positive integer k, there is a triangle-free k-chromatic graph [26, 36] (see also, for instance, [5, p. 371] or [35, p. 239]), and so Observation 3.3 describes a property of a non-trivial class of graphs.

Observation 3.3. *If G is a triangle-free graph with $\chi(G) \geq 5$, then G contains two disjoint subgraphs G_1 and G_2 such that $\chi(G_1) = 3$ and $\chi(G_2) \geq \chi(G) - 2$.*

Proof. Let G denote a triangle-free graph with $\chi(G) \geq 5$, and let G_1 denote a shortest odd cycle in G. Let the vertices of G_1 be labelled cyclically $v_1, \ldots, v_\ell$, and let G_2 denote the graph $G - V(G_1)$. We may assume $\chi(G_2) = \chi(G) - 3$. Now, assign the colour 2 to all even numbered vertices of G_1, and the colour 1 to all odd numbered vertices of G_1, except v_ℓ which is assigned the colour 3 — this, of course, gives a proper 3-colouring of G_1. Colour the vertices of G_2 with $\chi(G_2)$ colours all distinct from $1, 2$, and 3. We now have a proper $\chi(G)$-colouring of G, and so each colour class of G contains at least one vertex which is adjacent to at least one vertex in each other colour class. Let v denote such a vertex of $V(G_2)$. Since $v_\ell \in V(G_1)$ is the only vertex of G coloured 3, it follows that v is adjacent to v_ℓ. Since $\omega(G) = 2$, it now follows that v is adjacent to neither v_1 nor $v_{\ell-1}$. Hence v is adjacent to some even numbered vertex v_{e} (coloured 2) and some odd numbered vertex v_{o} (coloured 1) in $\{v_2, v_3, \ldots, v_{\ell-2}\}$. Now the $(v_{\mathrm{e}}, v_{\mathrm{o}})$-path P in G_1 not containing v_ℓ has odd length, in particular, $|E(P)| \leq \ell - 4$, since neither $v_{\ell-2}v_{\ell-1}$, $v_{\ell-1}v_\ell$, $v_\ell v_1$, nor $v_1 v_2$ is in $E(P)$. Hence $V(P) \cup \{v\}$ contains an induced odd cycle of length at most $(\ell - 4) + 2$, a contradiction. This completes the proof. □

Problem 3.2. *Prove that the Erdős-Lovász Tihany Conjecture holds for triangle-free graphs.*

4. Disjoint complete minors in 7-chromatic graphs

The Erdős-Lovász Tihany Conjecture remains unsettled for $(4, 4)$-graphs, but it follows easily from the abovementioned theorems by Dirac (Theorem 2.5) and Stiebitz (Theorem 2.4) that any $(4, 4)$-graph G with $\omega(G) < 7$ contains two disjoint subgraphs G_1 and G_2 such that $\eta(G_1) \geq 4$ and $\eta(G_2) \geq 4$.

Observation 4.1. *Every 7-chromatic graph with clique number at most 6 contains a $\overline{K_{4,4}}$ minor.*

Proof. Let G denote a 7-chromatic graph with clique number at most 6. It follows from Theorem 2.4 that G contains two disjoint subgraphs G_1 and G_2 such that $\chi(G_1) \geq 4$ and $\operatorname{col}(G_2) \geq 4$, and so the desired result follows directly from Theorem 2.5. □

Corollary 4.1. *Every 7-chromatic graph with clique number at most 6 contains a $\overline{K_{2,6}}$, $\overline{K_{3,5}}$, and $\overline{K_{4,4}}$ as a minor.*

Proof. Let G denote a 7-chromatic graph with clique number at most 6. By Observation 4.1, $G \geq \overline{K_{4,4}}$. By a theorem of Stiebitz [32], there are two disjoint subgraphs G_1 and G_2 of G with $\chi(G_1) \geq 3$ and $\chi(G_2) \geq 5$. Hence, by Observation 2.1 and Theorem 2.6, respectively, $G_1 \geq K_3$ and $G_2 \geq K_5$, that is, $G \geq \overline{K_{3,5}}$. Finally, we need to show $G \geq \overline{K_{2,6}}$. If G contains two disjoint subgraphs G_1 and G_2 with $\chi(G_1) \geq 2$ and $\chi(G_2) \geq 6$, then the desired result follows Theorem 2.7. Hence we may assume that G is a $(2,6)$-graph with $\omega(G) < 7 = \chi(G)$. This means that G is a non-complete double-critical 7-chromatic graph, and so, by Theorem 2.8, $G \geq K_7$. Let $\mathcal{B}$ denote a set of seven branch sets which together form a K_7 minor of G. If each branch set of $\mathcal{B}$ consists of a single vertex, then G contains K_7 as a subgraph, a contradiction. Hence some branch set $B \in \mathcal{B}$ has size at least two. Now $\mathcal{B} \setminus \{B\}$ form a K_6 minor of $G - B$, while $G[B] \supseteq K_2$, since $|B| \geq 2$ and $G[B]$ is connected. Hence $G \geq \overline{K_{2,6}}$. □

5. Disjoint complete minors in 8-chromatic graphs

There are three values of (s,t) to consider for 8-chromatic graphs, namely $(2,7)$, $(3,6)$, and $(4,5)$.

The case $(s,t) = (2,7)$ of Conjecture 1.3 remains unsettled, but at least we are able to settle the case $(s,t) = (2,7)$ of Conjecture 1.4 in the affirmative. (Our proof method does not settle the case $(s,t) = (2,7)$ of Conjecture 1.3, since Hadwiger's Conjecture remains unsettled for 7-chromatic graphs.)

Theorem 5.1. *Every $(2,7)$-graph with clique number at most 7 contains a $\overline{K_{2,7}}$ minor.*

Proof. Let G denote a $(2,7)$-graph with clique number at most 7. Then G is a non-complete double-critical 8-chromatic graph. Now, according to Proposition 2.1 (i), G has minimum degree $\delta(G)$ at least 9. Suppose $\delta(G) \geq 12$. Then, according to Theorem 2.10, $G \geq K_8$. Let $\mathcal{B}$ denote a set of eight branch sets which together form a K_8 minor of G. If each branch set of $\mathcal{B}$ consists of a single vertex, then G contains K_8 as a subgraph, a contradiction.

Hence some branch set $B \in \mathcal{B}$ has size at least two. Now $\mathcal{B} \setminus \{B\}$ form a K_7 minor of $G - B$, and $G[B] \geq K_2$, since $|B| \geq 2$ and $G[B]$ is connected. Hence we may assume $\delta(G) \in \{9, 10, 11\}$, and let x denote a vertex of G of minimum degree.

Suppose that $V(G) \setminus N[x]$ is an independent set of G. We must have $\chi(G[N(x)]) \geq 7$, since otherwise $\chi(G) \leq \chi(G - N(x)) + \chi(G[N(x)]) \leq 1 + 6 < \chi(G)$. Since $\delta(G) \geq 9$ and $\omega(G) < 8 = \chi(G)$, it follows that there is some vertex $y \in N(x)$ which has at least one neighbour, say z, in $V(G) \setminus N[x]$. Define $S' := N(x) \setminus \{y\}$ and $S := S' \cup \{x\}$. Then $\chi(G[S']) \geq 6$, since $\chi(G[N(x)]) \geq 7$. Hence, according to Theorem 2.7, $G[S'] \geq K_6$. This implies $G[S] \geq K_7$. Moreover, $G - S \geq K_2$, since $y, z \in V(G) \setminus S$ with $yz \in E(G)$, and so the desired partition exists.

Suppose that $G[V(G) \setminus N[x]]$ contains at least one edge. We show that the induced subgraph $G[N(x)]$ has minimum degree at least 6. Given any vertex $y \in N(x)$, let $A(y)$ denote the set of vertices of $N(x) \setminus \{y\}$ which are not adjacent to y, and let $B(y)$ denote the set of common neighbours of x and y. Then, clearly,

$$\deg(x, G) = |A(y)| + |B(y)| + 1 \quad \text{and} \quad |A(y)| = |N(x)| - 1 - \deg(y, G[N(x)]).$$

According to Proposition 2.1 (iii), $|B(y)| \geq 6$, since G is a non-complete double-critical 8-chromatic graph. Hence

$$\deg(x, G) \geq (\deg(x, G) - 1 - \deg(y, G[N(x)])) + 7$$

which implies $\deg(y, G[N(x)]) \geq 6$. Since, in addition, $|N(x)| \leq 11$, it now follows from Theorem 2.11 that the induced subgraph $G[N(x)]$ contains a K_6 minor. Hence $G[N[x]]$ contains a K_7 minor, and so, since $G - N[x]$ contains at least one edge, $G \geq \overline{K_{2,7}}$, as desired. □

Corollary 5.1. *If Hadwiger's Conjecture is true for all* 7*-chromatic graphs, then every* 8*-chromatic graph with clique number at most* 7 *contains a* $\overline{K_{2,7}}$ *minor.*

Proof. Let G denote an arbitrary but fixed 8-chromatic graph with clique number at most 7. If G is a $(2,7)$-graph then, by Theorem 5.1, $G \geq \overline{K_{2,7}}$. Otherwise, there are two disjoint subgraphs G_1 and G_2 of G such that $\chi(G_1) \geq 2$ and $\chi(G_2) \geq 7$. Obviously, $G_1 \geq K_2$ and, by assumption, $G_2 \geq K_7$. Hence $G \geq \overline{K_{2,7}}$. □

Theorem 5.2. *Every* 8*-chromatic graph with clique number at most* 7 *contains a* $\overline{K_{4,5}}$ *minor.*

Proof. In order to obtain a contradiction, suppose that the statement is false, and let G denote a counterexample with the least possible number of vertices. Then G is vertex-critical, and so $\delta(G) \geq \chi(G)-1 = 7$. If G contains two disjoint subgraphs G_1 and G_2 such that $\chi(G_1) \geq 4$ and $\chi(G_2) \geq 5$, then $G_1 \geq K_4$ and $G_2 \geq K_5$ by Theorem 2.5 and Theorem 2.6, a contradiction. Hence we may assume that G is a $(4,5)$-graph. In addition, $\omega(G) < 8 = \chi(G)$ and so, by Lemma 2.2, $\omega(G) \leq 4$.

Suppose $\delta(G) \geq 10$. Then, by Theorem 2.3, there are two disjoint subgraphs G_1 and G_2 of G such that $\delta(G_1) \geq 3$ and $\delta(G_2) \geq 6$, and so, by Theorem 2.5 and Theorem 2.9, $G_1 \geq K_4$ and $G_2 \geq K_5$, a contradiction. Hence we must have $9 \geq \delta(G) \geq 7$. Let u denote a vertex of G of minimum degree.

Suppose $\delta(G) = 7$, and let $u_1, \ldots, u_7$ denote the neighbours of u. Since $\omega(G)$ is at most 4, $G[N(u)]$ contains at least two non-adjacent vertices. Let u_1 and u_2 denote two non-adjacent vertices of $G[N(u)]$, and let G' denote the graph obtained from G by contracting $U := \{u, u_1, u_2\}$ into a single vertex u'. Since $G' - u' = G - \{u, u_1, u_2\}$ and $\chi(G - \{u, u_1, u_2\}) \leq 7$, it follows that G' is 8-colourable. Suppose G' is 8-chromatic. Now, the fact that G is a minimum counterexample implies that G' must contain a clique of order 8. This, however, implies $\omega(G) = 7$ which contradicts the fact that G has clique number at most 4. Hence $\chi(G') \leq 7$. Let $k := \chi(G')$, and let φ denote a k-colouring of G'. We may assume $\varphi(u') = 1$ and $\varphi(\{u_3, \ldots, u_7\}) \subseteq \{2,3,4,5,6\}$. Now, let $\psi : V(G) \to [7]$ denote the mapping defined as follows: $\psi(w) := \varphi(w)$ for all $w \in V(G) \backslash U$, $\psi(u_1) := \psi(u_2) := 1$, and $\psi(u) = 7$. Then ψ is a 7-colouring of G, which contradicts the fact that G is 8-chromatic.

Suppose $\delta(G) = 8$. If $G[N(u)]$ contains an independent 3-set, then we obtain a contradiction as in the case $\delta(G) = 7$. Hence $\alpha(G[N(u)]) = 2$. Since the graph $G[N(u)]$ has 8 vertices and independence number 2, it follows from Theorem 2.1 (i) that $G[N(u)]$ contains a triangle, say $Z \subseteq N(u)$ with $G[Z] \simeq K_3$. Define $Z' := Z \cup \{u\}$. Then $G[Z'] \simeq K_4$ and $\chi(G[Z']) = 4$. Moreover, $\chi(G - Z') \leq 4$, since G is a $(4,5)$-graph. In fact, $\chi(G - Z')$ must be equal to 4, since $\chi(G[Z']) + \chi(G - Z') \geq \chi(G) = 8$. Let $\varphi_1 : Z' \to [4]$ denote 4-colouring of $G[Z']$, and let $\varphi_2 : V(G - Z') \to [8] \setminus [4]$ denote a 4-colouring of $G - Z'$. Let φ denote the 8-colouring of G obtained by combining φ_1 and φ_2. Then, since $\chi(G) = 8$, it follows that each colour class contains a vertex adjacent to at least one vertex in each of the other colour classes (see, for instance, [32, Lemma 3.1]), in particular, there is a vertex z of colour 5 adjacent to each of the vertices of colour $1, 2, 3$, and 4. This means that $Z'' := Z' \cup \{z\}$ induces a complete 5-graph in G, a contradiction.[2] This

[2] We shall refer to this argument as the *standard argument*.

completes the case $\delta(G) = 8$.

Suppose $\delta(G) = 9$. If $G[N(u)]$ contains an independent 4-set, then we obtain a contradiction as in the case $\delta(G) = 7$. Now follows from Theorem 2.1 (ii) that $G[N(u)]$ contains a triangle, and we obtain a contradiction as in the case $\delta(G) = 8$ using the standard argument. This completes the proof. □

Theorem 5.3. *Every 8-chromatic graph with clique number at most 7 contains a $\overline{K_{3,6}}$ minor.*

Proof. In order to obtain a contradiction, suppose that the statement of the theorem is false, and let G denote a counterexample with the least possible number of vertices. Then G is vertex-critical, and so $\delta(G) \geq \chi(G) - 1 = 7$. If G contains two disjoint subgraphs G_1 and G_2 with $\chi(G_1) \geq 3$ and $\chi(G_2) \geq 6$, then $G_1 \geq K_3$ and $G_2 \geq K_6$ by Observation 2.1 and Theorem 2.7, a contradiction. Hence we may assume that G is a $(3, 6)$-graph. Moreover, $\omega(G) < 8 = \chi(G)$ and so, by Lemma 2.2, $\omega(G) \leq 5$. If G contains an even induced cycle C_e, then $\chi(G - V(C_e)) \geq \chi(G) - \chi(C_e) = 8 - 2 = 6$ in which case we obtain a contradiction by Theorem 2.7 and the fact that $C_\mathrm{e} \geq K_3$. Hence G is an even-hole-free graph, and so, according to Theorem 2.2, G contains a bisimplicial vertex, say u. If $\deg(u) \geq 9$, then $G[N[u]]$ contains a clique of order at least 6, a contradiction. Hence $\deg(u) \leq 8$. If $\deg(u) = 7$ then we obtain a contradiction just as in the case $\delta(G) = 7$ in proof of Theorem 5.2. Hence $\deg(u) = 8$ and $G[N(u)]$ is coverable by two 4-cliques A and B. Let the vertices of A and B be denoted a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4, respectively. No vertex of A (B) is adjacent to every vertex of B (A), since that would imply that G contains a 6-clique.

(i) Suppose that there is at least one (A, B)-edge in G; we may, without loss of generality, assume $a_3b_4 \in E(G)$ and $a_4b_4 \notin E(G)$. We contract the vertices a_4, b_4, and u into a single vertex u', and let G' denote the resulting graph. If $\chi(G') \geq 8$, then $\chi(G') = 8$ and it follows from the minimality of the counterexample G that $\omega(G') = 8$ and so $\omega(G) \geq 7$, a contradiction. Hence $\chi(G') \leq 7$. Define $k := \chi(G')$ and let $\varphi : V(G') \to [k]$ denote a k-colouring of G'. We obtain a 7-colouring of G, i.e. a contradiction, unless φ applies seven distinct colours to $a_1, a_2, a_3, b_1, b_2, b_3$, and u'. We may assume $\varphi(u') = 7$, and $\varphi(a_i) = i$ and $\varphi(b_i) = i + 3$ for $i \in [3]$. If, for some $i \in [3]$, there is no $(3, i + 3)$-Kempe chain starting at a_3 and containing b_i, then we may exchange the colours 3 and $i+3$ in a $(3, i+3)$-Kempe chain starting at a_3 and so, after the obvious assignment of colours, we obtain a 7-colouring of G, a contradiction. This means in particular that, for each $i \in [3]$, there is a (a_3, b_i)-path Q_i in G with all internal vertices in $V(G) \setminus N[u]$. Contracting $Q_1 - b_1$, $Q_2 - b_2$, and $Q_3 - b_3$ into a single vertex Q we obtain a minor H of G in which $\{a_1, a_2, a_4\}$ induce a K_3 and $\{b_1, b_2, b_3, b_4, Q, u\}$ induce a K_6. (Here we use the fact that a_3 is adjacent to b_4 in G and so Q is also adjacent to b_4 in G.) This contradicts the assumption that G is a counterexample.

(ii) Suppose that there is no (A, B)-edge in G. Since u is not a cutvertex of G, G contains an (A, B)-path Q with all internal vertices contained in $V(G) \setminus N[u]$; we may, without loss of generality, assume $Q \cap A = \{a_3\}$ and $Q \cap B = \{b_4\}$. Now the argument from case (i) can be copied to obtain a $\overline{K_{3,6}}$ minor in G, a contradiction. This contradiction implies that no counterexample exists, and the proof is complete. □

6. Disjoint complete minors in 9-chromatic graphs

Theorem 6.1. *Every* 9*-chromatic graph with clique number at most* 8 *contains* $\overline{K_{4,6}}$ *or* $\overline{K_{5,5}}$ *as a minor.*

Proof. In order to obtain a contradiction, suppose that the statement of the theorem is false, and let G denote a counterexample with the fewest possible number of vertices. Then G is vertex-critical and so $\delta(G) \geq \chi(G) - 1 = 8$. If G contains two disjoint subgraphs G_1 and G_2 such that (a) $\chi(G_1) \geq 5$ and $\chi(G_2) \geq 5$ or (b) $\chi(G_1) \geq 4$ and $\chi(G_2) \geq 6$, then we obtain a contradiction by applying Theorem 2.5, Theorem 2.6, and Theorem 2.7. Hence we may assume that G is a $(4, 6)$-graph and a $(5, 5)$-graph. Since $\omega(G) < \chi(G) = 9$, it follows from Lemma 2.2 that $\omega(G) \leq 4$. If G contains a 4-clique K, then by the standard argument — the one occurring in the proof of Theorem 5.2 — $\chi(G - V(K)) = 5$, and so G contains a 5-clique, a contradiction. Hence $\omega(G) \leq 3$.

Suppose $\delta(G) \geq 12$. Then it follows from Theorem 2.3 that there is a partition (S, T) of $V(G)$ such that $\delta(G[S]) \geq 3$ and $\delta(G[T]) \geq 8$. Now, by Theorem 2.9, $G[S] \geq K_4$ and $G[T] \geq K_6$, a contradiction. Hence we must have $11 \geq \delta(G) \geq 8$. Let u denote a vertex of G of minimum degree.

If $\delta(G) = 8$, then we obtain a contradiction as in the case $\delta(G) = 7$ in the proof of Theorem 5.2.

Suppose $\delta(G) = 9$. If, in addition, $\alpha(G[N(u)]) \geq 3$, then we again obtain a contradiction as in the case $\delta(G) = 7$ in the proof of Theorem 5.2. Hence $\alpha(G[N(u)]) = 2$, and so, by Theorem 2.1 (ii), $\omega(G[N(u)]) \geq 4$, a contradiction. A similar argument applies in the case $\delta(G) = 10$.

Suppose $\delta(G) = 11$. If $\alpha(G[N(u)]) \geq 5$, then we obtain a contradiction as in the case $\delta(G) = 7$ in the proof of Theorem 5.2. Hence $\alpha(G[N(u)]) \leq 4$, and so, since $\chi(G[N(u)]) \geq 11/\alpha(G[N(u)])$, we obtain $\chi(G[N(u)]) \geq 3$. This, of course, implies $\chi(G[N[u]]) \geq 4$, and so we may assume that $\chi(G - N[u]) \leq 5$. Suppose $G[N(u)]$ contains an even induced cycle C_e. Then we obtain $\chi(G[V(C_e) \cup \{u\}]) = 3$, $G[V(C_e) \cup \{u\}] \geq K_4$, and $\chi(G - (V(C_e) \cup \{u\})) \geq \chi(G) - \chi(G[V(C_e) \cup \{u\}]) = 6$ and so, by Theorem 2.7, $G - (V(C_e) \cup \{u\}) \geq K_6$, a contradiction. Hence $G[N(u)]$ contains no even induced cycle, and so, by Theorem 2.2, $\chi(G[N(u)]) \leq 2\omega(G[N(u)]) - 1 \leq 3$. Altogether, $\chi(G[N(u)]) = 3$. The fact that G is 9-chromatic and $G[N(u)]$ is 3-colourable implies that $G - N[u]$ is non-empty, in particular, $\chi(G - N(u))$ is equal to $\chi(G - N[u])$. Thus,

$$\chi(G - N[u]) = \chi(G - N(u)) \geq \chi(G) - \chi(G[N(u)]) = 9 - 3 = 6$$

and so we have a contradiction. This completes the proof. □

7. Subdivisions, complete minors and vertex partitions

What happens if we replace the chromatic number χ by the Hadwiger number η in the premise and conclusion of the Erdős-Lovász Tihany Conjecture? Then the corresponding statement is true whenever $s = 2$, but false in general. For instance, any subdivision of K_5 which is not isomorphic to K_5 has Hadwiger number 5, clique number at most 4, but does not contain two disjoint K_3 minors, since such a graph does not contain two disjoint cycles. One way of excluding such counterexamples would be to require the Hadwiger number η to be strictly greater than the Hajós number h, which gives us the following statement.

($\star$) For any pair of integers $s, t \geq 2$ and any graph G with $h(G) < \eta(G) = s + t - 1$ there are disjoint subgraphs G_1 and G_2 of G with $\eta(G_1) \geq s$ and $\eta(G_2) \geq t$.

The statement ($\star$) is also not true in general. Figure 1 shows counterexamples for the cases $(s, t) = (3, 5)$ and $(s, t) = (4, 4)$. The statement ($\star$) is, of course, true for $s = 2$, but it is also true for $(s, t) \in \{(3, 3), (3, 4)\}$.

In case $(s, t) = (3, 3)$, we are looking for two disjoint (that is, vertex-disjoint) cycles in a graph. The question of whether a graph contains two disjoint cycles was studied already by Dirac, Erdős, and Pósa [9, 13, 28] in the early 1960s. For instance, Pósa [28] and Dirac (unpublished, see [13, p. 4]) proved that if G is a graph with $n(G) \geq 6$ and $m(G) \geq 3n(G) - 5$ then G contains two disjoint cycles.

(a) The graph H_0. (b) The graph H_1.

Figure 1: The graphs H_0 and H_1 are counterexamples to statement ($\star$) for $(s,t) = (3,5)$ and $(s,t) = (4,4)$, respectively.

Lovász [22] obtained a complete characterisation of the graphs without two disjoint cycles (see also [23, p. 425]). Suppose G is a graph with no two disjoint cycles. We remove all vertices of degree one and "suppress" vertices of degree 2, then the resulting multigraph G' is one of the following types: $G' - v$ is a tree for some vertex $v \in V(G')$; a triangle T – possibly with multiple edges – and an arbitrary number of vertices joined to all three vertices of T; a cycle C with each vertex of C joined by one or more edges to an extra vertex; the complete 5-graph; and subgraphs of the already mentioned graphs. The case $(s,t) = (3,3)$ of ($\star$) follows from this.

The following observation shows that ($\star$) is true for $(s,t) = (3,4)$.

Observation 7.1. *Any graph with Hadwiger number* 6 *and Hajós number at most* 5 *contains two disjoint subgraphs* G_1 *and* G_2 *such that* $\eta(G_1) \geq 3$ *and* $\eta(G_2) \geq 4$*.*

To the best of our knowledge, no characterisation of the graphs without two disjoint complete minors of order 3 and 4, respectively, have been published.

Proof. Suppose G is a counterexample with the minimum number of edges. Let $\mathcal{B}$ denote the set of branch sets of a K_6 minor of G. Then each branch set from $\mathcal{B}$ induces a tree in G and there is exactly one edge between any pair of branch sets of $\mathcal{B}$. We choose the branch sets in $\mathcal{B}$ such that each vertex of G is in some branch set from $\mathcal{B}$. The fact that G is a minimum counterexample implies that G contains no vertices of degree less than 3.

Suppose that $G[B]$ with $B \in \mathcal{B}$ is not a singleton; then $G[B]$ contains at least two leaves. Since $\delta(G) \geq 3$, each leaf ℓ of $G[B]$ is adjacent to vertices $v_\ell \in B'_\ell$ and $w_\ell \in B''_\ell$ with $B'_\ell, B''_\ell \in \mathcal{B} \setminus B$. This, however, implies the existence of a leaf of $G[B]$ with degree exactly 3 in G. Using this latter fact, it is easy to see that G contains the desired subgraphs G_1 and G_2; a contradiction. Hence each induced subgraph $G[B]$ with $B \in \mathcal{B}$ is a singleton, and so $G \simeq K_6$. This is a contradiction. $\square$

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