Ramsey functions for quasi-progressions with large diameter

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A k-term quasi-progression of diameter d is a sequence

$$x_1 < \cdots < x_k$$

of positive integers for which there exists a positive integer l such that $l \leq x_j - x_{j-1} \leq l + d$, for all $j = 2, \ldots, k$. Let $Q(d, k)$ be the least positive integer such that every 2-coloring of $\{1, \ldots, Q(d, k)\}$ contains a monochromatic k-term quasi-progression of diameter d. We prove that

$$Q(k - i, k) = 2ik - 4i + 2r - 1,$$

if $k = mi + r$ for integers m, r such that $3 \leq r < \frac{i}{2}$ and $r - 1 \leq m$. We also prove that, if $k \geq 2i \geq 1$, then

$$Q(k - i, k) = \begin{cases} 2ik - 4i + 3 & \text{if } k \equiv 0 \text{ or } 2 \pmod{i} \\ 2ik - 2i + 1 & \text{if } k \equiv 1 \pmod{i} \end{cases}$$

These results partially settle several conjectures due to Landman [Ramsey Functions for Quasi-Progressions, Graphs and Combinatorics 14 (1998) 131–142].

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1. Introduction

Several renowned open conjectures in combinatorics and number theory involve arithmetic progressions. Van der Waerden famously proved in 1927 that for each positive integer k there exists a least positive integer $w(k)$ such that any 2-coloring of $1, \ldots, w(k)$ produces a monochromatic k-term arithmetic progression. The best known upper bound for $w(k)$ is due to Gowers and is quite large. Ron Graham [2] conjectures $w(k) \leq 2^{k^2}$, for all k. The

best known lower bound is due to Berlekamp: $w(p+1) \geq p2^p$, for primes p. Another conjecture involving arithmetic progressions is one of Erdős' most famous (still open) conjectures: if the sum of the reciprocals of the members of a set of positive integers S diverges, then S contains arbitrarily long arithmetic progressions. Many researchers have worked on this or special cases of this conjecture, including the recent and now famous theorem by Green and Tao which proves that the primes contain arbitrarily long arithmetic progressions.

Motivated by Erdős' conjecture, Brown, Erdős, and Freedman [1] introduced quasi-progressions: a k-term *quasi-progression of diameter d* is a sequence $x_1 < \cdots < x_k$ of positive integers for which there exists a positive integer l such that $l \leq x_j - x_{j-1} \leq l + d$, for all $j = 2, \ldots, k$. Arithmetic progressions are quasi-progressions with diameter zero. Brown et al. consider the question of when a set of positive integers contains arbitrarily long quasi-progressions of a given diameter. A main result of theirs is that quasi-progressions behave similarly to arithmetic progressions, at least with respect to Erdős' conjecture, in that Erdős' conjecture is equivalent to the statement: if the sum of the reciprocals of the members of a set of positive integers S diverges, then there exists a diameter d such that S contains arbitrarily long quasi-progressions of diameter d.

Analogous to the van der Waerden function $w(k)$, Landman [3] introduced a Ramsey function for quasi-progressions. Let $Q(d, k)$ be the least positive integer such that every 2-coloring of $\{1, \ldots, Q(d, k)\}$ contains a monochromatic k-term quasi-progression of diameter d. This function produces a lower bound for $w(k)$ since

$$w(k) = Q(0, k) \geq Q(1, k) \geq Q(2, k) \geq Q(3, k) \geq \cdots$$

So, it is of great interest to find bounds on $Q(d, k)$ for various d, especially small values of d. Of particular interest is the rate of growth of $Q(1, k)$. Is it merely polynomial or is it at least exponential in k? Vijay [5] has recently established an exponential lower bound for $Q(1, k)$, so quasi-progressions of small diameter behave similarly to arithmetic progressions, at least with respect to these Ramsey functions. An interesting open problem is to determine the largest diameter d for which $Q(d, k)$ is at least exponential (in k).

Landman established several bounds on $Q(d, k)$ and made several conjectures which we resolve in this paper. Our results, like Landman's, focus on large diameter; that is, $d = k - i$, for some positive integer i satisfying $k \geq 2i \geq 1$. The main difficulty is the upper bound on $Q(k-i, k)$. Specifically, how can we tailor an argument that handles the large number of extremal

2-colorings which seem to defy uniform description (cf. Landman's data at the end of his paper)? In Section 2 we introduce super blocks, an equivalence relation that imposes sufficient structure on extremal 2-colorings to extract long monochromatic quasi-progression fragments via a greedy strategy. Through the super block lens, the extremal 2-colorings coalesce. The super block argument is used in Section 3 where we show how to splice monochromatic fragments together to produce long monochromatic quasi-progressions. This yields an upper bound on $Q(k-i,k)$; the bound is often sharp. One consequence is that, if $k \geq 2i$ and $k = mi + r$ for integers m, r such that $1 < r < i$, then

$$Q(k-i,k) \leq 2ik - 4i + 2r - 1.$$

This improves bounds given by Landman.

Our main result is proven in Section 4:

$$Q(k-i,k) = 2ik - 4i + 2r - 1,$$

if $k = mi + r$ for integers m, r such that $3 \leq r < \frac{i}{2}$ and $r - 1 \leq m$. This disproves Conjecture 1 of Landman's paper for these values of r. Residues $r \geq i/2$ that are not considered in this result appear to be more difficult. The techniques here are inadequate to resolve those values of $Q(k-i,k)$. However, we also prove that, if $k \geq 2i \geq 1$, then

$$Q(k-i,k) = \begin{cases} 2ik - 4i + 3 & \text{if } k \equiv 0 \text{ or } 2 \pmod{i} \\ 2ik - 2i + 1 & \text{if } k \equiv 1 \pmod{i} \end{cases}$$

thus proving Landman's conjecture in these cases.

2. Super blocks

In this section we develop notation, concepts and tools to describe the structure of extremal strings which avoid long monochromatic quasi-progressions.

A *k-term progression* is an increasing sequence of k positive integers $x_1 < x_2 < \cdots < x_k$. Given a k-term progression $P = \{x_j\}_{j=1}^k$, the *differences* in P are the elements in the set

$$D(P) = \{x_j - x_{j-1} : j = 2, \ldots, k\}.$$

The *low-difference* of P, denoted $\delta(P)$ (or simply δ), is the minimum element in $D(P)$; the *high-difference* of P, denoted $\Delta(P)$ (or simply Δ), is

the maximum. The diameter of P is $d = \Delta - \delta$. Observe that arithmetic progressions are quasi-progressions with diameter $d = 0$.

Consider now 2-colorings of the positive integers. A quasi-progression is a *good progression* if it is a monochromatic quasi-progression with length k and diameter at most d. Define $Q(d, k)$ to be the least positive integer such that every 2-coloring of $\{1, \ldots, Q(d, k)\}$ contains a good progression. Motivated by conjectures of Landman [3], we consider $Q(d, k)$ for values of d and k satisfying $d = k - i$ and $k \geq 2i$, where i is some fixed positive integer. So, for the rest of this paper $d = k - i$ and a *good progression* means a monochromatic quasi-progression with length k and diameter at most $k - i$. To understand the structure of extremal 2-colorings avoiding good progressions, we now introduce two important substructures: blocks and super blocks.

Let $C = c_1 c_2 \ldots c_\ell$ be a binary string of length ℓ. A *substring* of C is a string of the form $c_p c_{p+1} \cdots c_q$, for some positive integers $1 \leq p \leq q \leq \ell$. A *block* of C is a maximal monochromatic substring of C. We employ the usual shorthand notation in which, for $x \in \{0, 1\}$, the shorthand x^n represents the string $\underbrace{x \cdots x}_{n}$. There is a natural partition of C into blocks: without loss of generality, the first block of C is a block of 1's so

$$C = 1^{\alpha_1} 0^{\alpha_2} \cdots 1^{\alpha_{b-1}} 0^{\alpha_b},$$

where α_j's are positive integers, except possibly α_b which may be zero (in which case the final block of C is actually a block of 1's). Note that $\sum_{j=1}^{b} \alpha_j = \ell$.

Now consider an extremal coloring C; that is, suppose that $\ell = Q(k - i, k) - 1$ and C represents a 2-coloring of the integers $1, \ldots, \ell$ with no good progression. For convenience, blocks of C of length at most $k - i$ are *minor* blocks; longer blocks are *major*. There are two important facts that motivate this dichotomy: 1) quasi-progressions with low-difference 1 and diameter $k - i$ can jump over any intermediate minor blocks, and 2) "greedy monochromatic jumping" (in which jumps of length at least δ but at most $\delta + k - i$ are taken, for some choice of δ) can not get stuck in substrings that avoid major blocks of one color (see later Theorems 2.3 and 2.4). A consequence of observation 1) is that, in a substring in which only minor blocks of one color appear, all of the integers with the other color in this substring form a monochromatic progression with low-difference 1 and high-difference $k - i + 1$ (that is, a monochromatic quasi-progression with diameter $k - i$).

Now we turn to the task of identifying monochromatic substructures of C with the property that a greedy strategy can guarantee dense monochromatic progressions beginning and ending at endpoints of the substructure. To make this precise, we define the equivalence relation $\sim$ on the integers $1, \ldots, \ell$ so that $x \sim y$ if and only if x and y are contained in a monochromatic quasi-progression P of C such that $D(P) \subseteq \{1, \ldots, k - i + 1\}$ (the transitivity of $\sim$ follows from the fact that the union of two intersecting monochromatic quasi-progressions with differences in $\{1, \ldots, k - i + 1\}$ is another such quasi-progression). The equivalence classes under $\sim$ are called *super blocks*. Super blocks are not necessarily substrings. Suppose that C has t super blocks $B_1, \ldots, B_t$. We naturally order super blocks this way: $B_p < B_q$ if and only if $\min B_p < \min B_q$, where $\min B_p$ denotes the smallest integer in B_p (that is, the left-most one). A super block is *major* if it contains all of the elements from a major block of C; otherwise it is *minor*. The *extremes* of a super block are its minimum and maximum elements.

Example 1. Consider $k = 12, i = 6$. Because $Q(6, 12) = 123$, an extremal string in this case has length 122. There are several extremal strings, one is shown below:

$$C = 1^6 0^{10} 1^1 0^1 1^{10} 0^{11} 1^{11} 0^{10} 1^1 0^1 1^{10} 0^{10} 1^1 0^1 1^{10} 0^{11} 1^{11} 0^6.$$

This string has 18 blocks, but only 12 super blocks of which exactly two are minor. The cardinalities of the super blocks are (in this order)

$$6, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 6.$$

Theorem 2.1 (Super Block Upper Bound). *If C is a binary string that represents a 2-coloring with no good progression, then every super block of C has cardinality at most $k - 1$.*

Proof. The union of two intersecting monochromatic quasi-progressions of C using differences from $\{1, \ldots, k - i + 1\}$ is also a monochromatic quasi-progression of C using differences from $\{1, \ldots, k - i + 1\}$. It follows that all of the elements of a super block are contained in a single monochromatic quasi-progression of C using differences from $\{1, \ldots, k - i + 1\}$. Because C contains no good quasi-progression, each super block contains fewer than k elements. $\square$

The following theorem lists basic facts about super blocks.

Theorem 2.2 (Super Block Fundamentals). *Suppose C is a binary string that represents a 2-coloring with no good progression. If C has t super blocks $B_1 < \cdots < B_t$, then*

   (i) *all elements in a super block have the same color (that is, super blocks are monochromatic),*

  (ii) $B_1, \ldots, B_t$ *form a partition of C,*

 (iii) *consecutive super blocks have opposite color,*

 (iv) *super blocks $B_2, \ldots, B_{t-1}$ contain exactly one major block (in particular they are major super blocks),*

  (v) *an extreme element of a super block is adjacent to either the end of the string, a minor super block, or the major block of neighboring major super block, and*

 (vi) *a substring of C consisting of all characters between (and including) the extreme elements of a super block contains exactly one major block.*

Proof. i) two elements are in relation $\sim$ if they are in a common monochromatic quasi-progression. Therefore their color is identical. ii) the equivalence classes of an equivalence relation form a partition. iii)–vi) the boundary of a super block is reached at the end of the string or at an obstructing major block of the opposite color. Thus each super block with a neighbor, must contain a major block that defines the boundary of that neighbor. So super blocks alternate color. If a super block contained two major blocks, then its size would exceed $k - 1$, contradicting Theorem 2.1.   □

Note that, by (ii), the length of C is $\ell = \sum_{j=1}^{t} |B_j|$. The basic approach for an upper bound on the length of C is based on this partition — we seek to bound the cardinality of each of the super blocks. To accomplish this, we need to argue that segments of a long monochromatic quasi-progression can be strung together using fragments from each super block. The technique relies on the following two fundamental theorems.

Theorem 2.3 (Greedy Major Super Block Jumping). *Assume $k \geq 2i$. If B is a major super block of C and $1 \leq \delta \leq i$, then B contains a monochromatic quasi-progression P such that*

   i) *P has length at least $\lceil \frac{|B|}{\delta} \rceil$,*

  ii) *the low difference of P is at least δ,*

 iii) *the diameter of P is at most $k - i$, and*

 iv) *both extremes of B are in P.*

Proof. Without loss of generality, B has color 1. Let S denote the binary substring of C consisting of all characters between the minimum and maximum elements of B. By definition, S begins and ends with a 1. Let us suppose that $p\delta < |B| \leq (p+1)\delta$, for some positive integer p. We must show that there is a monochromatic quasi-progression of length at least $p+1$ (that

is, a progression that satisfies (i) above) with the additional properties (ii)–(iv). First, create a monochromatic quasi-progression this way: start with the left-most 1 of S and repeatedly jump right to the first available 1 that is distance of at least δ, but no more than $\delta + k - i$ from the last chosen 1. Note that there can never be an obstruction to jumping to the next available 1 unless we reach the end of S because, if a jump to the next 1 required a length more than $\delta + k - i$, then the last $k - i + 1$ skipped elements would be a major block of 0's in S which is impossible by Theorem 2.2 part (vi). This means that when this greedy jumping reaches the end of S, it must land on a 1 that is distance of at most $\delta - 1$ of the rightmost 1 of S. Also notice that each jump can pass over at most $\delta - 1$ ones. Thus each 1 in our constructed progression "consumes" at most δ ones, itself plus the at most $\delta - 1$ ones that are skipped by the next jump. But, since there are more than $p\delta$ ones and we have not wasted any 1's because we started at the beginning of S, our progression must have at least $p + 1$ ones. The only problem is that this progression may not end at the maximum element of B. We now address this problem.

Let x denoted the leftmost 1 of S and y the rightmost 1 of S. Suppose that our currently constructed progression from S is $x = x_1 < \cdots < x_q$, for some $q \geq p + 1$. In a manner similar to the construction of this sequence, construct a new progression starting at y and greedily jumping leftward toward x. Suppose that this second progression is $y_h < \cdots < y_2 < y_1$; that is, this progression begins at the rightmost element $y_1 = y$ of S, jumps leftwards greedily until it reaches y_h and no further jumps are possible. A consequence of the next claim is that the x-progression and the y-progression have the same length (i.e. $h = q$).

Claim. $y_j - x_{q+1-j} < \delta$, *for* $j = 1, \ldots, q$.

We prove this by induction on j. The basis case is true because $y_1 = y$ and as noted in the paragraph above, the progression of x's must end within δ of y. Now suppose that $y_j - x_{q+1-j} < \delta$, for some j. Because the progression of y's must end within δ of x, if $q + 1 - j > 1$, the element y_{j+1} must exist. The distance $y_j - y_{j+1}$ must be at least δ so $y_{j+1} < x_{q+1-j}$. In particular, $x_{q-j} \leq y_{j+1} < x_{q+1-j}$. Because the x-sequence did not jump from x_{q-j} to y_{j+1}, it follows that $y_{j+1} - x_{q-j} \leq \delta - 1$, as desired.

Observe that if $y_j = x_{q+1-j}$, for some $j \in \{1, \ldots, q\}$ then the progression

$$x_1, \ldots, x_{q+1-j}, y_{j-1}, \ldots, y_1$$

satisfies the conclusion of the theorem.

So, we have proven that we may assume that these two sequences inter-
lace:

$$x_1 < y_q < x_2 < y_{q-1} < \cdots < x_{q-1} < y_2 < x_q < y_1,$$

and $y_j - x_{q+1-j} < \delta$, for $j = 1, \ldots, q$.

Now let T denote the substring of S corresponding to the major block of
1's in B. Because T is a major block, T is a substring of 1's with length at
least $k-i+1$. In particular, since $k \geq 2i$, the length of T is at least $i+1$, which
is larger than δ. Now observe that among the differences between consecutive
elements of the progression $x_1, \ldots, x_q$, there must be a difference of exactly
δ because the first greedy jump that this progression makes into T must
either have length exactly δ or it hits the first element of T. In the latter
event, the following jump must have length δ because T contains at least
$\delta + 1$ ones.

So, there is some $j \in \{1, \ldots, q-1\}$ such that $x_{q+1-j} - x_{q-j} = \delta$. Because
$y_j - x_{q+1-j} < \delta$, it follows that $y_j - x_{q-j} \leq 2\delta \leq k - i + \delta$. Therefore, the
progression

$$x_1, \ldots, x_{q-j}, y_j, \ldots, y_1$$

satisfies the conclusion of the theorem.   □

Theorem 2.4 (Greedy Minor Super Block Jumping)**.** *Assume $k \geq 2i$. If B
is a minor super block of C, x is an extreme of B, and $1 \leq \delta \leq i$, then B
contains a monochromatic quasi-progression P such that*

   i) P has length at least $\lceil \frac{|B|}{\delta} \rceil$,
  ii) the low difference of P is at least δ,
 iii) the diameter of P is at most $k - i$, and
 iv) $x \in P$.

Proof. We argue essentially the same way as in the proof of Theorem 2.3.
Without loss of generality, B has color 1 and x is the leftmost element of B
(that is, $x = \min B$). Let S denote the binary substring of C consisting of
all characters between the minimum and maximum elements of B. By defi-
nition, S begins and ends with a 1. Let us suppose that $p\delta < |B| \leq (p+1)\delta$,
for some positive integer p. We must show that there is a monochromatic
quasi-progression of length at least $p+1$ (that is, a progression that satisfies
(i) above) with the additional properties (ii)–(iv). Create such a monochro-
matic quasi-progression this way: start with x and repeatedly jump right to
the first available 1 that is distance of at least δ, but no more than $\delta + k - i$
from the last chosen 1. Note that there can never be an obstruction to jump-
ing to the next available 1 unless we reach the end of S because, if a jump

to the next 1 required a length more than $\delta + k - i$, then the last $k - i + 1$ skipped elements would be a major block of 0's in S which is impossible by Theorem 2.2 part (vi). This means that when this greedy jumping reaches the end of S, it must land on a 1 that is distance of at most $\delta - 1$ of the right most 1 of S. Also notice that each jump can pass over at most $\delta - 1$ ones. Thus each 1 in our constructed progression "consumes" at most δ ones, itself plus the at most $\delta - 1$ ones that are skipped by the next jump. But, since there are more than $p\delta$ ones and we have not wasted any 1's because we started at the beginning of S, our progression must have at least $p + 1$ ones.   □

The next theorem brings together the previous two theorems and is a significant tool in later proofs.

Theorem 2.5. *Assume $k \geq 2i$. Suppose C is a binary string that represents a $\{red, blue\}$-coloring of positive integers with no good progression. If C has no red major blocks of cardinality at least $k - i + \delta$, for some $1 \leq \delta \leq i$, and $B_1, \ldots, B_h$ are the blue super blocks of C, then C contains a monochromatic quasi-progression P with diameter at most $k - i$, low-difference at least δ, and length at least $\sum_{j=1}^{h} \lceil \frac{|B_j|}{\delta} \rceil$.*

Proof. Apply Theorem 2.3 to super blocks $B_2, \ldots, B_{h-1}$ to obtain monochromatic quasi-progression fragments $P_2, \ldots, P_{h-1}$ with length at least $\lceil \frac{|B_j|}{\delta} \rceil$, low-difference δ, diameter $k - i$ and that contain both extremes of each of their major super blocks. Similarly, apply Theorem 2.4 to super blocks B_1 and B_h to obtain monochromatic quasi-progression fragments P_1 and P_h with length at least $\lceil \frac{|B_j|}{\delta} \rceil$, low-difference δ, diameter $k - i$ and that contain the maximum and minimum, respectively, of B_1 and B_h. Because C has no red major blocks of cardinality at least $k - i + \delta$, jumps of length at most $k - i + \delta$ (and at least δ) can be made to join the extremes of these fragments into the desired monochromatic quasi-progression P.   □

3. Some upper bounds

This section establishes upper bounds on $Q(k - i, k)$. In many cases the bounds are sharp. The proofs rely heavily on the super block results from the previous section.

Theorem 3.1. *If $k \geq 2i$ and $k \equiv 0 \ (mod \ i)$, then $Q(k - i, k) = 2ik - 4i + 3$.*

Proof. The lower bound $Q(k - i, k) \geq 2ik - 4i + 3$ follows from Corollary 1 of Landman's paper [3]; so it suffices to prove the upper bound. Suppose

that $\ell = Q(k-i,k)-1$ and C is a binary string of length ℓ that represents a 2-coloring of the integers $1, \ldots, \ell$ with no good progression. We must prove that $\ell \leq 2ik - 4i + 2$. Assume that $k = mi$, for some $m \geq 2$.

We claim that there can not be i major super blocks of the same color. To see this, suppose to the contrary, that $B_1, \ldots, B_p$ $(p \geq i)$ are all of the blue major super blocks of C. Apply Theorem 2.5 with $\delta = i$ to the substring of C between the $\min B_1$ and the $\max B_i$. This theorem guarantees a monochromatic quasi-progression of length at least

$$\sum_{j=1}^{p} \left\lceil \frac{|B_j|}{i} \right\rceil \geq \sum_{j=1}^{p} \left\lceil \frac{k-i+1}{i} \right\rceil \geq \sum_{j=1}^{p} m \geq mp \geq k,$$

a contradiction. So C has at most $i - 1$ major super blocks of each color.

Suppose that C has $i - 1$ major super blocks of the same color. We now claim that the total number of elements in minor blocks of that color is at most $k - i$. To prove this, suppose that C has $i - 1$ blue major super blocks $B_1, \ldots, B_{i-1}$ and two minor super blocks B_0 and B_i (it is clear that there are at most two blue minor super blocks, since there is at most one at each end). Again, apply Theorem 2.5 with $\delta = i$ to the substring of C between the $\min B_0$ and the $\max B_i$. This theorem guarantees a monochromatic quasi-progression of length at least

$$\sum_{j=0}^{i} \left\lceil \frac{|B_j|}{i} \right\rceil = (i-1)m + \left\lceil \frac{|B_0|}{i} \right\rceil + \left\lceil \frac{|B_i|}{i} \right\rceil.$$

Since this sum is at most $k - 1 = mi - 1$, it follows that $|B_0| + |B_i| \leq k - i$. Consequently, the blue super blocks have cardinalities that sum to at most $(i-1)(k-1) + (k-i)$. The same argument applies to the red super blocks. Therefore, the length of C is at most

$$2(i-1)(k-1) + 2(k-i) = 2ik - 4i + 2,$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 3.2. *If $k \geq 2i+1$ and $k \equiv 1 \pmod{i}$, then $Q(k-i,k) = 2ik-2i+1$.*

Proof. The lower bound $Q(k-i,k) \geq 2ik - 2i + 1$ follows from Corollary 1 of Landman's paper [3]; so it suffices to prove the upper bound. Let C be binary string realizing an extremal 2-coloring with no good progression. We must prove that the length of C is at most $2i(k-1)$. Let $B_1, \ldots, B_p$ be the blue super blocks of C and $R_1, \ldots, R_q$ the red super blocks. Because

the super blocks form a partition, the length of C is $\sum_{j=1}^{p} |B_j| + \sum_{j=1}^{q} |R_j|$. However, applying Theorem 2.5 with $\delta = i$ to the substring of C between the min B_1 and the max B_p, we find that C contains a monochromatic quasi-progression of length $\sum_{j=0}^{p} \lceil \frac{|B_j|}{i} \rceil$. Since this can not exceed $k-1$, it follows that $\sum_{j=1}^{p} |B_j| \le i(k-1)$. A symmetric argument shows $\sum_{j=1}^{q} |R_j| \le i(k-1)$. Therefore the length of C is at most $2i(k-1)$, as desired. $\qquad\square$

The next theorem gives a general upper bound that we shall show in the next section is sharp when $k = mi + r$ for integers m, r such that $3 \le r < \frac{i}{2}$ and $r - 1 \le m$.

Theorem 3.3. *If $k \ge 2i$ and $k = mi + r$ for integers m, r such that $1 < r < i$, then*

$$Q(k - i, k) \le 2ik - 4i + 2r - 1.$$

Proof. Suppose that $\ell = Q(k - i, k) - 1$ and C is a binary string of length ℓ that represents a 2-coloring of the integers $1, \ldots, \ell$ with no good progression. We must prove that $\ell \le 2ik - 4i + 2r - 2$. We argue by contradiction: assume that $\ell > 2ik - 4i + 2r - 2$.

For $j = 0, 1$, let α_j denoted the number of super blocks of C that have size greater than mi. Apply Theorem 2.5 with $\delta = i$ to the shortest substring containing all super blocks of color j:

$$\alpha_j(m + 1) + \sum_{\substack{B \text{ has color } j \\ |B| \le mi}} \left\lceil \frac{|B|}{i} \right\rceil \le k - 1.$$

It follows that, for $j = 0, 1$,

$$\sum_{\substack{B \text{ has color } j \\ |B| \le mi}} |B| \le i(k - 1 - \alpha_j(m + 1)).$$

Therefore, the length of C can be bounded as follows:

$$\begin{aligned}
\ell &= \sum_{j=0}^{1} \left(\sum_{\substack{B \text{ has color } j \\ |B| > mi}} |B| \right) + \sum_{j=0}^{1} \left(\sum_{\substack{B \text{ has color } j \\ |B| \le mi}} |B| \right) \\
&\le (\alpha_0 + \alpha_1)(k - 1) + i(k - 1 - \alpha_0(m + 1)) + i(k - 1 - \alpha_1(m + 1)) \\
&= 2ik - 2i - \alpha(i + 1 - r),
\end{aligned}$$

where $\alpha = \alpha_0 + \alpha_1$. Because we are assuming that $\ell > 2ik - 4i + 2r - 2$, we may conclude that $\alpha < 2$. Without loss of generality, $\alpha_0 = 0$ and $\alpha_1 \le 1$.

Because $\alpha_0 = 0$, the coloring contains no super blocks of 0's with cardinality larger than mi. Now apply Theorem 2.5 with $\delta = i - r + 1$ to the substring of C containing all super blocks of color 1:

$$\sum_{B \text{ has color } 1} \left\lceil \frac{|B|}{i - r + 1} \right\rceil \le k - 1.$$

Consequently the number of 1's in C is at most $(k-1)(i-r+1)$. Applying Theorem 2.5 with $\delta = i$ to the substring of C containing all super blocks of color 0 we find

$$\sum_{B \text{ has color } 0} \left\lceil \frac{|B|}{i} \right\rceil \le k - 1.$$

Therefore the number of 0's in C is at most $(k-1)i$. It follows that the length of C can be bounded as follows:

$$\begin{aligned}
\ell = \ & (\#1\text{'s in } C) + (\#0\text{'s in } C) \\
\le \ & (k-1)(i-r+1) + (k-1)i \\
= \ & 2ik - 2i + (k-1)(1-r),
\end{aligned}$$

which is at most $2ik - 4i + 2r - 2$ since $r > 1$ and $k > 2i$. This contradicts that $\ell > 2ik - 4i + 2r - 2$. $\qquad\square$

4. A general lower bound

In this section we exhibit an extremal 2-coloring of positive integers that avoids monochromatic k-term quasi-progressions of diameter $k - i$ for many values of k and i. To describe this coloring we first introduce some notation.

Recall that a block of a binary string is a maximal length monochromatic substring. A *segment* of a binary string is a maximal length substring in which all blocks have the same length. Its segments naturally partition a binary string. Therefore a binary string C can be abbreviated by an expression involving positive integers of the form $a_1^{b_1} \cdots a_s^{b_s}$, which indicates that the jth segment consists of b_j blocks of length a_j (we assume that the first block is a block of 1's). We adopt this notation in this section. Note that C has length $\sum_{j=1}^s a_j b_j$.

Theorem 4.1. *Suppose that $k = mi + r$ for integers m, r such that $3 \le r < \frac{i}{2}$. If $r - 1 \le m$, then the following 2-coloring of the integers from 1*

to $2ik - 4i + 2r - 2$ contains no monochromatic k-term quasi-progression of diameter $k - i$:

$$(i(r-2))^1 \, (mi)^{i-1} \, (k-1)^2 \, (mi)^{i-1} \, (i(r-2))^1.$$

Proof. Let C denote the binary string corresponding to this coloring. We assume that C begins with a block of 1's. Observe that C is a palindrome and, because $r - 1 \leq m$, the initial block of 1's is shorter than the others. For a positive integer δ, let Q_δ be a longest monochromatic quasi-progression in C with low-difference δ, and let ℓ be the length of Q_δ. We must show $\ell \leq k - 1$. Because C is a palindrome, we may assume that Q_δ consists of elements of color 1. If $\delta < i - r + 1$, then Q_δ can not jump across blocks of length mi or longer, so Q_δ has length at most $k - 1$ in this case. So it suffices to consider values of $\delta \geq i - r + 1$. However, because C has only interior blocks of length mi or $k - 1$, we can restrict our attention of $\delta = i - r + 1$ (which permits leaps over blocks size mi but not $k - 1$) or $\delta = i$ (which permits leaps over all block sizes). Upper bounds on ℓ for other values of δ follow immediately from the upper bounds on these two values of δ.

Assume first that $\delta = i$. The quasi-progression Q_δ can use at most $\lceil \frac{|B|}{i} \rceil$ elements from a block B. Consequently, the length ℓ can be bounded this way:

$$
\begin{aligned}
\ell &\leq \left\lceil \frac{i(r-2)}{i} \right\rceil + (i-1) \left\lceil \frac{mi}{i} \right\rceil + \left\lceil \frac{k-1}{i} \right\rceil \\
&= (r-2) + (i-1)m + (m+1) \\
&= k - 1,
\end{aligned}
$$

as desired.

Assume now that $\delta = i - r + 1$. Clearly the quasi-progression Q_δ can use at most $\lceil \frac{|B|}{i-r+1} \rceil$ elements from a block B. There are two cases according to the parity of i. In each case the length of Q_δ is bounded from above:

Case 1 (i is even). In this case the extremal length occurs when Q_δ uses 1's from the small beginning block, $(i-2)/2$ intermediate blocks of size mi, and the large block of $k - 1$ ones, so

(1) $$\ell \leq \left\lceil \frac{i(r-2)}{i-r+1} \right\rceil + \frac{i-2}{2} \left\lceil \frac{mi}{i-r+1} \right\rceil + \left\lceil \frac{k-1}{i-r+1} \right\rceil.$$

Case 2 (i is odd). In this case the extremal length occurs when Q_δ uses 1's from $(i-1)/2$ intermediate blocks of size mi, and the large block of $k - 1$

ones, so

$$(2) \qquad \ell \le \frac{i-1}{2} \left\lceil \frac{mi}{i-r+1} \right\rceil + \left\lceil \frac{k-1}{i-r+1} \right\rceil.$$

Consider now the following upper bounds which, when substituted into inequalities (1) and (2), will show that $\ell \le k-1$.

Bound 1. $\left\lceil \frac{mi}{i-r+1} \right\rceil \le \min\{2m-1, 2m+\frac{2(r-1-m)}{i-1}\}$.

It suffices to prove that $\frac{mi}{i-r+1} \le 2m-1+\frac{2(r-1-m)}{i-1}$. Assume the contrary: $\frac{mi}{i-r+1} > 2m-1+\frac{2(r-1-m)}{i-1}$. Applying the assumption that $2r+1 \le i$ while solving for m, we find that

$$\frac{(i-r+1)(i-2r+1)}{i(i-2r-1)+4(r-1)} > m.$$

Because we have assumed that $m \ge r-1$, it follows that

$$\frac{(i-r+1)(i-2r+1)}{i(i-2r-1)+4(r-1)} > r-1.$$

Consequently,

$$0 > (r-1)\left[i(i-2r-1)+4(r-1)\right] - (i-r+1)(i-2r+1)$$
$$= (r-2)(i-1)\left(i - \frac{2r^2-5r+3}{r-2}\right).$$

Because $r > 2$ and $i > 1$, we conclude that $i < \frac{2r^2-5r+3}{r-2}$. However, since $i > 2r$ we find that

$$2r < \frac{2r^2-5r+3}{r-2}$$

which implies that $r < 3$, a contradiction.

Bound 2. $\left\lceil \frac{k-1}{i-r+1} \right\rceil \le 2m$.

Assume, to the contrary, that $\frac{k-1}{i-r+1} > 2m$. Substituting $k = mi+r$ and solving for m produces the inequality $\frac{r-1}{i-2r+2} > m$. Because we have assumed that $m \ge r-1$, it follows that $\frac{r-1}{i-2r+2} > r-1$ which implies that $i < 2r-1$, a contradiction.

Bound 3. $\left\lceil \frac{i(r-2)}{i-r+1} \right\rceil \le \frac{i}{2}+r-2$.

It suffices to prove that $\frac{i(r-2)}{i-r+1} \leq \frac{i}{2} + r - 3$. Assume, to the contrary, that $\frac{i(r-2)}{i-r+1} > \frac{i}{2} + r - 3$. Equivalently,

$$0 > 2\left(\frac{i}{2} + r - 3\right)(i - r + 1) - 2i(r - 2) = (i + r - 3)(i - 2r + 2),$$

a contradiction.

To complete the proof we show, using these bounds applied to inequalities (1) and (2), that $\ell \leq k - 1$. Consider inequality (1):

$$\ell \leq \left\lceil \frac{i(r-2)}{i-r+1} \right\rceil + \frac{i-2}{2} \left\lceil \frac{mi}{i-r+1} \right\rceil + \left\lceil \frac{k-1}{i-r+1} \right\rceil$$
$$\leq \left(\frac{i}{2} + r - 2\right) + \left(\left(\frac{i-2}{2}\right)(2m - 1)\right) + (2m)$$
$$= k - 1.$$

Now consider inequality (2):

$$\ell \leq \frac{i-1}{2} \left\lceil \frac{mi}{i-r+1} \right\rceil + \left\lceil \frac{k-1}{i-r+1} \right\rceil$$
$$\leq \left(\left(\frac{i-1}{2}\right)\left(2m + \frac{2(r-1-m)}{i-1}\right)\right) + (2m)$$
$$= k - 1.$$

$\square$

Theorem 4.2. *If $k = mi + r$ for integers m, r such that $3 \leq r < \frac{i}{2}$ and $r - 1 \leq m$, then*

$$Q(k - i, k) = 2ik - 4i + 2r - 1.$$

Proof. Theorem 3.3 proves the upper bound and Theorem 4.1 proves the lower bound. $\square$

Theorem 4.3. *If $k \geq 2i$ and $k \equiv 2 \pmod{i}$, then $Q(k - i, k) = 2ik - 4i + 3$.*

Proof. Theorem 3.3 gives the upper bound $Q(k-i, k) \leq 2ik - 4i + 3$. Arguing in a manner similar to the proof of Theorem 4.1, one can show that

$$(k - 2)^{i-1} (k - 1)^2 (k - 2)^{i-1}.$$

is a 2-coloring of $1, \ldots, 2ik - 4i + 2$ that avoids monochromatic k-term quasi-progressions of diameter $k - i$. $\square$

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