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A variation of the Stern-Brocot tree

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We study of a variation of the Stern-Brocot tree, in which not one but two fractions are inserted between each existing pair. Relating this tree to the original one gives rise to a permutation of the natural numbers.

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1. The Stern-Brocot tree, and a variation

The Stern-Brocot tree (or rather half of it) can be defined as follows. Start with two fractions $0/1$ and $1/1$, forming an ordered set S_0. (Throughout this paper, "fraction" means "fraction in lowest terms".) At stage k, $(k = 1, 2, \ldots)$, form a new set S_k by inserting between each pair of adjacent fractions in S_{k-1}, say p/q and r/s, the fraction $(p+r)/(q+s)$. Name the (ordered) set of fractions that are introduced at this stage R_k. Thus $R_1 = (1/2)$, $R_2 = (1/3, 2/3), R_3 = (1/4, 2/5, 3/5, 3/4), R_4 = (1/5, 2/7, 3/8, 3/7, 4/7, 5/8, 5/7, 4/5)$ etc. R_k has 2^{k-1} elements. It is well known (see e.g. [1]) that every proper fraction appears (exactly once) in some R_k, and that adjacent fractions $p/q, r/s$ satisfy

$$|qr - ps| = 1. \tag{1}$$

We define a new tree (first noticed in [2, Section 9]) starting with $S'_0 = S_0$. At the k-th stage insert two fractions between each existing adjacent pair in S'_{k-1}, namely between p/q and r/s (where p is even and r is odd), insert $(p+r)/(q+s)$ and $(p+2r)/(q+2s)$. Notice that we may have either $p/q < (p+r)/(q+s) < (p+2r)/(q+2s) < r/s$ or the same with all the inequalities reversed. It is easy to see that every adjacent pair of fractions in S'_k satisfy (1) and that the numerators of successive fractions in S'_k are alternately even and odd, so that the insertion rule is well-defined. Successive generations of insertions are denoted $R'_1, R'_2, \ldots$. Thus R'_k has 2.3^{k-1} elements.

Explicitly,

$$R'_1 = (1/2, 2/3), \qquad R'_2 = (1/3, 2/5, 4/7, 3/5, 3/4, 4/5),$$
$$R'_3 = (1/4, 2/7, 4/11, 3/8, 2/7, 4/9, 6/11, 5/9, 7/12, 10/17, 8/13, 5/8, 5/7, 8/11, 10/13, 7/9, 5/6, 6/7).$$

Lemma 1. *For every proper fraction x, there is a k such that x appears in R'_k.*

Proof. Define the "*ndsum*" of a fraction p/q to be $p+q$. An easy induction shows that for $k \geq 1$ the minimum *ndsum* in the row R'_k is $k+2$ (attained by the first element, which is $1/(k+1)$). The minimum *ndsum* in S'_0 is 1, attained by $0/1$. Suppose the fraction a/b, where $a+b \geq 2$, does not appear in any R'_k. Consider the row R'_{a+b}. There must be a fraction p/q in this row and a fraction r/s in S'_{a+b} such that $|qr - ps| = 1$ and $p/q < a/b < r/s$, or the same with both inequalities reversed. Suppose the inequalities are as shown. Then $aq - bp > 0$, so $aq - bp \geq 1$, and similarly $br - as \geq 1$. Thus

$$(p+q)(br-as) + (r+s)(aq-bp) \geq p+q+r+s. \tag{2}$$

But the l.h.s. of (2) equals $(a+b)(qr-ps) = a+b$, and the r.h.s. is at least $(a+b+2)+1$, so $a+b \geq a+b+3$ which is a contradiction. When the inequalities are reversed, the argument is similar. □

2. Relating the two trees

We study the relation between the sets $\{R'_k\}$ and $\{R_k\}$. We find that (as far as we have computed, namely R'_6 and R_{12}) there is a sequence p, starting

Sequence p

1, 2, 5, 3, 4, 8, 17, 9, 10, 20, 11, 6, 7, 14, 29, 15, 16, 32, 65, 33, 34, 68, 35, 18, 19,

such that for each k, and for $i = 1, 2, \ldots, 2.3^{k-1}$, the fraction $R'_k(i)$ appears as $R_{k'(i)}(p(i))$ for some $k'(i)$. We write $k'(i) = k + r_k(i)$, and set $n_k = 2^{k'-1}$, which is the sequence of lengths of the rows k' of R in which these fractions appear. Thus for $k=3$, the rows of the following matrix M_3 are

the numerators of fractions in R'_3
the corresponding denominators
the m such that each such fraction appears in R_{m+3}
the length of the row R_{m+3} (this is n_{m+3})
the position of this fraction in R_{m+3} (this is a prefix of p).

1	2	4	3	3	4	6	5	7	10	8	5	5	8	10	7	5	6
4	7	11	8	7	9	11	9	12	17	11	8	7	11	13	9	6	7
0	1	2	1	1	2	3	2	2	3	2	1	1	2	3	2	2	3
4	8	16	8	8	16	32	16	16	32	16	8	8	16	32	16	16	32
1	2	5	3	4	8	17	9	10	20	11	6	7	14	29	15	16	32

Rows 3 and 5 of the first six columns of this matrix give the corresponding results for R'_2, while the fourth row is twice the fourth row for R'_2.

We have studied similar matrices through $k = 6$, finding that for each k, the fifth row of M_k contains the first 2.3^{k-1} elements of the sequence we have called p. The third row contains numbers in the range $(0, k)$, with successive entries equal or consecutive.

The following lemma shows how the sequence for R'_{k+1} can be obtained from that for R'_k.

Lemma 2. *Given the finite sequences p_k and n_k that describe the relation of R'_k to the rows of S, the sequences for row R'_{k+1} are as follows.*

$$p_{k+1} = (p_k, \mathrm{rev}(3n_k + 1 - p_k), 3n_k + p_k),$$
$$n_{k+1} = (2n_k, \mathrm{rev}(4n_k), 4n_k)$$

*where "*rev*" means "the reverse of".*

Proof. The n and p sequences for R'_{k+1} are unchanged if we replace the starting fractions S_0 and S'_0 by $(0/1, 1/2)$. So the (finite) p sequence for R'_k is the same as the first third of the p sequence for R'_{k+1}, while the rows for R_{m+1} are twice as long as those for R_m. Similarly, the final third of the p-sequence for R'_{k+1}, which relate to the interval $(2/3, 1/1)$, are the same as the sequence for R'_k, translated by $3/4$ of the length, which is four times the length for R_m. Finally, for the middle third, which relates to the interval $(1/2, 2/3)$, we have to read the R_k values backwards (because the numerator of $1/2$ is odd and the numerator of $2/3$ is even) and count backwards from $3/4$ of the lengths. □

This lemma makes it easy to compute p as far as desired. However it has not led us to a proof that the sequence p is a permutation of the natural numbers. We will show that another sequence, pp, which we have checked agrees with p through $354,294$ terms, is indeed a permutation.

3. The sequences *b* and *pp*

To approach the sequence pp, we must first define another sequence $b(\mathcal{N})$.

Algorithm B. $b(1) = 1$. For $k \geq 1$,

$$(b(3k-1),\ b(3k),\ b(3k+1)) = (4i-1, 2i, 4i+1)$$

where $i = b(k)$. Thus the sequence b begins

$$1, 3, 2, 5, 11, 6, 13, 7, 4, 9, 19, 10, 21, 43, 22, 45, 23, 12, 25, 51, \\ 26, 53, 27, 14, 29, \ldots.$$

Theorem 1. *The sequence $b(\mathcal{N})$ is a permutation of $\mathcal{N}$.*

Proof. Suppose m is the smallest integer that does not appear as an element of $b(\mathcal{N})$. It is impossible that m is even, since $m/2$ does appear, and for some k we have $b(k) = m/2$. Then m must appear at $b(3k)$. If m is odd, set $i = \text{round}(m/4)$. Then i appears at some point k, $b(k) = i < m$, so that m appears as an element of the triad centered at $3k$. Thus all integers must appear. A similar argument shows that no integer can appear twice. Suppose m is the smallest integer that appears twice. If m is even, we have $b(3k_1) = b(3k_2) = m$, with $k_1 \neq k_2$. Then $b(k_1) = b(k_2) = m/2$ so that the integer $m/2$ appears twice before m does. Thus m cannot be even. If m is odd, suppose first that the smallest violation is $b(3k_1 - 1) = b(3k_2 - 1) = 4i - 1$, with $k_1 \neq k_2$. Then $b(3k_1) = b(3k_2) = 2i$ so that $b(k_1) = b(k_2) = i$, and i appears twice before $4i - 1$ does. Similarly if $m = 4i + 1$. □

We define another sequence pp by:

Algorithm PP. $pp(1) = 1, pp(2) = 2$. For $k = 1, 2, \ldots$

$$(pp(4k-1), pp(4k), pp(4k+1), pp(4k+2)) = (6i-1, 3i, 3i+1, 6i+2)$$

where $i = b(k)$.

Theorem 2. *The sequence $pp(\mathcal{N})$ is a permutation of $\mathcal{N}$.*

Proof. Since the sequence b is a permutation of $\mathcal{N}$, it is clear that numbers of the form $3i$ and $3i + 1$ appear just once in pp, in positions $4k$ and $4k + 1$, and numbers of the form $3i - 1$ appear in positions $4k - 1$ and $4k + 2$, where $i = b(k)$. □

We have verified that the sequences p and pp agree through their first 354,294 terms. Of course, this result does not prove anything about the sequence p, merely that it agrees with the facts as far as we have computed them. We have not been able to prove that Algorithms P and PP generate the same sequence.

We think it remarkable that (it appears) Algorithm B and Algorithm P are so closely related, since b generates blocks of length 4 in PP, while Algorithm P generates the sequence p in blocks of length $2, 4, 12, 36, \ldots$ with the first half of each block involving reading previous blocks backwards.

4. Generalizations

Once we have the sequence b in hand, we can generate many permutations of $\mathcal{N}$ by constructions similar to that in Algorithm PP. For example,

Algorithm AA. $aa(1,2,3) = (1,2,3)$. For $k \geq 1$,

$$\begin{aligned}&(aa(6k-2), aa(6k-1), aa(6k), aa(6k+1), aa(6k+2), aa(6k+3))\\ &\quad = (8i-2, 8i-1, 4i, 4i+1, 8i+2, 8i+3)\end{aligned}$$

where $i = b(k)$.

This particular sequence happens to be identical to one that makes no reference to the sequence b, but is generated by the following

Algorithm A. Set $a(1) = 1$. For $n \geq 1$:

$$a(2n) = (1 + a(2n-1))/2 \quad \text{if this value has not yet appeared} \tag{3}$$

$$= 2a(2n-1) \quad \text{else} \tag{4}$$

$$a(2n+1) = 1 + a(2n). \tag{5}$$

The proof of this equality is left for another occasion.

There are other ways of defining a modified Stern-Brocot tree, but we have not found any as elegant as the one we have presented.

References

[1] Graham, R.L., Knuth, D.E. and Patashnik, O. (1994). *Concrete Mathematics*. Addison-Wesley, Boston, MA. MR1397498

[2] Guettler, G. and Mallows, C. (2010). A generalization of Apollonian packing of circles. *J. Combinatorics* **1** 1–27. MR2675919

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