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Extremal results regarding K_6-minors in graphs of girth at least 5

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We prove that every 6-connected graph of girth ≥ 6 has a K_6-minor and thus settle Jorgensen's conjecture for graphs of girth ≥ 6. Relaxing the assumption on the girth, we prove that every 6-connected n-vertex graph of size $\geq 3\frac{1}{5}n - 8$ and of girth ≥ 5 contains a K_6-minor.

Whenever possible, notation and terminology are that of [2]. Throughout, a graph is always simple, undirected, and finite. G always denotes a graph. We write $\delta(G)$ and $d_G(v)$ to denote the minimum degree of G and the degree of a vertex $v \in V(G)$, respectively. A vertex of degree k is called *k-valent*. We write $\kappa(G)$ to denote the vertex connectivity of G. The *girth* of G is the length of a shortest circuit in G. Finally, the cardinality $|E(G)|$ is called the *size* of G and is denoted $\|G\|$; $|V(G)|$ is called the *order* of G and is denoted $|G|$.

1. Introduction

A conjecture of Jorgensen postulates that *the 6-connected graphs not containing K_6 as a minor are the apex graphs*, where a graph is apex if it contains a vertex removal of which results in a planar graph. The 6-connected apex graphs contain triangles. Consequently, if Jorgensen's conjecture is true, then a 6-connected graph of girth ≥ 4 contains a K_6-minor. Noting that the extremal function for K_6-minors is at most $4n - 10$ [4] (where n is the order of the graph), our first result in this spirit is that

Theorem 1.1. *A graph of size $\geq 3n - 7$ and girth at least 6 contains a K_6-minor.*

So that,

Theorem 1.2. *Every 6-connected graph of girth ≥ 6 contains a K_6-minor.*

This settles Jorgensen's conjecture for graphs of girth ≥ 6. Relaxing the assumption on the girth in Theorem 1.1, we prove the following:

Theorem 1.3. *A 6-connected graph of size $\geq 3\frac{1}{5}n - 8$ and girth at least 5 contains a K_6-minor.*

Remark. In our proofs of Theorems 1.1 and 1.3, the proofs of claims (1.1.A–B) and (1.3.A–D) follow the approach of [3].

2. Preliminaries

Let H be a subgraph of G, denoted $H \subseteq G$. The boundary of H, denoted by $\mathrm{bnd}_G H$ (or simply $\mathrm{bnd} H$), is the set of vertices of H incident with $E(G) \setminus E(H)$. By $int_G H$ (or simply $intH$) we denote the subgraph induced by $V(H) \setminus bndH$. If $v \in V(G)$, then $N_H(v)$ denotes $N_G(v) \cap V(H)$.

Let $k \geq 1$ be an integer. By *k-hammock* of G we mean a connected subgraph $H \subseteq G$ satisfying $|bndH| = k$. A hammock H coinciding with its boundary is called *trivial*, *degenerate* if $|H| = |bndH| + 1$, and *fat* if $|H| \geq |bndH| + 2$. A proper subgraph of H that is a k-hammock is called a *proper* k-hammock of H. A fat k-hammock is called *minimal* if all its proper k-hammocks, if any, are trivial or degenerate. Clearly,

(2.1) every fat k-hammock contains a minimal fat k-hammock.

Let H be a fat 2-hammock with $bndH = \{u, v\}$. By *capping* H we mean $H + uv$ if $uv \notin E(H)$ and H if $uv \in E(H)$. In the former case, uv is called a *virtual* edge of the capping of H. The set $bndH$ is called the *window* of the capping.

Let now $\kappa(G) = 2$ and $\delta(G) \geq 3$. By the standard decomposition of 2-connected graphs into their 3-connected components [1, Section 9.4], such a graph has at least two minimal fat 2-hammocks whose interiors are disjoint and that capping of each is 3-connected. Such a capping is called an *extreme* 3-connected component.

A k-(vertex)-disconnector, $k \geq 1$, is called *trivial* if its removal isolates a vertex. Otherwise, it is called *nontrivial*. A graph is called *essentially* k-connected if all its $(k-1)$-disconnectors are trivial. If each $(k-1)$-disconnector D isolates a vertex and $G - D$ consists of precisely 2 components (one of which is a singleton) then G is called *internally* k-connected.

Suppose $\kappa(G) \geq 1$ and that $D \subseteq V(G)$ is a $\kappa(G)$-disconnector of G. Then, $G[C \cup D]$ is a fat $\kappa(G)$-hammock for every non-singleton component C of $G - D$. In particular, we have that

Lemma 2.1. *If $\kappa(G) \geq 1$, $\delta(G) \geq 3$, and $D \subseteq V(G)$ is a nontrivial $\kappa(G)$-disconnector of G, then G has at least two fat minimal $\kappa(G)$-hammocks whose interiors are disjoint.*

Lemma 2.2. *If $\kappa(G) \geq 1$, $\delta(G) \geq 3$, $e \in E(G)$, and G has a nontrivial $\kappa(G)$-disconnector, then G has a minimal fat $\kappa(G)$-hammock H such that if $e \in E(H)$, then e is spanned by $bndH$.*

Let H be a k-hammock. By *augmentation* of H we mean the graph obtained from H by adding a new vertex and linking it with edges to each vertex in $bndH$.

Lemma 2.3. *Suppose $\kappa(G) = 3$ and that H is a minimal fat 3-hammock of G. Then, an augmentation of H is 3-connected.*

Proof. Let H' denote the augmentation and let $\{x\} = V(H') \setminus V(H)$. Assume, to the contrary, that H' has a minimum disconnector D, $|D| \leq 2$. If $H' - D$ has a component containing x, then H has a nontrivial $|D|$-hammock; contradicting the assumption that $\kappa(G) = 3$. Hence, $x \in D$. As x is 3-valent, $H' - D$ has a component C containing a single member of $bndH'$ $(= N_{H'}(x))$, say u. Since $\delta(G) \geq 3$, $|N_C(u) \setminus D| \geq 1$ so that $(D \setminus \{x\}) \cup \{u\}$ is a disconnector of H of size ≤ 2 not containing x and hence also a disconnector of G; contradiction. □

Lemma 2.4. *Suppose $\kappa(G) = 3$ and that H is a triangle free minimal fat 3-hammock of G such that $e \in E(G[bndH])$. Then, an augmentation of $H - e$ is 3-connected.*

Proof. Let H' be the augmentation of $H - e$, let $\{x\} = V(H') \setminus V(H)$, and let $e = tw$ such that $t, w \in N_{H'}(x)$. By Lemma 2.3, $\kappa(H' + e) \geq 3$. Suppose that $\kappa(H') < 3$, then H' contains a 2-disconnector, say $\{u, v\}$, so that $H' = H_1 \cup H_2$, $H'[\{u, v\}] = H_1 \cap H_2$ and such that $x \in V(H_i)$ for some $i \in \{1, 2\}$. Unless $x \in \{u, v\}$, then $t, w \in V(H_i)$. Thus, if $x \notin \{u, v\}$, then $\{u, v\}$ is a 2-disconnector of $H' + e$; contradiction.

Suppose then that, without loss of generality, $x = u$. Thus, since x is 3-valent, there exists an $i \in \{1, 2\}$ such that $|N_{H_i}(x) \setminus \{v\}| = 1$. As $\{x, v\}$ is a minimum disconnector of H', it follows that $H_i - \{x, v\}$ is connected so that $N_{H_i}(x) \cup \{v\}$ is the boundary of a 2-hammock of G; such must be trivial as $\kappa(G) = 3$, implying that $|V(H_i)| = \{x, v, z\}$, where $z \in \{t, w\}$.

We may assume that x is not adjacent to v; for otherwise, $|N_{H_{3-i}}(x) \setminus \{v\}| = 1$ so that the minimality of the disconnector $\{x, v\}$ implies that $H_{3-i} - \{x, v\}$ is connected and consequently that $N_{H_{3-i}}(x) \cup \{v\}$ is the boundary of a 2-hammock of G; since such must be trivial we have that H is a triangle (consisting of $\{t, v, w\}$) contradicting the assumption that H is triangle-free.

Hence, since H is triangle free and since each member of $\{v\} \cup N_{H_{3-i}}(x)$ has at least two neighbors in H_{3-i}, $\{v\} \cup N_{H_{3-i}}(x)$ is the boundary of a proper fat 3-hammock of H; contradiction to H being minimal. □

The maximal 2-connected components of a connected graph are called its *blocks*. Such define a tree structure for G whose leaves are blocks and are called the *leaf* blocks of G [2].

We conclude this section with the following notation. Let $H \subseteq G$ be connected (possibly H is a single edge). By G/H we mean the contraction minor of G obtained by contracting H into a single vertex. We always assume that after the contractions the graph is kept simple; i.e., any multiple edges resulting from a contraction are removed.

3. Truncations

Let $\mathcal{F}$ be a family of graphs (possibly infinite). A graph is $\mathcal{F}$-*free* if it contains no member of $\mathcal{F}$ as a subgraph. A graph G is *nearly* $\mathcal{F}$-free if it is either $\mathcal{F}$-free or has a *breaker* $x \in V(G) \cup E(G)$ such that $G - x$ is $\mathcal{F}$-free. A breaker that is a vertex is called a *vertex-breaker* and an *edge-breaker* if it is an edge.

An $\mathcal{F}$-*truncation* of an $\mathcal{F}$-free graph G is a minor H of G that is nearly $\mathcal{F}$-free such that either $H \subseteq G$ (and then it has no breaker) or H contains a breaker x such that $H - x \subseteq G$. In the former case, the truncation is called *proper*; in the latter case, the truncation is *improper* with x as its breaker and $H - x$ as its *body*. An improper truncation is called an *edge-truncation* if its breaker is an edge and a *vertex-truncation* if its breaker is a vertex. A vertex-truncation is called a 3-*truncation* if its breaker is 3-valent.

Lemma 3.1. *Let $\mathcal{F}$ be a graph family such that $K_3 \in \mathcal{F}$ and let G be $\mathcal{F}$-free with $\delta(G) \geq 3$. Then G has an essentially 4-connected $\mathcal{F}$-truncation H such that:*

(3.1.1) $|H| \geq 4$*; and*

(3.1.2) *if H is a vertex-truncation then it is a 3-truncation and $|H| \geq 5$.*

Proof. Let $\mathcal{H}$ denote the 3-connected truncations of G.

(3.1.A) *$\mathcal{H}$ is nonempty. In particular, $\mathcal{H}$ contains a truncation H with $|H| \geq 4$ so that if improper then it is an edge-truncation with edge-breaker e such that $\kappa(H - e) = 2$.*

Proof. We may assume that G is connected. Let B be a leaf block of G (possibly $B = G$). If $\kappa(B) \geq 3$, then (3.1.1) follows (by setting $H = B$) as B is a proper truncation of G. Assume then that $\kappa(B) = 2$ and let H be an extreme 3-connected component of B with window $\{x, y\}$. Now, $H \in \mathcal{H}$ with possibly xy an edge-breaker. If H is improper, then $\kappa(H - xy) = 2$. Note that $\delta(G) \geq 3$ implies that $|H| \geq 4$ in both cases. □

If $\mathcal{H}$ contains a proper or an edge-truncation that is essentially 4-connected, then (3.1.1) follows. Suppose then that

$$\text{(3.1)} \qquad \mathcal{H} \text{ has no essentially 4-connected proper or edge-truncations.}$$

(3.1.B) *Assuming (3.1), then $\mathcal{H}$ contains a truncation that if improper then it is a 3-truncation of order ≥ 5.*

Proof. Let $H \in \mathcal{H}$ such that if improper then H and e are as in (3.1.A). By (3.1) and Lemma 2.2, H has a minimal fat 3-hammock H' such that if $e \in E(H')$, then e is spanned by the boundary of H'. Let H'' be the graph obtained from an augmentation of H' by removing e if it is spanned by $bndH'$. Let $\{x\} = V(H'') \setminus V(H')$.

By Lemmas 2.3 and 2.4, $\kappa(H'') \geq 3$ so that $H'' \in \mathcal{H}$ with x as a potential 3-valent vertex-breaker and (3.1.B) follows.

Finally, note that $|intH'| \geq 2$ so that $|H''| \geq 5$. □

Next, we show the following.

(3.1.C) *If $\mathcal{H}$ contains a 3-truncation X of order ≥ 5, then $\mathcal{H}$ contains essentially 4-connected 3-truncations Y such that $5 \leq |Y| \leq |X|$.*

Proof. Let $H^* \in \mathcal{H}$ be a 3-truncation of order ≥ 5 with the order of its body minimized. We show that H^* is essentially 4-connected. Let x denote the vertex-breaker of H^*. By the minimality of H^*,

$$\begin{gathered}\text{any minimal fat } 3- \text{ hammock } T \text{ of } H^* \\ \text{with } x \notin V(T) \text{ satisfies } T = H^* - x\end{gathered} \tag{3.2}$$

(so that $bndT = N_{H^*}(x)$).

Assume now, towards contradiction, that H^* is not essentially 4-connected so that it contains nontrivial 3-disconnectors and at least two minimal fat 3-hammocks that may meet only at their boundary, by Lemma 2.1. By (3.2), existence of at least two such hammocks implies that x belongs to every nontrivial 3-disconnector and thus to the boundary of every minimal fat 3-hammock. As x is 3-valent, there is a minimal fat 3-hammock T of H^* with x on its boundary such that $N_T(x) = \{y\}$. As T is a minimal fat 3-hammock, $V(T)$ consists of x, y, the two members of $bndT \setminus \{x\}$, and an additional vertex u. As $\delta(G) \geq 3$, $uy \in E(T)$, u is adjacent to both members of $bndT \setminus \{x\}$ and y is adjacent to at least one member of $bndT \setminus \{x\}$. Hence, $K_3 \subseteq T - x \subseteq H^* - x$ so that x is not a breaker; contradiction. □

Assuming (3.1), then, by (3.1.B), there are 3-connected 3-truncations of G of order ≥ 5 so that an essentially 4-connected 3-truncation of G exists by (3.1.C). ☐

Lemma 3.2. *Let $\mathcal{F}$ be a graph family such that $\{K_3, K_{2,3}\} \subseteq \mathcal{F}$, then G has an internally 4-connected $\mathcal{F}$-truncation satisfying (3.1.1–2) and if such is a vertex-truncation then it is a 3-truncation.*

Proof. Let $\mathcal{T}$ denote the essentially 4-connected truncations of G that are either proper, or edge-truncations, or 3-truncations; $\mathcal{T}$ is nonempty by Lemma 3.1. Let $\alpha(\mathcal{T})$ denote the least k such that $\mathcal{T}$ contains a proper truncation of order k or an improper edge-truncation of order k. Let $\beta(\mathcal{T})$ denote the least k such that $\mathcal{T}$ contains an improper 3-truncation with its body of order k. Let $H \in \mathcal{T}$ such that $|H| = \min\{\alpha(\mathcal{T}), \beta(\mathcal{T}) + 1\}$ and let x denote its breaker if improper.

We show that H is internally 4-connected. To see this, assume, to the contrary, that H is not internally 4-connected and let D be a 3-disconnector of H such that $H - D$ consists of ≥ 3 components at least one of which is a singleton (since H is essentially 4-connected). Let $\mathcal{C}$ denote the non-singleton components of $H - D$. Since $K_{2,3} \in \mathcal{F}$, $|\mathcal{C}| \geq 1$.

Suppose $J = H[C \cup D]$ is a 3-hammock of H, for some $C \in \mathcal{C}$, that does not meet x in its interior (if x exists). By the choice of H,

$$\text{for each fat 3-hammock } X \text{ of } J \text{ either} \quad x \in bndX or x \in E(H[bndX]). \tag{3.3}$$

Indeed, for otherwise, an augmentation of a minimal fat 3-hammock of X is a 3-truncation of order $\geq$5 of G that belongs to $\mathcal{H}$ and has order $<$$|H|$, where $\mathcal{H}$ is as in the proof of Lemma 3.1; existence of such a 3-truncation of G implies that G has an essentially 4-connected 3-truncation of order $\geq$5, by (3.1.C), and such has order $<$$|H|$ contradicting the choice of H. Consequently, the assumption that the interior of J does not meet x implies that

$$\text{if } J \text{ exists, then } x \in D \cup E[H[D]]. \tag{3.4}$$

Suppose now that J has a minimal fat 3-hammock J' (possibly $J' = J$) with $x \in bndJ'$ so that $x \in D$, by (3.4). $|D| = \kappa(H)$ imply that x is incident with each component of $H - D$ so that $|N_{intJ'}(x)| = 1$, as x is 3-valent. The minimality of J' then implies that $|intJ'| = 2$ so that $J' - x$ contains a K_3 (see proof of (3.1.C) for the argument) and thus x is not a breaker of H; contradiction.

Suppose next that J' is a minimal fat 3-hammock of J whose boundary vertices span x (as an edge). Then, an augmentation of $J' - x$ belongs to $\mathcal{H}$, by Lemma 2.4, and such contains an essentially 4-connected 3-truncation of G, by (3.1.C), of order $< |H|$. Hence,

$$J \text{ (if exists) has no minimal fat 3-hammock } J' \text{ with } x \in bndJ' \cup E[H[bndJ']]. \tag{3.5}$$

If J exists, then (3.3) and (3.5) are contradictory. Thus, to obtain a contradiction and hence conclude the proof of Lemma 3.2 we show that a 3-hammock such as J exists. This is clear if $|\mathcal{C}| \geq 2$ as then at least one member of $\mathcal{C}$ does not meet x. Suppose then that $|\mathcal{C}| = 1$ so that $H - D$ consists of two singleton components, say $\{u, v\}$, and the single member C of $\mathcal{C}$. $D \cup \{u, v\}$ induce a $K_{2,3}$, say K. Since $K_{2,3} \in \mathcal{F}$ and x is a breaker, K contains x so that C does not; hence, $H[C \cup D]$ is the required 3-hammock. □

For $k \geq 4$, a graph that is nearly $\{K_3, C_4, \ldots, C_{k-1}\}$-free is called *nearly k-long*. That is, G is nearly k-long if either it has girth $\geq k$ or it has a *breaker* $x \in V(G) \cup E(G)$ such that $G - x$ has girth $\geq k$.

A nearly 5-long graph is nearly $\{K_3, C_4\}$-free; such is also nearly $\{K_3, K_{2,3}\}$-free. In addition, a 3-connected nearly 5-long truncation has order ≥ 5. Consequently, we have the following consequence of Lemma 3.2.

Lemma 3.3. *A graph with girth $\geq k \geq 5$ and $\delta \geq 3$ has an internally 4-connected nearly k-long truncation of order ≥ 5 and if such is a vertex-truncation then it is a 3-truncation.*

4. Nearly long planar graphs

For a plane graph G, we denote its set of faces by $F(G)$ and by X_G its infinite face.

Lemma 4.1. *Let G be a 2-connected plane graph of girth ≥ 6, and let $S \subseteq V(G)$ be the 2-valent vertices of G. Then, $|S| \geq 6$.*

Proof. By Euler's formula:

$$|E(G)| = |V(G)| + |F(G)| - 2. \tag{4.1}$$

Since G is 2-connected, every vertex in $V(G) \setminus S$ is at least 3-valent so that

$$2|E(G)| \geq 3(|V(G)| - |S|) + 2|S|. \tag{4.2}$$

As G is of girth ≥ 6 and 2-connected (and hence every edge is contained in exactly two distinct faces) then:

$$2|E(G)| \geq 6|F(G)|. \tag{4.3}$$

Substituting (4.1) in (4.2),

$$\begin{aligned} &2(|V(G)| + |F(G)| - 2) \geq 3(|V(G)| - |S|) + 2|S| \\ &\quad \Rightarrow |V(G)| \leq 2|F(G)| + |S| - 4. \end{aligned} \tag{4.4}$$

Substituting (4.1) in (4.3),

$$2(|V(G)| + |F(G)| - 2) \geq 6|F(G)| \Rightarrow |V(G)| \geq 2|F(G)| + 2. \tag{4.5}$$

From (4.4) and (4.5),

$$2|F(G)| + 2 \leq 2|F(G)| + |S| - 4 \Rightarrow |S| \geq 6. \tag{4.6}$$

Hence, the proof follows. □

From Lemma 4.1 we have that:

Lemma 4.2. *A nearly* 6*-long internally* 4*-connected graph is nonplanar.*

Lemma 4.3. *Let G be a nearly* 5*-long internally* 4*-connected planar graph and suppose that if G has a vertex-breaker, then it also has a vertex-breaker which is a* 3*-valent vertex. Then, $|G| \geq 11$.*

Proof. Define $S \subseteq V(G) \cup E(G)$ as follows. If G is of girth ≥ 5 set $S := \emptyset$; otherwise set $S := \{x\}$, where $x \in V(G) \cup E(G)$ is a breaker of G so that if $x \in V(G)$ then x is 3-valent. Then, $G - S$ is 2-connected, and has at most three 2-valent vertices. Hence,

$$2|E(G)| \geq 3(|V(G)| - 3) + 6. \tag{4.7}$$

As $G - S$ is of girth ≥ 5 and G is 2-connected then:

$$2|E(G)| \geq 5|F(G)|. \tag{4.8}$$

Substituting (4.1) in (4.7),

$$\begin{aligned} &2(|V(G)| + |F(G)| - 2) \geq 3(|V(G)| - 3) + 6 \\ &\quad \Rightarrow |F(G)| \leq (|V(G)| + 1)/2. \end{aligned} \tag{4.9}$$

Substituting (4.1) in (4.8),

$$(4.10) \qquad 2(|V(G)| + |F(G)| - 2) \geq 5|F(G)| \Rightarrow |F(G)| \geq (2|V(G)| - 2)/3.$$

From (4.9) and (4.10),

$$(4.11) \qquad (|V(G)| + 1)/2 \leq (2|V(G)| - 2)/3 \Rightarrow |V(G)| \geq 11.$$

Hence, the proof follows. □

Lemma 4.4. *A* 2*-connected plane graphs* G *satisfying the following does not exist.*

(4.4.1) G *has girth* ≥ 5*;*
(4.4.2) *each member of* $V(G) - V(X_G)$ *is at least* 4*-valent; and*
(4.4.3) G *has a set* $S \subseteq V(X_G)$, $|S| \leq 3$ *(possibly* $S = \emptyset$*) with each of its members* 2*-valent and each member of* $V(X_G) - S$ *at least* 3*-valent.*

Proof. Assume towards contraction that the claim is false. We will use the Discharging Method to obtain a contradiction to Euler's formula. The discharging method starts by assigning numerical values (known as charges) to the elements of the graph. For $x \in V(H) \cup F(H)$, define $ch(x)$ as follows.

(CH.1) $ch(v) = 6 - d_H(v)$, for any $v \in V(H)$.
(CH.2) $ch(f) = 6 - 2|f|$, for any $f \in F(H) - \{X_H\}$.
(CH.3) $ch(X_H) = -5\frac{2}{3} - 2|X_H|$.

Next, we observe that

$$(4.12) \qquad \sum_{x \in V(H) \cup F(H)} ch(x) = \frac{1}{3}.$$

Indeed, we have the following.

$$\begin{aligned}
\sum_{x\in V(H)\cup F(H)} ch(x) = & -5\frac{2}{3} - 2|X_H| + \sum_{f\in F(H)-X_H}(6-2|f|) \\
& + \sum_{v\in V(H)}(6-d(v)) \\
= & -5\frac{2}{3} - 2|X_H| + 6(|f(H)|-1) + \sum_{f\in F(H)-X_H}(-2|f|) \\
& + \sum_{v\in V(H)}(6-d(v)) \\
= & -5\frac{2}{3} + 6(|f(H)|-1) - 2(2|E|) + 6|V(H)| - 2|E(H)| \\
= & \, 6(F(H)-E(H)+V(H)) - 11\frac{2}{3} = \frac{1}{3}.
\end{aligned}$$

Next the charges are locally redistributed according to the following discharging rules:

(DIS.1) If v is of degree two, then v sends $3\frac{1}{5}$ to X_G and $\frac{4}{5}$ to the other face incident to it.
(DIS.2) If v is of degree three, then v sends $1\frac{5}{8}$ to X_G and $\frac{4}{5}$ to every other face incident to it.
(DIS.3) If v is of degree at least four then v sends $\frac{4}{5}$ to each incident face.

For $x \in V(G) \cup F(G)$, let $ch^*(x)$ (denoted as the modified charge) be the resultant charge after modification of the initial charges according to (DIS.1–3). We obtain a contradiction to (4.12) by showing that $ch^*(x) \le 0$ for every $x \in V(H) \cup F(H)$. This is clearly implied by the following claims proved below.

(A) *$ch^*(v) \le 0$, for each $v \in V(H)$.*
(B) *$ch^*(f) \le 0$, for each $f \in F(H) - \{X_H\}$.*
(C) *$ch^*(X_H) \le 0$.*

Observe that according to DIS.(1)–(3), faces do not send charge and vertices do not receive charge.

To prove (A), it is sufficient to consider vertices v satisfying $d_G(v) \ge 5$. Indeed, if $d_H(v) \ge 6$, then $ch(v) = ch^*(v) \le 0$ by (CH.1). If $2 \le d_G(v) \le 3$, then it is easily seen by (CH.1) and (DIS.1–2) that $ch^*(v) = 0$. If $4 \le d_G(v) \le 5$, then, by (CH.1) and (DIS.3), $ch^*(v) = 6 - d_H(v) - \frac{4}{5}d_G(v) \le 0$.

Next, we prove (B). Let $f \in F(H) - \{X_{_H}\}$. By (DIS.1–3), f receives a charge of $\frac{4}{5}$ from every vertex incident to it. Hence, togther with (CH.2), $ch^*(f) = 6 - 2|f| + \frac{4}{5}|f| \leq 0$. (The last inequality follows as $|f| \geq 5$.)

Finally, we prove (C). Let $S_1 \subseteq V(X_G)$ be the set of vertices of X_G of degree three, and let $S_2 = V(X_G) - (S \cup S_1)$. By (CH.3), (DIS.1–3) and as $|S| \leq 3$, we see that $ch^*(f) = -5\frac{2}{3} - 2|X_G| + 3\frac{1}{5}|S| + 1\frac{5}{8}|S_1| + \frac{4}{5}|S_2| \leq -5\frac{2}{3} - 2|X_G| + 3 \times 3\frac{1}{5} + 1\frac{5}{8}(|X_G| - 3) = -\frac{3}{8}|X_G| - \frac{11}{12} \leq 0$. □

5. K_5-minors in internally 4-connected graphs

By V_8 we mean C_8 together with 4 pairwise overlapping chords. By TG we mean a subdivided G.

The following is due to Wanger.

Theorem 5.1. [6, Theorem 4.6] *If G is 3-connected and $TV_8 \subseteq G$ then either $G \cong V_8$ or G has a K_5-minor.*

The following structure theorem was proved independently by Kelmans [7] and Robertson [8].

Theorem 5.2. [7] *Let G be internally 4-connected with no minor isomorphic to V_8. Then G satisfies one of the following conditions:*

(5.2.1) *G is planar;*

(5.2.2) *G is isomorphic to the line graph of $K_{3,3}$;*

(5.2.3) *there exist a $uv \in E(G)$ such that $G - \{u, v\}$ is a circuit;*

(5.2.4) *$|G| \leq 7$;*

(5.2.5) *there is an $X \subseteq V(G)$, $|X| \leq 4$ such that $\|G - X\| = 0$.*

From Theorems 5.1 and 5.2 we deduce that

Lemma 5.3. *A nearly 5-long internally 4-connected nonplanar G has a K_5-minor.*

Proof. We may assume that $G \ncong V_8$ and that G has no V_8-minor. The former since V_8 is not nearly 5-long and the latter by Theorem 5.1. Hence, G satisfies one of (5.2.1–5). As G is nonplanar, by assumption, and the line graph of $K_{3,3}$ has a K_5-minor (and is not nearly 5-long) it follows that G satisfies one of (5.2.3–5).

If G is of girth ≤ 4, let $a \in V(G) \cup E(G)$ be a breaker of G; otherwise (if G has girth ≥ 5) let a be an arbitrary vertex of G. If $a \in V(G)$, put $b := a$; otherwise let b be some end of a. By definition, $G - b$ has girth ≥ 5.

(5.3.A) *$G - \{u, v\}$ is not a circuit for any $u, v \in V(G)$ so that G does not satisfy* (5.2.3).

Proof. Suppose not; and let $C := G - \{u, v\} = \{x_0, \ldots, x_{k-1}\}$, where $k \geq 3$ is an integer. Suppose first that $b \in \{u, v\}$ and assume, without loss of generality, that $u = b$. Then, $k \geq 5$. As v is at least 3-valent, there exists $0 \leq i \leq k-1$ so that $vx_i \in E(G)$. Since $G - b$ has girth ≥ 5, $vx_{i+1}, vx_{i+2} \notin E(G)$ (subscripts are read modulo k). Since x_{i+1} and x_{i+2} are at least 3-valent in G, each is adjacent to u. But then $\{u, x_i, x_{i+3}\}$ is a 3-disconnector of G separating $\{x_{i+1}, x_{i+2}\}$ from $\{v, x_{i+4}\}$ (note that since $k \geq 5$, $x_{i+1}, x_{i+2} \neq x_{i+4}$); a contradiction to G being internally 4-connected.

Suppose then that $x_i = b$, for some $0 \leq i \leq k-1$. Hence, exactly one of v and u is adjacent to x_{i+1} and exactly one to x_{i+2} (this is true since every vertex of C is adjacent to v or u, and if say, v, is adjacent to both x_{i+1} and x_{i+2} then $G - b$ contains a triangle). If $x_{i+3} \neq x_i$, then x_{i+3} is adjacent to one of u and v. If $x_i = x_{i+3}$, then C is a circuit of length three, and $V(G) = 5$. Both cases contradict the fact that G is nearly 5-long. □

(5.3.B) $|G| \geq 8$ *so that* G *does not satisfy* (5.2.4).

Proof. For suppose $|G| \leq 7$. As G is internally 4-connected, $G - b$ is 2-connected. Since $G - b$ is of girth ≥ 5, then $G - b$ contains an induced circuit C of length ≥ 5. Hence $|G| \geq 6$. If $|G| = 6$, then $G = C \cup b$ and then G is planar; a contradiction. If $|G| = 7$ then G is a circuit plus two vertices and we get a contradiction to (5.3.A). Hence, $V(G) \geq 8$. □

To reach a contradiction we show that (5.2.5) is not satisfied by G. Suppose it is satisfied and let X be as in (5.2.5) and let $Y = V(G) - X$. As $V(G) \geq 8$, then $|Y| \geq 4$ and every vertex of Y is adjacent to at least three vertices in X. But then it is easily seen that G is of girth ≤ 4 but contains no edge- or vertex-breaker; a contradiction. □

Let G be a plane graph. By a *jump* over G we mean a path P internally-disjoint of G whose ends are not cofacial in G.

Lemma 5.4. *Let* G *be an internally* 4*-connected nearly* 5*-long plane graph and let* P *be a jump over* G*. Then,* G *has a* K_5*-minor with every branch set meeting* $V(G)$*.*

Proof. Put $G' := G \cup P$. (By possibly contracting P) we may assume that P is an edge e with both ends in G. It suffices now to show that G' has a K_5-minor. Suppose G' has no such minor. We may assume that $G' \ncong V_8$, since V_8 with any edge removed is not internally 4-connected, and that G' has no V_8-minor, by Theorem 5.1. Since G' is nonplanar, $|G'| \geq |G| \geq 11$, by Lemma 4.3, and since the line graph of $K_{3,3}$ has a K_5-minor, we have that G'

satisfies (5.2.3) or (5.2.5). We show that both options lead to a contradiction to the definition of G.

Suppose (5.2.3) is satisfied. Set $C := G' - \{u, v\} = \{x_0, \ldots, x_{k-1}\}$, where $k \geq 9$ is an integer. If $e \notin E(C)$, then a contradiction is obtained by showing that $G - e - \{v, u\}$ cannot be a circuit. The proof is exactly the same as the proof of (5.3.A) with $G - e$ instead of G.

Hence we may assume that $e \in E(C)$; so let $e = x_i x_{i+1}$, for some $0 \leq i \leq k-1$ (subscript are read modulo k). Observe that $d_{G'}(x_i), d_{G'}(x_{i+1}) \geq 4$. Hence, in G, each of x_i and x_{i+1} is adajcent to both u and v.

By assumption that (5.2.3) is satisfied, $uv \in E(G)$, and we see that one of u or v is a breaker, say u. Hence, $vx_{i+2}, vx_{i+3} \notin E(G)$. But then, since and $d_G(x_{i+1}), d_G(x_{i+2}) = 3$, the set $\{u, x_{i+1}, x_{i+4}\}$ is a 3-disconnector of G (note that since $k \geq 9$, x_{i+1}, x_{i+4} are distinct) separating $\{x_{i+2}, x_{i+3}\}$ from $\{x_{i+5}, x_{i+6}\}$; a contradiction. Hence (5.2.3) is not satisfied.

Suppose (5.2.5) is satisfied. As $V(G) \geq 11$, it is easily seen that $G(= G' - e)$ is of girth ≤ 4 but has no edge- or vertex-breaker; a contradiction. This concludes the proof. □

By *society* we mean a pair (G, Ω) consisting of a graph G and a cyclic permutation Ω over a finite set $\overline{\Omega} \subseteq V(G)$. Let $\overline{\Omega} = \{v_1, \ldots, v_k\}$, $k \geq 4$. Two pairs of vertices $\{s_1, t_1\} \subseteq \overline{\Omega}$ and $\{s_2, t_2\} \subseteq \overline{\Omega}$ are said to *overlap* along (G, Ω) if $\{s_1, s_2, t_1, t_2\}$ occur in $\overline{\Omega}$ in this order along Ω.

Two vertex disjoint paths P and P' of G that are both internally-disjoint of $\overline{\Omega}$ are said to form a *cross* on (G, Ω) if their ends are in $\overline{\Omega}$ and these overlap along (G, Ω).

Lemma 5.5. [9, Lemma (2.4)] *Let (G, Ω) be a society. Then either*

(5.5.1) *(G, Ω) admits a cross in G, or*
(5.5.2) *$G = G_1 \cup G_2$, $G_1 \cap G_2 = G[D]$, $|D| \leq 3$ such that $\overline{\Omega} \subseteq V(G_1)$ and $|V(G_2) \setminus V(G_1)| \geq 2$, or*
(5.5.3) *G can be drawn in a disc with $\overline{\Omega}$ on the boundary in order Ω.*

Let C be a circuit in a plane graph G. Then the clockwise ordering of $V(C)$ induced by the embedding of G defines a cyclic permutation on $V(C)$ denoted Ω_C and we do not distinguish between the cyclic shifts of this order. Then, (G, Ω_C) is a society with $\overline{\Omega_C} = V(C)$. Throughout, we omit this notation when dealing with such societies of circuits of plane graphs and instead say that C is a society of G.

Lemma 5.6. *Let G be a 3-connected plane graph of order ≥ 5 and let P and P' be vertex disjoint paths that are internally-disjoint of G and whose ends* *are contained in a facial circuit f of G. If $P \cup P'$ form a cross on f, then $G \cup P \cup P'$ contains a K_5-minor with every branch set meeting $V(G)$.*

Proof. Clearly, $V(G) \neq V(f)$. Since the facial circuits of a 3-connected plane graph are it induced nonseparating circuits [5], we have that $G - V(f)$ is connected so that $f \cup P \cup P'$ have a K_4-minor which is completed into a K_5-minor by adding a fifth branch set that is $G - V(f)$ (as f is an induced circuit). □

6. Proof of Theorem 1.1

Let $\mathcal{H} = \{H \subseteq G : H \text{ is connected}, |G/H| \geq 5, \text{ and } \|G/H\| \geq 3|G/H| - 7\}$. $\mathcal{H}$ contains every member of $V(G)$ as a singleton and thus is nonempty. Let $H_0 \in \mathcal{H}$ be maximal in $(\mathcal{H}, \subseteq)$, $H_1 = G[N_G(H_0)]$, and let $G_0 = G/H_0$, where $z_0 \in V(G_0)$ represents H_0. Let $G_1 = G_0 - z_0$ and note that $G_1 \subseteq G$.

$|G_0| = 5$ implies that $\|G_0\| \geq 8$ so that $\|G_1\| \geq 4$ and contains a k-circuit with $k < 5$; contradiction to the assumption that G has girth at least 6. Thus, we may assume that

(1.1.A) $|G_0| \geq 6$.

Let $x \in V(H_1)$ and put $G_0' = G_0/z_0x$. $|G_0'| \geq 5$, by (1.1.A). Thus, the maximality of H_0 in $(\mathcal{H}, \subseteq)$ implies that $\|G_0'\| \leq 3|G_0'| - 8$. Thus, $\|G_0\| - \|G_0'\| \geq 3|G_0| - 7 - 3(|G_0| - 1) + 8 \geq 4$; implying that z_0x is common to at least three triangles so that $d_{H_1}(x) \geq 3$. It follows then that

(1.1.B) $\delta(H_1) \geq 3$.

Let H be an internally 4-connected nearly 6-long truncation of H_1, by Lemma 3.3. Such is nonplanar by Lemma 4.2 and has a K_5-minor by Lemma 5.3. Consequently, G_0 has a K_6-minor. This concludes the proof of Theorem 1.1.

7. Proof of Theorem 1.3

In a manner similar to that presented in the proof of Theorem 1.1, let $\mathcal{H} = \{H \subseteq G : H \text{ is connected}, |G/H| \geq 5, \text{ and } \|G/H\| \geq 3\frac{1}{5}|G/H| - 8\}$ (such is nonempty) and let H_0, H_1, G_0, z_0, G_1 be as in the proof of Theorem 1.1.

$|G_0| = 5$ implies that $\|G_0\| \geq 8$ so that $\|G_1\| \geq 4$ and contains a k-circuit with $k < 5$; contradiction to the assumption that G has girth at least 5. Thus, we may assume that

(1.3.A) $|G_0| \geq 6$.

Let $x \in V(H_1)$ and put $G_0' = G_0/z_0x$. $|G_0'| \geq 5$, by (1.3.A). Thus, the maximality of H_0 in $(\mathcal{H}, \subseteq)$ implies that $\|G_0'\| \leq 3\frac{1}{5}|G_0'| - 9$. Thus, $\|G_0\| - \|G_0'\| \geq 3\frac{1}{5}|G_0| - 8 - 3\frac{1}{5}(|G_0| - 1) + 9 \geq 4$; implying that z_0x is common to at least three triangles so that $d_{H_1}(x) \geq 3$. It follows then that

(1.3.B) $\delta(H_1) \geq 3$;

implying that

(1.3.C) $\delta(G_0) \geq 4$.

Next, we prove that

(1.3.D) $\kappa(G_0) \geq 5$.

To see (1.3.D), let $T \subseteq V(G)$ be a minimum disconnector of G_0 and assume, towards contradiction, that $|T| \leq 4$. As $\kappa(G) \geq 6$, $z_0 \in T$. Let then $y = |N_{G_0}(z_0) \cap T|$ and let $\mathcal{C}$ denote the components of $G_0 - T$. Choose $C \in \mathcal{C}$ and put $H_1 = G_0[C \cup T]$ and $H_2 = G_0 - C$.

Let H_i' be the graph obtained from G_0 by contracting H_{3-i} into z_0 (note that minimality of T implies that each of its members is incident with each member of $\mathcal{C}$), for $i = 1, 2$. As $|H_i| \geq 5$, by (1.3.C), then $|H_i'| \geq 5$, for $i = 1, 2$. The maximality of H_0 in $(\mathcal{H}, \subseteq)$ then implies that $\|H_i'\| \leq 3\frac{1}{5}|H_i'| - 9$.

As $z_0x \in E(H_i')$ for each $x \in T' = T \setminus \{z_0\}$, for $i = 1, 2$, it follows that

$$\|G_0\| + y + 2(|T'| - y) + \|G_0[T']\| \leq \|H_1'\| + \|H_2'\| \leq 3\frac{1}{5}(|G_0| + |T|) - 18. \tag{7.1}$$

As $\|G_0\| \geq 3\frac{1}{5}|G_0| - 8$, we have that

$$8 + \|G_0[T']\| \leq 1\frac{1}{5}|T| + y. \tag{7.2}$$

Now, $|T| \leq 4$ (by assumption), so that $y \leq 3$, and $\|G_0[T']\| \geq 0$. Consequently, the right-hand size of (7.2) does not exceed 7.8. This contradiction establishes (1.3.D).

Let $\mathcal{B}$ denote the bridges of H_1 in G_1. We may assume that $\mathcal{B}$ is nonempty. Otherwise, G_1 coincides with H_1 so that H_1 is a nonplanar 4-connected graph of girth ≥ 5 and thus containing a K_5-minor by 5.3. Consequently, G_0 has a K_6-minor and 1.3 follows.

Let H be an internally 4-connected nearly 5-long truncation of H_1, by Lemma 3.3. We may assume that H is planar for otherwise H has a K_5-minor, by Lemma 5.3, so that G_0 has a K_6-minor and Theorem 1.3 follows. Let x denote the breaker of H, if such exists in H. Let $\mathcal{B}_1 = \emptyset$ if x does not exist (so that $H \subseteq G$) or is an edge-breaker. Otherwise (i.e., if x is a vertex-breaker), $\mathcal{B}_1$ denotes the members of $\mathcal{B}$ with attachment vertices in the subgraph of H_1 contracted into x. Put $\mathcal{B}_2 = \mathcal{B} \setminus \mathcal{B}_1$.

Fix an embedding of H in the plane. No member of $\mathcal{B}$ defines a jump over H for otherwise the union of H and such a jump has has a K_5-minor with every branch set meeting $V(H)$, by Lemma 5.4. Hence, every member of $\mathcal{B}$ has all of its attachment vertices confined to a single face of H.

By *patch* we mean a face f of H together with all members of $\mathcal{B}$ attaching to $V(f)$. Patches not meeting x in case it is a vertex-breaker are called *clean* (so that if x does not exist or is an edge-breaker, then every patch is clean). f is called the *rim* of the patch. If $\mathcal{P}$ is a patch with rim f, then by $(\mathcal{P}, \Omega_f)$ we mean a society with $\overline{\Omega_f} = V(f)$ and Ω_f is the clockwise order on $V(f)$ defined by the embedding of f in the plane.

(1.3.E) *Let H' denote the union of H and all members of $\mathcal{B}_2$. Then, H' is planar.*

To see (1.3.E) it is sufficient to show that every clean patch is planar. Indeed, since any two faces of H meet either at a single vertex or at a single edge, the union of any number of planar patches results in a planar graph.

Let $\mathcal{P}$ be a clean patch with rim f. If $(\mathcal{P}, \Omega_f)$ contains a cross, then the union of H and such a cross has a K_5-minor, by Lemma 5.6, with every branch set meeting $V(H)$; so that G_0 has a K_6-minor and Theorem 1.3 follows. Assume then that $(\mathcal{P}, \Omega_f)$ has no cross and is nonplanar. Then, $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, $\mathcal{P}_1 \cap \mathcal{P}_2 = \mathcal{P}[D]$ and $|D| \leq 3$ such that $V(f) \subseteq V(\mathcal{P}_1)$ and $|V(\mathcal{P}_2) \setminus V(\mathcal{P}_1)| \geq 2$, by Lemma 5.5. Hence, $\{z_0\} \cup D$ is a k-disconnector of G_0 with $k \leq 4$; contradicting (1.3.D). It follows that $\mathcal{P}$ is planar so that (1.3.E) follows.

If x is a vertex-breaker, then let C be the vertices of H cofacial with x. 4-connectivity of G_1 implies that every vertex in $H' - \{x\} - C$ is at least 4-valent in $H' - x$. As x is 3-valent in this case, by (3.1.3), we have that $H' - x$ is a 2-connected planar graph of girth $\geq$5 has an embedding in the plane with each vertex not in $X_{H'-x}$ at least 4-valent, and each vertex in $X_{H'-x}$ at least 3-valent except for at most 3 vertices which are at least 2-valent. By Lemma 4.4, $H' - x$ does not exist; contradiction.

This concludes our proof of Theorem 1.3.

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