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A note on Lovász removable path conjecture

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Lovász [8] conjectured that for any natural number k, there exists a smallest natural number $f(k)$ such that, for any two vertices s and t in any $f(k)$-connected graph G, there exists an s-t path P such that $G - V(P)$ is k-connected. This conjecture is proved only for $k \leq 2$. Here, we strengthen the result for $k = 2$ as follows: for any integers $l > 0$ and $m \geq 0$, there exists a function $f(l, m)$ such that, for any distinct vertices $s, t, v_1, \ldots, v_m$ in any $f(l, m)$-connected graph G, there exist l internally vertex disjoint s-t paths $P_1, \ldots, P_l$ such that for any subset $I \subset \{1, \ldots, l\}$, $G - \cup_{i \in I} V(P_i)$ is 2-connected and $\{v_1, v_2, \ldots, v_m\} \subset V(G) - \cup_{1 \leq i \leq l} V(P_i)$.

1. Introduction

The following conjecture is due to Lovász [8] which is still open for $k \geq 3$:

Conjecture 1.1. *For any natural number k, there exists a least natural number $f(k)$ such that, for any two vertices s, t in any $f(k)$-connected graph G, there exists an s-t path P such that $G - V(P)$ is k-connected.*

This conjecture has been proved for $k \leq 2$. A theorem of Tutte [11] shows that $f(1) = 3$. When $k = 2$, we have $f(2) = 5$ by a result of Chen, Gould and Yu [2] and, independently, of Kriesell [6]. Later, Kawarabayashi, Lee and Yu [4] characterized the 4-connected graphs G in which there exist two vertices $s, t \in V(G)$ such that $G - V(P)$ is not 2-connected for any s-t path P in G.

Conjecture 1.1 is equivalent to asking whether there exists a function $g(k)$ such that for any $g(k)$-connected graph and for any edge $st \in E(G)$, there exists a cycle C containing st such that $G - V(C)$ is k-connected. Lovász [8] also made a weaker conjecture: any $(k+3)$-connected graph contains a cycle C such that $G - V(C)$ is k-connected, which was confirmed by Thomassen [10]. Another weaker version of Conjecture 1.1 was proposed by Kriesell [7]: there exists a function $h(k)$ such that for any $h(k)$-connected graph G and for any two vertices $s, t \in V(G)$, there exists an induced s-t path P in G such that $G - E(P)$ is k-connected. This weaker version was established by Kawarabayashi, Lee, Reed and Wollan [3]. In [3], the authors further conjectured that there exists a function $F(k)$ such that for any $F(k)$-connected graph G and for any three distinct vertices $s, t, u \in V(G)$, G contains an s-t path P and a k-connected subgraph H such that $u \in V(H)$ and $V(H) \cap V(P) = \emptyset$; they also show that this conjecture implies Conjecture 1.1. In this sense, it is useful to find an s-t path that avoids a highly connected subgraph containing a specific vertex, which partially motivates our work.

Conjecture 1.1 asks for one removable path. In [2], Chen, Gould and Yu showed that in any $(22l+2)$-connected graph, there exist l internally vertex disjoint paths between any two given vertices such that the deletion of any one of these paths results in a connected graph. Recently, Kawarabayashi and Ozeki [5] strengthened this result as follows: for any $(3l+2)$-connected graph G and for any two vertices $s, t \in V(G)$, there exist l internally vertex disjoint s-t paths $P_1, \ldots, P_l$ such that $G - \cup_{i=1}^{l} V(P_i)$ is 2-connected; they also pointed out that if G is $(2l+1)$-connected, then one can find l internally vertex disjoint paths $P_1, \ldots, P_l$ between any two given vertices such that $G - \cup_{i=1}^{l} V(P_i)$ is connected.

In this note, we use a short argument to prove the following:

Theorem 1.2. *For any integers $l > 0$ and $m \geq 0$, let $f(l, m) = 30l + 10m + 2$. Then for any distinct vertices $s, t, v_1, \ldots, v_m$ in any $f(l, m)$-connected graph G, there exist l internally vertex disjoint s-t paths $P_1, \ldots, P_l$ such that for any subset $I \subset \{1, \ldots, l\}$, $G - \cup_{i \in I} V(P_i)$ is 2-connected and $\{v_1, v_2, \ldots, v_m\} \subset V(G) - \cup_{1 \leq i \leq l} V(P_i)$.*

2. Proof of Theorem 1.2

We begin with some definitions. A *linkage* is a graph in which every connected component is a path. A *linkage problem* in a graph G is a set of pairs of vertices of G, for example, $\mathcal{L} = \{\{s_1, t_1\}, \ldots, \{s_k, t_k\}\}$. A *solution* to the linkage problem $\mathcal{L}$ is a set of pairwise internally vertex disjoint paths $P_1, \ldots, P_k$ such that the ends of P_i are s_i and t_i, and if $x \in V(P_i) \cap V(P_j)$ for $i \neq j$ then $x = s_i$ or $x = t_i$. The graph G is *k-linked* if every linkage problem with k pairwise disjoint pairs of vertices has a solution.

Bollobás and Thomason [1] proved that every $22k$-connected graph is k-linked. Here we use the following improved bound by Thomas and Wollan [9]:

Lemma 2.1. *Every $10k$-connected graph is k-linked.*

We also need the following lemma.

Lemma 2.2. *For any distinct vertices* $s_1, \dots, s_l, t_1, \dots, t_l, v_1, \dots, v_m$ *in* $(30l+10m)$*-connected graph* G*, there exist* l *internally vertex disjoint paths* $P_1, \dots, P_l$ *in* G *and a* 2*-connected subgraph* H *of* $G - \cup_{1\le i\le l} V(P_i)$ *such that the ends of* P_i *are* s_i *and* t_i *for* $1 \le i \le l$*,* $\{v_1, \dots, v_m\} \subset V(H)$*,* $|V(H)| \ge 2l+m$*, and every vertex in* $\{s_1, \dots, s_l, t_1, \dots, t_l\}$ *has at least one neighbor in* H*.*

Proof. We may find a neighbor a_i of s_i and a neighbor b_i of t_i, for $1 \le i \le l$, such that $a_1, \dots, a_l, b_1, \dots, b_l, s_1, \dots, s_l, t_1, \dots, t_l, v_1, \dots, v_m$ are pairwise distinct, since G is $(30l+10m)$-connected. Now we look at the following linkage problem in G:

$$\begin{aligned}\mathcal{L} = \{&\{s_1, t_1\}, \dots, \{s_l, t_l\}, \{a_1, a_2\}, \{a_2, a_3\}, \dots, \{a_l, b_1\},\\ &\{b_1, b_2\}, \dots, \{b_l, v_1\}, \{v_1, v_2\}, \dots, \{v_m, a_1\}\},\end{aligned}$$

which has $3l+m$ pairwise disjoint pairs of vertices. By Lemma 2.1, we have a solution of $\mathcal{L}$: a collection of $3l+m$ paths $\{P_1, \dots, P_{3l}, Q_1, \dots, Q_m\}$, where, for $1 \le i \le 3l+m$, the ends of the ith path of this collection (in the order listed) are the two vertices of the ith pair in $\mathcal{L}$ (in the order listed). Let $H = (\cup_{l+1\le i\le 3l} P_i) \cup (\cup_{1\le j\le m} Q_j)$, which is a cycle through $a_1, \dots, a_l, b_1, \dots, b_l, v_1, \dots, v_m$. Note that $|V(H)| \ge 2l+m$. Then $P_1, \dots, P_l$ and H satisfy the conclusion of the lemma. □

Now, we are ready to give the proof of Theorem 1.2.

Proof. We may assume that $l \ge 2$ or $m \ge 1$; otherwise, $l = 1, m = 0$ and the theorem follows from known results. So we have that $l+m \ge 2$. Let $G' = G - \{s, t\}$. Since G is $(30l+10m+2)$-connected, G' is $(30l+10m)$-connected. We may fix l neighbors of s, say $s_1, s_2, \dots, s_l$, and l neighbors of t, say $t_1, t_2, \dots, t_l$, such that $s_1, \dots, s_l, t_1, \dots, t_l, v_1, \dots v_m$ are distinct.

By Lemma 2.2, there is a collection $\mathscr{P} = \{P_1, \dots, P_l\}$ of paths in G' such that $\{v_1, \dots, v_m\}$ is contained in a 2-connected subgraph $D(\mathscr{P})$ of $G' - \cup_{i=1}^{l} V(P_i)$, $|V(D(\mathscr{P}))| \ge 2l+m$ and any vertex of $\{s_1, \dots, s_l, t_1, \dots, t_l\}$ has a neighbor in $D(\mathscr{P})$. We call such collection $\mathscr{P}$ feasible. We may choose $D(\mathscr{P})$ to be a maximal 2-connected subgraph of $G' - \cup_{1\le i\le l} V(P_i)$, and if there is no ambiguity we simply call it D. Without loss of generality, we assume that the ends of P_i are s_i and t_i for any $1 \le i \le l$. If $D = G' - \cup_{1\le i\le l} V(P_i)$, then $\{s, ss_1\} \cup P_1 \cup \{t_1 t, t\}, \dots, \{s, ss_l\} \cup P_l \cup \{t_l t, t\}$ satisfy the conclusion of Theorem 1.2. Thus we may assume $D \ne G' - \cup_{1\le i\le l} V(P_i)$, and let $C_1, \dots, C_q$ be the components of $G' - \cup_{1\le i\le l} V(P_i) - V(D)$. By the maximality of D, D contains at most one neighbor of $V(C_i)$ for $1 \leq i \leq q$. Without loss of generality, we assume that

$$|V(C_1)| \geq |V(C_2)| \geq \cdots \geq |V(C_q)|.$$

We choose a feasible collection $\mathscr{P} = \{P_1, \ldots, P_l\}$ in G' such that

(1) $|V(D(\mathscr{P}))|$ is maximum, and
(2) subject to (1), $|V(C_1)|, |V(C_2)|, \ldots, |V(C_q)|$ are as large as possible with the larger order components having priority, and
(3) subject to (2), $|V(\cup_{1 \leq i \leq l} P_i)|$ is as small as possible.

Note that by Lemma 2.2, $|V(D(\mathscr{P}))| \geq 2l + m$. Now we consider $G^0 := G'[(\cup_{1 \leq i \leq l} P_i) \cup C_q]$. We claim that there exist a subset $J \subset \{1, 2, \ldots, l\}$ and $\{a_j, b_j\} \subset V(P_j)$ for all $j \in J$ such that $G'[(\cup_{j \in J} a_j P_j b_j) \cup C_q]$ is connected and it is separated from the other vertices of G^0 by $\{a_j, b_j : j \in J\}$. The existence of J follows by taking $G'[(\cup_{j \in J} a_j P_j b_j) \cup C_q]$ to be the component of G^0 containing C_q. Without loss of generality, we assume that $b_j \in a_j P_j t_j$ for $j \in J$ (possible $a_j = b_j$). We pick $J, \{a_j, b_j : j \in J\}$ such that

(4) if $J' \subset J$ and $\{a'_j, b'_j\} \subset V(a_j P_j b_j)$ for $j \in J'$ are such that $G'[C_q \cup (\cup_{j \in J'} a'_j P_j b'_j)]$ is connected and separated from the other vertices of G^0 by $\{a'_j, b'_j : j \in J'\}$, then $J' = J$ and for $j \in J$, $a'_j = a_j$ and $b'_j = b_j$.

In this sense, we call $J, \{a_j, b_j : j \in J\}$ minimal. We may assume that $J = \{1, \ldots, r\}$, $r \leq l$. Let $G^1 := G'[(\cup_{j \in J} a_j P_j b_j) \cup C_q]$, and $N_q := V(D) \cap N(C_q)$ (hence $|N_q| \leq 1$). We will prove the following claim, for any $k \in J$ and any $x, y \in V(a_k P_k b_k - \{a_k, b_k\})$, where $y \in x P_k b_k - \{b_k\}$ (possible $x = y$).

Claim. *There exist r vertex disjoint paths in $G^1 - V(xP_k y)$ from $A := \{a_j : 1 \leq j \leq r\}$ to $B := \{b_j : 1 \leq j \leq r\}$.*

Proof of Claim. Without loss of generality, we say $k = 1$. If not, then by Menger's Theorem there exists a cut of size $p \leq r - 1$ in $G^1 - V(xP_1 y)$, say $W := \{w_2, w_3, \ldots, w_{p+1}\}$, separating A from B. We see that $a_j P_j b_j$ has at least one vertex in W for $2 \leq j \leq r$; otherwise $a_j P_j b_j$ connects A and B. Thus $p = r - 1$ and we may assume that $w_j \in V(a_j P_j b_j)$ for $2 \leq j \leq r$. Now, $W \cup V(xP_1 y)$ is a cut in G^1 which separates A from B.

Let $D_1 = ((\cup_{2 \leq j \leq r} a_j P_j w_j) \cup a_1 P_1 x) - (W \cup \{x\})$, $D_2 = ((\cup_{2 \leq j \leq r} w_j P_j b_j) \cup y P_2 b_1) - (W \cup \{y\})$. We point out that at most one of $\{D_1, D_2\}$ contains a neighbor of C_q; otherwise, we can find a path in G^1 from A to B through C_q, disjoint from $W \cup V(xP_1 y)$, contradicting to the fact that $W \cup V(xP_1 y)$ is a cut in G^1 separating A from B. Without loss of generality, we assume that D_1 does not contain any neighbor of C_q. Thus $W \cup V(xP_1y)$ separates A from $C_q \cup B$.

We may assume that $x \neq y$. Otherwise we have $x = y$, then $W \cup \{x\}$ separates A from $C_q \cup B$, but $x \in V(a_1P_1b_1 - \{a_1, b_1\})$, so it contradicts (4), in particular the choice of A. Now, we consider $G^2 := G'[(\cup_{2\leq j\leq r} a_jP_jw_j) \cup a_1P_1y]$, and contract $xP_1y - \{x\}$ into a new vertex x', then call the resulting graph G^3. Note that xx' is an edge in G^3.

There exist r vertex disjoint paths from A to $W \cup \{x'\}$ in $G^3 - \{x\}$. Otherwise, by Menger's Theorem, there is a cut of size $t \leq r-1$ in $G^3 - \{x\}$, say $W' = \{w'_2, \ldots, w'_{t+1}\}$, separating A from $W \cup \{x'\}$. Clearly, $a_jP_jw_j$ has at least one vertex in W' for $2 \leq j \leq r$; so $t = r-1$ and we may assume that $w'_j \in V(a_jP_jw_j)$ for $2 \leq j \leq r$. Then, it means that $W' \cup \{x\}$ separates A from $W \cup V(xP_1y)$ in G^2; since $W \cup V(xP_1y)$ separates A from $C_q \cup B$ in G^1, $W' \cup \{x\}$ separates A from $C_q \cup B$ in G^1. But $x \in V(a_1P_1b_1) - \{a_1, b_1\}$, which contradicts (4), in particular $W' \cup \{x\}$ contradicts the choice of A.

Therefore, there exist r vertex disjoint paths in $G^2 - \{x\}$ from A to $W \cup \{u\}$, for some $u \in V(xP_1y) - \{x\}$, say P'_1 from $a_{\pi(1)}$ to u and P'_j from $a_{\pi(j)}$ to w_j for $2 \leq j \leq r$, where π is a permutation of $\{1, \ldots, r\}$. Then, we have a new collection $\mathscr{P}' = \{P'_1, \ldots, P'_l\}$, where $P'_1 = s_{\pi(1)}P_{\pi(1)}a_{\pi(1)} \cup a_{\pi(1)}P'_1u \cup uP_1t_1, P'_i = s_{\pi(i)}P_{\pi(i)}a_{\pi(i)} \cup a_{\pi(i)}P'_iw_i \cup w_iP_it_i$ for $2 \leq i \leq r$ and $P'_j = P_j$ for $r+1 \leq j \leq l$. We see that $\mathscr{P}'$ is a feasible collection of G' and satisfies (1) and (2), but $V(\cup_{1\leq i\leq l} P'_i) \subset V(\cup_{1\leq i\leq l} P_i) - \{x\}$, which contradicts (3). □

Let $N := N(G^1 - A \cup B) - V(G^1)$. By the choice of J, we see that $N \subset \cup_{i=1}^{q-1} V(C_i) \cup V(D)$.

We may assume that $N \subset V(D)$. If not, there exists C_h such that $1 \leq h \leq q-1$ and $V(C_h) \cap N \neq \emptyset$, then $x \in N(C_h) \cap V(a_kP_kb_k - \{a_k, b_k\}) \neq \emptyset$ for some $k \in J$. By Claim, there exist r vertex disjoint paths in $G^1 - \{x\}$ from A to B, say $a_{\pi(j)}P'_jb_j, 1 \leq j \leq r$, where π is a permutation of $\{1, \ldots, r\}$. Then we have a new collection $\mathscr{P}' = \{P'_1, \ldots, P'_l\}$, where $P'_i = s_{\pi(i)}P_{\pi(i)}a_{\pi(i)} \cup a_{\pi(i)}P'_ib_i \cup b_iP_it_i$ for $1 \leq i \leq r$ and $P'_j = P_j$ for $r+1 \leq j \leq l$. We see that $\mathscr{P}'$ is a feasible collection in G', such that $V(\cup_{1\leq i\leq l} P'_i) \subset V(\cup_{1\leq i\leq l} P_i \cup C_q) - \{x\}$, then $\mathscr{P}'$ either contradicts (1) or satisfies (1) but contradicts (2).

We may assume that there exists $k \in J$ such that $|V(D) \cap N(a_kP_kb_k - \{a_k, b_k\})| \geq 2$. Otherwise, for any $j \in J$, we have $|V(D) \cap N(a_jP_jb_j - \{a_j, b_j\})| \leq 1$; since $N = V(D) \cap N = \cup_{j\in J}(V(D) \cap N(a_jP_jb_j - \{a_j, b_j\}))$, we have $|N| \leq r \leq l$. Note that $|V(D)| \geq 2l + m$ and $l + m \geq 2$, so $|V(D - N \cup N_q)| \geq 2l + m - l - 1 \geq 1$, which means $D - N \cup N_q \neq \emptyset$. Note that $N \cup A \cup B \cup N_q$ is a cut of G' separating C_q from $D - N \cup N_q$, but

$|V(N\cup A\cup B\cup N_q)| \le 3l+1$, contradicting to the $(30l+10m)$-connectedness of G'.

Let $\{v_1,v_2\} \subset V(D)\cap N(a_kP_kb_k - \{a_k,b_k\})$ and $\{x,y\} \subset V(a_kP_kb_k - \{a_k,b_k\})$ such that $v_1 \ne v_2, v_1x \in E(G), v_2y \in E(G), y \in xP_kb_k - \{b_k\}$. By claim, there exist r vertex disjoint paths in $G^1 - V(xP_ky)$ from A to B, say $a_{\pi(j)}P'_jb_j, 1 \le j \le r$, where π is a permutation of $\{1,\dots,r\}$. Then we have a new collection $\mathscr{P}' = \{P'_1,\dots,P'_l\}$, where $P'_i = s_{\pi(i)}P_{\pi(i)}a_{\pi(i)} \cup a_{\pi(i)}P'_ib_i \cup b_iP_it_i$ for $1 \le i \le r$ and $P'_j = P_j$ for $r+1 \le j \le l$. We see that $\mathscr{P}'$ is a feasible collection in G', such that $V(\cup_{1\le i\le l}P'_i) \subset V(\cup_{1\le i\le l}P_i \cup C_q) - V(xP_ky)$. But $\mathscr{P}'$ contradicts (1), since $D(\mathscr{P}) \cup V(xP_ky) \subset D(\mathscr{P}')$. This completes the proof of Theorem 1.2. □

3. Concluding remarks

We note that in Theorem 1.2, those l internally vertex disjoint s-t paths $P_1,\dots,P_l$ are not induced; but we can strengthen the result by asking $P_i - \{s,t\}$ be induced for all $1 \le i \le l$. The function $f(l,m) = 30l + 10m + 2$ is likely not optimal since we use the result that $10k$-connected graph is k-linked, and $10k$ is not known to be optimal for the k-linkage problem. It is easy to see that an improvement on the k-linkage problem will give us a better function $f(l,m)$. We point out that a similar argument (after slight modification) gives a different and shorter proof of the theorem in [5] mentioned in Section 1.

Our result motivates us to propose the following question:

Question. For any integers $k,l > 0$ and $m \ge 0$, there exists a function $f(k,l,m)$ such that the following holds. For any distinct vertices $s,t,v_1,\dots,v_m$ in any $f(k,l,m)$-connected graph G, there exist l internally vertex disjoint s-t paths $P_1,\dots,P_l$ such that for any subset $I \subset \{1,\dots,l\}$, $G - \cup_{i\in I}V(P_i)$ is k-connected and $\{v_1,v_2,\dots,v_m\} \subset V(G) - \cup_{1\le i\le l}V(P_i)$.

We see that when $l=1, m=0$ this question is equivalent to Conjecture 1.1, and Theorem 1.2 shows that this question is true when $k=2$.

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