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The Erdős-Faber-Lovász conjecture – the uniform regular case

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We consider the Erdős-Faber-Lovász (*EFL*) conjecture for hypergraphs that are both regular and uniform. This paper proves that for fixed degree, there can be only finitely many counterexamples to EFL on this class of hypergraphs. The theorem is a direct application of a graph theoretic result of Alon, Krivelevich and Sudakov from 1999. This result combined with the known results for dense hypergraphs shows that any counterexample to *EFL* must be somewhere in the range between sparse and dense values.

1. Introduction

Definition 1.1. A hypergraph D is linear if any pair of vertices is contained in at most one hyperedge. We say a linear hypergraph D is a (n, d, r) linear set system if the vertex set X has size n, each hyperedge (called a "block") contains exactly d vertices, and each element $x \in X$ is contained in exactly r blocks. We say that D is well-colorable if the blocks can be colored with n colors so that no two intersecting blocks have the same color.

In the case that every pair is contained in some block, D is a $(n, d, 1)$ block design.

We shall prove the following theorem.

Theorem 1.1. *Let D be a (n, d, r) linear set system. Then*

(1) if $r \leq d + 1$, D is well-colorable;
(2) there is a universal constant C so that for all $d \geq C$, if $n \geq Cd^2$, D is well-colorable.

2. Remarks

In 1972, Paul Erdős, László Lovász and I got together at a tea party in my apartment in Boulder, Colorado. This was a continuation of the discussions we had had a few weeks before in Columbus, Ohio, at a conference on hypergraphs. We talked about various conjectures for linear hypergraphs analogous to Vizing's theorem for graphs (see [7]). Finding tight bounds in general seemed difficult so we created an elementary conjecture that we thought would be easy to prove. We called this the n sets problem: given n sets, no two of which meet more than once and each with n elements, color the elements with n colors so that each set contains all the colors. In fact, we agreed to meet the next day to write down the solution. Thirty-eight years later, this problem is still unsolved in general. (See [8] for a survey of what is known.)

To see that Theorem 1.1 is an instance of the n sets problem, consider the hypergraph H which is the dual of D. The dual is formed by taking the blocks of D as a vertex set. For every vertex x in D, a hyperedge in H is formed consisting of all those blocks containing x. Then H is a collection of n sets each with r elements, each element is in exactly d sets, and any two sets meet at most once. The instance of the n sets problem is created by padding each of the sets with $n - r$ isolated elements. A well-coloring of the blocks of D corresponds to a coloring of the vertices of H so that all vertices within an edge have distinct colors.

We know that if $n \leq d^2$ (the dense case) then coloring is possible by a greedy algorithm (Sánchez-Arroyo [9]). What we deal with here is the sparse case, $n \geq Cd^2$. We show that if d is large enough, we have to look for counterexamples in a middle ground of n not too small and not too large. Note that for fixed d there are infinitely many (n, d, r) linear set systems (for example, the $(n, d, 1)$ block designs) and only finitely many of them fail to satisfy the hypothesis of Theorem 1.1.

3. Proof of Theorem 1.1

The proof relies on the following theorem of Alon, Krivelevich and Sudakov [2], which extends a previous result of Ajtai, Komlós, and Szemerédi [1].

Theorem 3.1 (AKS). *The chromatic number of any graph with maximum degree Δ in which the number of edges in the induced subgraph on the set of all neighbors of any vertex does not exceed $\frac{\Delta^2}{f}$ is at most $c(\frac{\Delta}{\log f})$ for some fixed constant c.*

Let us assume that D is not well-colorable. We work with the clique graph G of the hypergraph H. The clique graph uses the ν vertices of H, so that two vertices form an edge if they are in the same r-element set in H. Any proper coloring of the vertices of G is a coloring of the vertices H so vertices within an edge have distinct colors, and vice versa. Note that the sets of H become cliques of size r in G, so the degree of a vertex in G is exactly $\Delta = d(r-1)$. By Brooks' Theorem, G can be colored in no more than Δ colors, so we only have to deal with cases where $\Delta > n$. Let us call this the Brooks consequence:

$$n \le \Delta - 1 = d(r-1) - 1 \tag{1}$$

Now we state some facts about (n, d, r) linear set systems:

$$n - 1 \ge r(d-1) \tag{2}$$

$$nr = \nu d \tag{3}$$

Note that the Brooks consequence and inequality (2) imply the first statement in the theorem, and so we can assume that $r > d+1$. We proceed by bounding the number N_x of edges in the neighborhood of the vertex x in G. This number is made of up two types of edges. First, there are the edges which do not contain x in each of the d cliques of size r at x. The second type of edges are the edges which are formed by the at most $n-d$ blocks in H which do not contain x. Each of these sets can intersect each of the sets containing x at most once. Thus each of these sets can contribute at most a clique of edges of size d. This gives

$$N_x \le d\binom{r-1}{2} + (n-d)\binom{d}{2}. \tag{4}$$

We write this quantity in terms of Δ. We have these rearrangements of inequalities (1) and (2) and equation (3):

$$r = \frac{\Delta}{d} + 1$$

$$n \le \Delta - 1$$

$$n \ge \Delta\left(1 - \frac{1}{d}\right) + d$$

We substitute these into the inequality (4) to get

$$\begin{aligned} N_x &\le \frac{d}{2}\left(\frac{\Delta}{d} - 1\right)\frac{\Delta}{d} + \Delta\frac{d(d-1)}{2} \\ &= \frac{\Delta^2}{2d} - \frac{\Delta}{2} + \frac{\Delta}{2}(d-1)d \le \frac{\Delta^2}{2d} + \frac{\Delta d}{2}(d-1). \end{aligned}$$

Let

$$\frac{1}{f} = \frac{1}{2d} + \frac{d}{2\Delta}(d-1)$$

which gives

$$\frac{1}{2d} + \frac{(d-1)^2}{2(n-d)} \leq \frac{1}{f} \leq \frac{1}{2d} + \frac{d(d-1)}{2n}.$$

Then $N_x \leq \frac{\Delta^2}{f}$, so $\chi(G) \leq c\frac{\Delta}{\log f} \leq c\frac{d(n-d)}{(d-1)\log f}$ by AKS. We need

$$\chi(G) \leq c\frac{d(n-d)}{(d-1)\log f} \leq n$$

so we rearrange this equation to get the condition we desire as

$$\log f \geq c\frac{d(n-d)}{n(d-1)} = c\frac{1-d/n}{1-1/d}.$$

Since the case when $d = 1$ is uninteresting, we assume $d \geq 2$ to get

$$\frac{1-d/n}{1-1/d} \leq \frac{1}{1-1/d} \leq 2.$$

Thus we need $\log f \geq 2c$, meaning $f \geq e^{2c} = C > 1$.

Now we need to solve

$$\frac{1}{2d} + \frac{d(d-1)}{2n} \leq \frac{1}{C}.$$

Thus we definitely need $d > \frac{C}{2}$, which gives us

$$n \geq C\frac{d(d-1)}{2-C/d}. \tag{5}$$

Let $d \geq C$. Then $2 - \frac{C}{d} \geq 1$ so we have

$$Cd^2 \geq Cd(d-1) \geq C\frac{d(d-1)}{2-C/d}.$$

Thus equation (5) is satisfied and therefore $n \geq Cd^2$ implies that $\chi(G) \leq n$.

4. Matchings and block designs

A $(n, d, 1)$ block design is a (n, d, r) linear set system where inequality (2) is an equality (equivalently, each pair of points is contained in exactly one block). In this section, we explore a relationship between colorings of certain $(n, d, 1)$ block designs and matchings. Again, we work with the clique graph G of the dual hypergraph H of the block design D. We restrict our attention to the special case where $r = kd$ for some integer k (which implies that $\nu = kn$). We let G' denote the complement of G and R be the representative graph of the k-element cliques in G'. This means that the vertices of R are the k-element cliques of G', which are the k-element independent sets of G. Two vertices of R are connected if they intersect.

For each vertex x in G', let t_x be the number of k-cliques containing x. This number is a constant which can be computed from the parameters of the design. Let t be the total number of k-cliques in G', that is, the number of vertices in R. Then $t = \frac{\nu t_x}{k} = nt_x$. In addition, the largest clique in R has size t_x (see, for example, Proposition 3 in section 1.8 of [3]). Thus we can prove the following theorem.

Theorem 4.1. *If the graph R has chromatic number equal to the size of its largest clique, then G can be colored in n colors.*

Proof. Note that if the chromatic number of R is t_x then each color class has n vertices. These n vertices correspond to n vertex-disjoint k-cliques in G' and the vertices of each of these k-cliques are independent sets in G and so we have colored G in n colors. □

In the case $d = 2$, the graphs involved are dual to the complete graph and R is associated with the intersection graph of the maximum matchings. It can be correctly colored. We will prove that G is n chromatic when $k = 2$. The result follows from a theorem of Faudree, Gould, Jacobsen and Schelp (see [4]).

Theorem 4.2 (FGJS)**.** *If G is a 2-connected graph of order ν such that for all distinct nonadjacent vertices x and y*

$$|N(x) \cup N(y)| \geq \frac{2\nu - 1}{3}$$

then G contains a hamiltonian circuit.

Corollary 4.3. *If G is a connected graph of order ν such that for all distinct nonadjacent vertices x and y*

$$|N(x) \cup N(y)| \geq \frac{2\nu - 2}{3}$$

then G contains a hamiltonian path.

Proof. Add a vertex x to G which connects to every vertex, and note that the conditions of Theorem 4.2 are satisfied. Take the hamiltonian circuit and remove x to get a hamiltonian path in G. □

Theorem 4.4. *Let D be a $(n, d, 1)$ block design with $r = 2d$. The blocks can be colored with n colors so that no two intersecting blocks have the same color.*

Proof. We use the corollary to *FGJS* on G' to construct a hamiltonian path. Note that $|V(G')| = |V(G)| = \nu = 2n = 4d^2 - 4d + 2$. To compute the size of the union of the neighborhoods of two nonadjacent vertices of G', we see

$$\begin{aligned} |N_{G'}(x) \cup N_{G'}(y)| &= |(V(G) \backslash N_G(x)) \cup (V(G) \backslash N_G(y))| - 2 \\ &= |V(G)| - |N_G(x) \cap N_G(y)| - 2 \end{aligned}$$

where we subtract 2 to discount x and y. Note that x and y are nonadjacent in G' when they are adjacent in G. We want to bound their common neighborhoods. Each vertex in G is in exactly d cliques of size $2d$. The vertices x and y must be in one common clique E. They share $2d - 2$ common neighbors within E. Each is contained in $d - 1$ cliques other than E. Since each clique can have at most 1 vertex in common, this gives at most $(d-1)^2$ common neighbors outside of E, meaning

$$|N_G(x) \cap N_G(y)| \leq 2d - 2 + (d - 1)^2 = d^2 - 1.$$

This gives

$$|N_{G'}(x) \cup N_{G'}(y)| \geq (4d^2 - 4d + 2) - (d^2 - 1) - 2 = 3d^2 - 4d + 1.$$

In order to apply the corollary, we have to compare three times this number with $2\nu - 2 = 4d^2 - 4d$, that is, we need

$$9d^2 - 12d + 3 \geq 4d^2 - 4d$$

which yields

$$5d^2 - 8d + 3 \geq 0.$$

This inequality holds for all d.

We now must show G' is connected. Suppose $V(G') = V_1 \cup V_2$ is a disconnection. This means that G contains all edges between V_1 and V_2. Since $|V(G)| = 2n$, without loss of generality we may assume $|V_2| \geq n = 2d^2 - 2d + 1$. Since the degree of each vertex in G is $d(r-1) = 2d^2 - d < 2n - 1$, no vertex is connected to all others. Thus any vertex $x \in V_1$ has a non-neighbor y, which must also be in V_1. Any nonadjacent pair in G has exactly d^2 common neighbors. Since x and y are both adjacent to all of V_2, we must have that $d^2 \geq 2d^2 - 2d + 1$, which is a contradiction except when $d = 1$, the uninteresting case.

Thus G' is connected, so it has a hamiltonian path. To complete the proof, we use every second edge of a hamiltonian path in G' as a color class in G. This colors the vertices of G in n colors. □

5. Designs and sufficient coverings

If we can break the blocks of D into disjoint set systems

$$D = T_1 \cup T_2 \cup \ldots \cup T_k$$

and the blocks of T_i can be colored with c_i colors so that $\sum c_i \leq n$, then D can be well-colored. In particular, if each T_i is regular with degree $r_i > 1$, then Brooks' Theorem says that $c_i \leq d(r_i - 1)$. This partition then gives a coloring of D with no more than

$$\sum_{i=1}^{k} d(r_i - 1) = d(r - k)$$

colors. So if $k \geq r - \frac{n}{d} = r - \frac{\nu}{r}$, D can be well-colored. We call any partition of this type a *sufficient covering*.

For a design, $n - 1 = r(d - 1)$ so a sufficient covering has at least $\frac{r-1}{d}$ parts. For triple systems either $n = 6k + 1$ or $n = 6k + 3$. In both cases, the minimum number of parts is k. For the former case, a sufficient covering could have all parts of degree 3. For the latter case, a sufficient covering might have $k - 1$ parts of degree 3 and the remaining one of degree 4. A covering of this type has been conjectured for triple systems in [6] and a computer search [5] of STS(19) shows that all 11 billion of them have a sufficient covering of this type.

More generally, suppose $r = dk + t$ with $0 \leq t < d$. If r is divisible by d then covering by parts of degree d would be a sufficient covering. If r is not divisible by d then $r - 1 = dk + t - 1$. If the remainder $t = 1$, then $k - 1$ parts of degree d and one of degree $d + 1$ is sufficient. Otherwise, $k + 1$ parts are required, say k of degree d and one of degree t.

This leads us to pose the following question:

Does every $(n, d, 1)$ block design have a sufficient covering with all but one part of degree d?

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