# Bernhard Riemann's Paper on Fourier Series

## In Memory of John Coates

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Bernhard Riemann was born in 1826 in Breselenz, a small town in what was then the Kingdom of Hanover, roughly 50 miles South-East of Hamburg. His father was a protestant minister, just able to make a living. Bernhard grew up as the second of six children. His mother died when Riemann was in his early teenage years. He then lived briefly with his maternal grandmother in Hanover, unhappily and homesick. After his grandmother's death, he attended a boarding school in Lüneburg, then and still now a sleepy city more or less halfway between Breselenz and Hamburg. Riemann was a good, but not outstanding student. He put considerable effort into the study of theology and Hebrew, then a subject to be learned if one was to become a Lutheran minister, like his father. The director of his school, recognizing Riemann's interest in mathematics, gave him access to his own library, which included textbooks of mathematics, and in particular Legendre's book on the theory of numbers.

In 1846 Riemann became a student of theology at the University of Göttingen, but he attended also some mathematics lectures. With his father's permission – he would not have defied his father! – he switched to the subject of mathematics, then part of the Faculty of Philosophy. He attended lectures of Gauss. But lectures at German universities at the time were large affairs, with lecturers typically unaware of who their students were. There is no evidence at all that Gauss recognized Riemann's mathematical talent at that time.

In 1847 Riemann moved to the University of Berlin, attending lectures by Dirichlet, Eisenstein, Jacobi, and Steiner. At the time Germany was a loose federation of principalities and kingdoms, but political unrest was brewing, culminating in the creation of a more tightly organized German state which then included Austria, with Berlin as capital. In 1849 Riemann returned to Göttingen, where he received his PhD, with Gauss as "Doktorvater", i.e., PhD advisor, in 1851. The thesis studied branched coverings of regions in the complex plane and holomorphic mappings between them; hence the name "Riemann surface". Riemann was the first mathematician to use what is now called the "Dirichlet Principle". In his report on Riemann's PhD thesis, Gauss did mention the enormous talent of its author.

At Gauss' urging, Riemann was appointed "Privatdozent" in Göttingen, which might be loosely translated as assistant professor. He went through the process of "Habilitation", in effect a second, more advanced PhD thesis, a prerequisite for becoming "Professor", i.e., full professor. In this second thesis, Riemann introduced what we now call the "Riemann integral", and used it to rigorously introduce Fourier series of periodic functions. An additional requirement for the Habilitation was to prepare lectures on three subjects chosen by the lecturer, with the final choice made by the faculty. They chose "Über die Hypothesen welche der Geometrie zu Grunde liegen" – about the hypotheses underlying geometry. In his lecture, Riemann rigorously defined the dimension of Euclidean n-space endowed with what one now calls a "Riemannian metric", studied the geodesics on such a space and introduced the curvature tensor. He also related the curvature to the deviation of the area from that of a flat triangle. One of his contemporaries said that in the audience, only Gauss was able to fully appreciate the depth of these ideas.

As Freudenthal observed in [6], Riemann was the first mathematician to realize the distinction between the curvature of a space in terms of its embedding into Rn on the one hand, and in terms of the metric determined by the embedding on the other. But Riemann was not without critics: he had used what he called the "Dirichlet Principle" to get solutions of Poisson's equation as the minimum of a certain energy functional. As Weierstrass pointed out, the minimum was not unique. In the end David Hilbert gave a rigorous alternative argument using the calculus of variations. Weierstrass and Riemann became rivals in other ways, as well. Riemann had used Cauchy's approach to holomorphic functions, in terms of line integrals, whereas Weierstrass used the representability of holomorphic functions by convergent power series. It is generally recognized that Weierstrass introduced present day standards of rigor into mathematics. But Weierstrass was quite aware of Riemann's enormous talents: he withdrew a paper on Abelian functions when he saw Riemann's paper [9]. It should be noted that Riemann's style is informal and discursive, with results and their proofs not separated from the surrounding text. He died prematurely in 1866, with tuberculosis as cause. In his short life he managed to produce a remarkable amount of work. Richard Dedekind and Heinrich Martin Weber edited the Collected Work of Bernhard Riemann [11]. I shall concentrate on [10] in this note.

The paper [10] is Riemann's "Habilitation" thesis; the "Habilitation" was, and still is, a second Doctoral degree in Germany (and some other European countries) which is meant to demonstrate the author's ability to teach at the university level. Though complex numbers were introduced in the 16th century, they were used primarily to solve polynomial equations. Riemann then launches into a historical discussion: in the middle of the 18th century many mathematicians studied the theory of vibrating strings. The position $y(x,t)$ of a vibrating string, of uniform thickness, fixed at the endpoints $x = 0$ and $x = \ell$ is governed by the differential equation

$$\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2}, \tag{1}$$

with an appropriately chosen constant $\alpha$; this is an approximation, as Riemann points out. The function $y(x)$ must satisfy the initial conditions

$$y(0,t) = y(\ell,t) = 0 \tag{2}$$

for all $t$, of course.

Riemann credits d'Alembert with first describing the differential equation (1) and the matching initial conditions (2) into

$$\frac{\partial^2 y}{\partial u \partial v} = 0 \quad \text{with } u = x + \alpha t, \quad v = \alpha t, \tag{3}$$

then if one knows the function $f(x)$ for $-\ell \leq x < \ell$, one knows it for all values of $x$ by means of the equation

$$f(x) = f(x+2\ell). \tag{4}$$

Besides this partial differential equation, the solution $y(x,t)$ must satisfy the boundary conditions (2):

$$y(0,t) = y(\ell,t) = 0, \tag{5}$$

for all $t$, of course. Riemann defines a new function $\phi(x) = -f(-x)$ which transforms the boundary conditions (2) into

$$f(z) = -\phi(-z), \quad f(\ell+z) = -\phi(\ell-z), \tag{6}$$

which implies[1]

$$f(z) = -\phi(-z) = \phi\big(\ell + (\ell+z)\big) = f(2\ell+z), \tag{7}$$

so $f(z)$ is periodic with period $2\ell$. After d'Alembert had established the periodicity of $f(z)$, he studied functions that are periodic.

Leonard Euler, according to Riemann, gave a better approach to the study of periodic functions $f(z)$. If one defines auxiliary functions

$$g(x) = f(x) - f(-x) \quad \text{and} \quad h(x) = -\alpha\big(f'(x) + f'(-x)\big), \tag{8}$$

then if one knows the values of $g(\cdot)$ and of $h(\cdot)$ for any point $x$ between $-\ell$ and $\ell$, one can determine the values at other points by integration. Euler objected to d'Alembert's argument on the grounds that it required the existence of an analytic expression for $y(x,t)$. Before Euler could answer this objection, Daniel Bernoulli [1] used a very different method. Even earlier Taylor [13] had observed that the function

$$y(x,t) = \sin(n\pi x/\ell)\cos(n\pi\alpha t/\ell), \quad \text{with } n \in \mathbb{Z}_{\geq 0}, \tag{9}$$

[1] There is an apparent misprint: Riemann's third term in (7) is $-\phi(\ell-(\ell+z))$, which would make the identity a tautology.

provides a solutions for the equation (1). That, d'Alembert, explained why a vibrating string could generate tones not only at its ground frequency, but also at integral multiples of the ground frequency. Bernoulli pointed out that the vibrating string could vibrate at all these frequencies simultaneously: the function

$$y(x,t) = \sum_{n\in\mathbb{Z}_{\geq 0}} a_n \sin(n\pi x/\ell)\cos\big(n\pi\alpha(t-\beta_n)/\ell\big), \tag{10}$$

provides a solution for the equation (1). Bernoulli even conducted experiments to support his analytic expressions.

Leonhard Euler [3] countered d'Alembert by stating that the function $y(x,0)$ could be arbitrary, but only if any periodic function of $x$ could be expanded in a Fourier series – Fourier series had been introduced roughly a century earlier. At the time it was unknown whether arbitrary periodic functions could be expanded as Fourier series, or what conditions needed to be met for such an expression to be valid. Euler therefore dismissed Bernoulli's solution of the vibrating string problem.

Since the argument between Euler and d'Alembert remained open, Lagrange – not well known at the time – tried to obtain a solution by considering massless strings weighted with $N$ equal, equidistant weights, and then letting $N$ tend to $\infty$. But this limiting procedure was impossible to justify at the time. That seemed to vindicate d'Alembert. But opinions of mathematicians at the time remained divided.

Bernoulli's results made Euler undertake a new attack on the problem [4]. He pointed out that Bernoulli's solution was valid only if any periodic function could be expressed as a Fourier series. At the time, it was unknown whether any periodic function could be expressed as a Fourier series. Euler applied methods of calculus to decide this question. Lagrange [7] considered Euler's approach correct, but considered his arguments insufficient. D'Alembert, to support his own point of view, doubted that any periodic function could be expressed as a Fourier series. Lagrange [8], on the other hand, believed he could prove this assertion.

For roughly 50 years this question remained open, until Joseph Fourier [5] made the remarkable discovery that a continuous periodic function of period $2\pi$ could be expressed as

$$f(x) = a_1\sin x + a_2\sin 2x + \cdots + \frac{1}{2}b_0 + b_1\cos x + b_1\cos x + b_2\cos 2x + \cdots, \tag{11}$$

with coefficients determined by the equations

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \tag{12}$$

as he showed in [10]. For the first time in the development of Fourier series, Fourier was specific about the hypotheses that $f(x)$ needed to satisfy for such a series expansion was to converge: piecewise continuity was enough.

In 1807 he submitted a paper to the French academy which spelled out these conditions. Lagrange, quite old at the time, found this assertion so unexpected that he opposed it strenuously. Lagrange interpolated periodic functions by finite Fourier series, and it seems strange today that he thought he would understand periodic functions this way. Fourier series, of course, have been widely applied to describe periodic phenomena. It took a long time before the expression of periodic functions in terms of Fourier series was established rigorously.

Cauchy gave a lecture to the Paris Academy in February 1826 in which he tried to establish the expression of periodic functions in terms of Fourier series rigorously, but Dirichlet in 1929, in a paper in Crelle's Journal [2], pointed out that Cauchy's argument fell short of rigorous proof: Cauchy assumed that the function in question extended to a holomorphic function on the holomorphic plane, but that is not the case. It is now clear that much weaker hypotheses suffice – extendability to a half plane is enough, as Riemann himself proved in his "Inauguraldissertation", i.e., in his PhD thesis.

In the article [2] Dirichlet also rigorously proved the validity of Fourier expansions for integrable (in the sense of Riemann integration) functions with only finitely many extrema. Riemann then discusses the difference between absolutely and conditionally convergent series. That distinction, Riemann points out, was not understood in the previous century – probably, according to Riemann, because power series converge absolutely in the largest open interval in which they do converge.

Next Riemann discusses under which conditions a Fourier series converges. His answer is the best possible: the Fourier series of a function of bounded variation converges everywhere. As an additional criterion he mentions a criterion for the integrability functions that are piecewise continuous except for a finite number of points where they may become unbounded: if one removes intervals of length $\epsilon$ around each point of discontinuity, the resulting integral must have a limit as $\epsilon$ tends to zero. That limit is then considered to be the value of the integral. He then gives an example of a discontinuous function that has left and right limits at every point – which need not agree, of course – for which he is nonetheless able define a value of the integral. Finally a discussion of the case of functions continuous except at one point, where the function is allowed to tend to $\infty$; depending on the rate of becoming infinite, there may be a definite limit for the integral. All in all, this comes close to the notion of the Lebesgue integral with respect to the Euclidean measure on the real line!

Riemann then studies a particular example. He uses the notation $(x)$ for the difference between the real number $x$ and the nearest integer; if $x$ lies exactly in the middle between two integers, $(x) = 0$. He then considers the series

$$f(x) = \frac{(x)}{1} + \frac{(2x)}{4} + \frac{(3x)}{9} + \cdots = \sum_{1 \le n < \infty} \frac{(nx)}{n^2}; \tag{13}$$

it converges, as is easy to see, for all values of $x$. If $x = p/2n$, with two relatively prime numbers $p$ and $n$, the limit is described by the identities

$$
\begin{aligned}
\lim_{y\to x, y>x} f(y) &= f(x) - \frac{1}{2n^2}\left(1+\frac{1}{9}+\frac{1}{25}+\cdots\right) = f(x) - \frac{\pi^2}{16n^2}, \\
\lim_{y\to x, y<x} f(y) &= f(x) + \frac{1}{2n^2}\left(1+\frac{1}{9}+\frac{1}{25}+\cdots\right) = f(x) + \frac{\pi^2}{16n^2},
\end{aligned} \tag{14}
$$

but $f(x)$ is continuous at all other points.

The function $f(x)$ is therefore discontinuous at all rational points which, when they are expressed as expressed as quotients $x = p/2n$, with $p$ and $n$ relatively prime, therefore discontinuous infinitely often in any non-empty open interval, but with the number of jumps greater than any particular $\eta > 0$ necessarily finite. It is integrable (in the sense of Riemann integrability, of course). Indeed, that requires only the finiteness of its values, the two properties that it has left and right limits at each point, and that the number of jumps greater or equal to any particular quantity $\sigma > 0$ is finite. For if we apply these considerations, we see that in all intervals which do not contain such jumps, the variations are smaller than $\sigma$, and that the total length which do contain such jumps, can be made as small as one wishes by choosing the intervals appropriately.

It should be mentioned that functions which do not have infinitely many local maxima and minima – which does not include the example (14) – do have the two properties just mentioned except at points where they tend to $\infty$. These functions are therefore integrable in the sense of Riemann, again away from points where they tend to $\infty$, is not difficult to show directly.

Let us now consider the case of a function $f(x)$ which we want to integrate, a function that tends to $+\infty$ at a particular point. We might as assume that this happens at $x = 0$, so that as $x$ tends to zero from above, $f(x)$ grows eventually beyond any given value.

In that case it is not difficult to show that $xf(x)$ cannot stay larger than a pre-given quantity $c$. For if that were to be the case,

$$
\int_x^a f(x)dx > c\int_x^a \frac{dx}{dy} \tag{15}
$$

hence greater than $c(\log\frac{1}{x} - \log\frac{1}{a})$, a quantity that goes to $\infty$ as $x$ tends to zero. Consequently $xf(x)$ must tend to 0 as $x \to 0$ from above unless this function has infinitely many local maxima and minima; otherwise $f(x)$ would not be integrable near zero. On the other hand, if

$$
f(x)x^\alpha = (1-\alpha)f(x)\frac{dx}{dy}, \quad \text{where } y = x^{1-\alpha}, \text{ with } \alpha < 1, \tag{16}
$$

tends to 0 as $x$ tends to zero from above, then it is clear that the integral (15) tends to $-\infty$ as the lower limit of the integral goes to zero. When the integral does converge, one finds that the functions

$$(17)\qquad \begin{aligned} &f(x)x\log\frac{1}{x}=\frac{f(x)dx}{-d\log\log\frac{1}{x}},\quad f(x)x\log\frac{1}{x}\log\log\frac{1}{x}=\frac{f(x)dx}{-d\log\log\log\frac{1}{x}}\cdots,\\ &f(x)x\log\frac{1}{x}\log\log\frac{1}{x}\cdots\log^{n-1}\frac{1}{x}\log^{n}\frac{1}{x}=\frac{f(x)dx}{-d\log^{n+1}\frac{1}{x}}\end{aligned}$$

cannot remain bounded away from zero as $x$ tends to zero from above. Thus – assuming the expression does not have infinitely many local maxima an minima – tends to zero; on the other hand, the integral $\int f(x)dx$ does converge as the lower limit tends to zero from above, provided the expression

$$(18)\qquad f(x)x\log\frac{1}{x}\cdots\log^{n-1}\frac{1}{x}d\left(\log^{n}\frac{1}{x}\right)^{\alpha}=\frac{f(x)dx}{-(\log^{n}\frac{1}{x})^{1-\alpha}}\quad\text{with }\alpha>1,$$

tends to zero as $x\to 0$ from above.

On the other hand, if the function $f(x)$ has infinitely many local maxima and minima, one cannot say anything about how its order of growth near the origin as $x$ tends to zero from above. Indeed, if we assume that its order of growth is given – and that alone determines the qualitative behavior of $|f(x)|$ near the origin – then one can by appropriately determining its sign one can arrange that the integral $\int f(x)dx$ converges as the lower limit goes to $-\infty$. As example Riemann gives the function

$$(19)\qquad \frac{d(x\cos e^{\frac{1}{x}})}{dx}=\cos e^{\frac{1}{x}}+\frac{1}{x}e^{\frac{1}{x}}\sin e^{\frac{1}{x}}.$$

That, says Riemann, should serve as sufficient example for this phenomenon. He then turns to the main subject of this article, the representability of a periodic function by a trigonometric series.

Previous studies of periodic functions, according to Riemann, had the purpose to representing periodic functions occurring in nature by Fourier series; one could therefore prove the representability for an arbitrary periodic function, putting suitable restrictions on the function as necessary, if it served the purpose. For this study of periodic functions. Here, on the other hand, he only wants to impose those conditions that are required if one wants to represent a periodic function by a trigonometric series; he therefore looks first for necessary conditions, and then for those necessary conditions that are also sufficient. According to Riemann, previous studies showed that if a periodic function has suitable properties, it is representable by a Fourier series; now he considers the opposite question: if a periodic function is representable by a trigonometric series, what can one say about its change when the argument is varied? He initially considers the series

$$(20)\qquad a_1\sin x+a_2\sin 2x+\cdots+\frac{1}{2}b_0+b_1\cos x+b_2\cos 2x+\cdots$$

or, to shorten the formula, with $A_0 = \frac{1}{2}b_0$, $A_1 = a_1 \sin x + b_1 \cos x$, $A_2 = a_2 \sin 2x + b_2 \cos 2x, \dots$, he expresses the series as

$$A_0 + A_1 + A_2 + \cdots . \tag{21}$$

Riemann denote this expression by $\Omega$ and its value as a function by $f(x)$, a function that is defined only for those $x$ for which the series does converge.

To make the series converge, it is necessary the its terms tend to zero. If the coefficients $a_n$ and $b_n$ do go to zero as $n \to \infty$, the terms of the series $\Omega$ go to zero uniformly with respect to $x$; otherwise the series can converge only for certain values of $x$. It is then necessary to consider both cases separately.

Riemann first assumes that the terms of the series $\Omega$ tend to zero uniformly in $x$. Under this assumption, the series

$$F(x) = C + C'x + A_0\frac{x^2}{2} - A_1 - \frac{A_2}{4} - \frac{A_3}{9} \cdots , \tag{22}$$

which one obtains by integrating the terms of $\Omega$ twice, converges for all values of $x$; he denotes its value at $x$ by $F(x)$, and so $F(x)$ is therefore integrable in the sense of Riemann integration.

To show both – the convergence of the series and the continuity of the function $F(x)$ – Riemann denotes the sum of the terms up to, and including the term $-\frac{A_n}{(n+1)^2}$ by $N$, and the remainder of the series, i.e., the series

$$-\frac{A_{n+1}}{(n+1)^2} - \frac{A_{n+2}}{(n+2)^2} - \cdots \tag{23}$$

by $R$ and the greatest value of $A_m$ for $m > n$ by $\epsilon$. In that case the value of $R$ satisfies the bound

$$< \epsilon\left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots\right) < \frac{\epsilon}{n} \tag{24}$$

so the sum of the remaining terms can be made as small as one chooses, provided only that $n$ is chosen large enough; consequently the series (22) converges.

The function $\mathbb{F}(x)$ is continuous; i.e., the change in $F(x)$ can be made as small as one wants if one bounds the change in $x$ suitably. For the change in $F(x)$ is consists of the change in $R$ and the change in $N$; evidently one can first require $n$ to be large enough so that $R$ can be made as small as one wants, and then bound the change in $x$ suitably to bound the change of $N$ to make it also as small as one wants.

Riemann then formally states a "Lehrsatz" – i.e., theorem – to the effect that if the series $\Omega$, as defined just below (22), converges, then the expression

$$\frac{F(x+\alpha+\beta) - F(x+\alpha-\beta) - F(x-\alpha+\beta) + F(x-\alpha-\beta)}{4\alpha\beta} \tag{25}$$

converges to the same value as the series, provided $\alpha$ and $\beta$ tend to zero and the ratio of these to two quantities remains bounded from above and below. Indeed,

$$\frac{F(x+\alpha+\beta)-F(x+\alpha-\beta)-F(x-\alpha+\beta)+F(x-\alpha-\beta)}{4\alpha\beta} = A_0+A_1\frac{\sin\alpha}{\alpha}\frac{\sin\beta}{\beta}+A_2\frac{\sin 2\alpha}{2\alpha}\frac{\sin 2\beta}{2\beta}+A_3\frac{\sin 3\alpha}{3\alpha}\frac{\sin 3\beta}{3\beta}+\cdots, \tag{26}$$

or to deal first with the simpler case of $\beta=\alpha$,

$$\frac{F(x+2\alpha)-2F(x)+F(x-2\alpha)}{4\alpha^2}=A_0+A_1\left(\frac{\sin\alpha}{\alpha}\right)^2+A_2\left(\frac{\sin 2\alpha}{2\alpha}\right)^2+\cdots. \tag{27}$$

If $f(x)=A_0+A_1+A_2+\cdots$, and if $f(x)+\epsilon_n=A_0+A_1+A_2+\cdots+A_{n-1}$, then given any $\delta>0$ there must exist an $m=m(n)$ such that $n>m$ implies $\epsilon_n<\delta$. If one chooses $\alpha>0$ small enough so that $m\alpha<\pi$, and if one substitutes $A_n$ for $\epsilon_{n+1}-\epsilon_n$, one obtains the identity

$$\sum_{0\le n<\infty} A_n\left(\frac{\sin(n\alpha)}{n\alpha}\right)^2=f(x)+\sum_{1\le n<\infty}\epsilon_n\left\{\left(\frac{\sin(n-1)\alpha}{(n-1)\alpha}\right)^2-\left(\frac{\sin n\alpha}{n\alpha}\right)^2\right\}. \tag{28}$$

Riemann then expresses this series into three components, by putting together

(29)
1) the terms indexed from $n=1$ to $n=m$,
2) indexed from $n=m+1$ to the greatest integer $\le \dfrac{\pi}{\alpha}$, which he denotes by $s$,
3) from $s+1$ to $\infty$.

The first component consists of a finite number of continuous terms and can therefore be forced to be as small as one wants, by choosing $\alpha$ sufficiently small; the second component is, disregarding signs,

$$<\delta\left\{\left(\frac{\sin m\alpha}{m\alpha}\right)^2-\left(\frac{\sin s\alpha}{s\alpha}\right)^2\right\}; \tag{30}$$

since the $\epsilon_n$ are positive. In order to bound the third term, Riemann expresses its general term as a sum of two components,

$$\epsilon_n\left\{\left(\frac{\sin(n-1)\alpha}{(n-1)\alpha}\right)^2-\left(\frac{\sin(n-1)\alpha}{n\alpha}\right)^2\right\}, \tag{31}$$

and

$$\epsilon_n\left\{\left(\frac{\sin(n-1)\alpha}{n\alpha}\right)^2-\left(\frac{\sin(n\alpha)}{n\alpha}\right)^2\right\}=-\epsilon_n\frac{\sin((2n-1)\alpha)\sin\alpha}{(n\alpha)^2}. \tag{32}$$

It is therefore clear that this term is

$$(33)\qquad < \delta\left\{\left(\frac{1}{(n-1)^2\alpha^2}\right)^2 - \left(\frac{1}{n^2\alpha^2}\right)\right\} + \delta\frac{1}{n^2\alpha}.$$

Consequently the sum of terms indexed by $n = s+1$ to $\infty$ is

$$(34)\qquad < \delta\left\{\frac{1}{(s\alpha)^2} + \frac{1}{s\alpha}\right\},$$

the value of which tends to a limit

$$(35)\qquad \delta\left\{\frac{1}{\pi^2} + \frac{1}{\pi}\right\}$$

as $\alpha$ decreases.

As $\alpha$ tends to zero, the sum

$$(36)\qquad \sum_n \epsilon_n\left\{\left(\frac{\sin(n-1)\alpha}{(n-1)\alpha}\right)^2 - \left(\frac{\sin n\alpha}{n\alpha}\right)^2\right\}$$

goes to a limiting value which cannot be greater than

$$(37)\qquad \delta\left\{1 + \frac{1}{\pi} + \frac{1}{\pi^2}\right\},$$

and which therefore must be zero. Consequently the expression

$$(38)\qquad \frac{F(x+2\alpha) - 2,F(x) + F(x-2\alpha)}{4\alpha^2},$$

which equals

$$(39)\qquad f(x) + \sum \epsilon_n\left\{\left(\frac{\sin(n-1)\alpha}{(n-1)\alpha}\right)^2 - \left(\frac{\sin(n\alpha)}{n\alpha}\right)^2\right\},$$

and thus tends to $f(x)$ as $\alpha$ goes to zero. That establishes our theorem when $\beta = \alpha$. To prove this result in general, we set

$$(40)\qquad \begin{aligned} F(x+\alpha+\beta) - 2F(x) + F(x-\alpha-\beta) &= (\alpha+\beta)^2\big(f(x+\delta_1)\big),\\ F(x+\alpha-\beta) - 2F(x) + F(x-\alpha+\beta) &= (\alpha-\beta)^2\big(f(x+\delta_2)\big),\end{aligned}$$

which implies that

$$(41)\qquad \begin{aligned} &F(x+\alpha+\beta) - F(x+\alpha-\beta) - F(x-\alpha+\beta) + F(x-\alpha-\beta)\\ &\quad = 4\alpha\beta f(x) + (\alpha+\beta)^2\delta_1 - (\alpha-\beta)^2\delta_2.\end{aligned}$$

As consequence of what was just proved, both $\delta_1$ and $\delta_2$ go to zero as soon as $\alpha$ and $\beta$ tend to zero; it follows that

$$(42)\qquad \frac{(\alpha+\beta)^2}{4\alpha\beta}\delta_1 - \frac{(\alpha-\beta)^2}{4\alpha\beta}\delta_2$$

also tends to zero, provided the coefficients of $\delta_1$ and $\delta_2$ do not go to $\infty$; but that does not happen if the ratio $\frac{\beta}{\alpha}$ remains bounded. Consequently the expression

$$\frac{F(x+\alpha+\beta)+F(x+\alpha-\beta)-F(x-\alpha+\beta)+F(x-\alpha-\beta)}{4\alpha\beta} \tag{43}$$

converges to $f(x)$, as had to be proved. Riemann the states "Lehrsatz 2" – i.e., theorem 2 – as follows: the expression

$$\frac{F(x+2\alpha)+F(x-2\alpha)-2F(x)}{2\alpha} \tag{44}$$

converges to zero as $\alpha \to 0$.

In order to prove this theorem, Riemann divides the expression

$$\sum_n A_n \left(\frac{\sin n\alpha}{n\alpha}\right)^2 \tag{45}$$

into three groups, the first of which contains all terms up to a fixed index $m$, with $m$ chosen so that $n \geq m$ implies $A_n \leq \epsilon$; the second group all subsequent terms for which $n\alpha$ is bounded from above by a fixed quantity $c$; and the third containing all remaining terms. One can then see easily that, as $\alpha$ goes to zero, the sum of the terms in the first group remains finite, i.e., bounded by a fixed quantity $Q$; the sum of the terms in the second group is bounded by $\epsilon\frac{c}{\alpha}$; and the sum of the terms in the third group is bounded by

$$\epsilon \sum_{c<n\alpha} \frac{1}{n^2\alpha^2} < \frac{\epsilon}{ac}. \tag{46}$$

Consequently

$$\frac{F(x+2\alpha)+F(x-2\alpha)-2F(x)}{2\alpha}, \tag{47}$$

which equals

$$2\alpha \sum_n A_n \left(\frac{\sin n\alpha}{n\alpha}\right)^2 < 2\alpha \sum_n A_n \left(\frac{\sin n\alpha}{n\alpha}\right)^2 \tag{48}$$

remains bounded by

$$2\big(Q\alpha+\epsilon\big(c+c^{-1}\big)\big), \tag{49}$$

and that implies "Lehrsatz 2".

Riemann then states "Lehrsatz 3". He considers two constants $b$ and $c$, the larger one $c$, and a function $\lambda(x)$ on the interval, which has a continuous first derivative. The function $\lambda(x)$ and its first derivative are continuous $b$ vanish at the boundary points, and whose second derivative does not have infinitely many local maxima and minima. In that case the integral

$$\mu^2 \int_b^c F(x)\cos\big(\mu(x-\alpha)\big)\lambda(x)dx \tag{50}$$

tends to zero as $\mu \to +\infty$.

If one substitutes for $F(x)$ its expression as a series, one obtains for the expression (50) the series $(\boldsymbol{\Phi})$,

$$\mu^2 \int_b^c \left(C + C'x + A_0\frac{x^2}{2}\right)\cos\big(\mu(x-\alpha)\big)\lambda(x)dx - \sum_{1\le n<\infty} \frac{\mu^2}{n^2}\int_b^c A_n \cos\big(\mu(x-a)\big)\lambda(x)dx. \tag{51}$$

The term $A_n\cos(\mu(x-a))$ can be expressed as a linear combination of

$$\cos\big((\mu+n)(x-a)\big),\quad \sin\big((\mu+n)(x-a)\big),\quad \cos\big((\mu-n)(x-a)\big),\quad \sin\big((\mu-n)(x-a)\big). \tag{52}$$

If in this sum one denotes the first two components by $B_{\mu+n}$ and the sum of the last two components by $B_{\mu-n}$, one finds $A_n\cos(\mu(x-a)) = B_{\mu+n} + B_{\mu-n}$, and thus obtains the identities

$$\frac{d^2B_{\mu+n}}{dx^2} = -(\mu+n)^2 B_{\mu+n},\qquad \frac{d^2B_{\mu-n}}{dx^2} = -(\mu-n)^2 B_{\mu-n}, \tag{53}$$

and both $B_{\mu+n}$ and $B_{\mu-n}$ tend to zero as $n \to \infty$ uniformly with respect to $x$. The general term of the series $\boldsymbol{\Phi}$,

$$-\frac{\mu^2}{n^2}\int_b^c F(x)\cos\big(\mu(x-a)\big)\lambda(x)dx, \tag{54}$$

therefore equals

$$\frac{\mu^2}{n^2(\mu+n)^2}\int_b^c \frac{d^2B_{\mu+n}}{d^2x}\lambda(x)dx + \frac{\mu^2}{n^2(\mu-n)^2}\int_b^c \frac{d^2B_{\mu-n}}{d^2x}\lambda(x)dx, \tag{55}$$

or by twice integrating by parts, first regarding $\lambda(x)$, then $\lambda'(x)$ as constant,

$$\frac{\mu^2}{n^2(\mu^2+n)^2}\int_b^c B_{(\mu+n)}\lambda''(x)dx + \frac{\mu^2}{n^2(\mu^2-n)^2}\int_b^c B_{(\mu\pm n)}\lambda''(x)dx, \tag{56}$$

since both $\lambda(x)$ and $\lambda'(x)$ vanish at the upper and lower limits of the integral.

It is not difficult to see that $\int_b^c B_{\mu\pm\nu}\lambda''(x)dx$ tends to zero as $\mu$ goes to infinity, independently of $n$; for this expression is a combination of the integrals

$$\int_b^c \cos(\mu\pm n)(x-a)\lambda''(x)dx,\qquad \int_b^c \sin(\mu\pm n)(x-a)\lambda''(x)dx, \tag{57}$$

and as $\mu \pm n$ goes to $\infty$, these integrals tend to zero, not because $n$ grows indefinitely, but because the coefficients of this expression go to zero. To prove our theorem, it therefore suffices if the sum

$$\sum_n \frac{\mu^2}{n^2(\mu - n^2)}, \tag{58}$$

summed over all values of $n$ which satisfy the conditions $n < -c'$, $c'' < n < \mu - c'''$, $\mu + c^{IV} < n$, remains bounded no matter how $c$ is chosen. For disregarding terms for which

$$-c' < n < c'', \quad \mu - c''' < n < \mu + c^{IV}, \tag{59}$$

which are finite in number, and each of which tends to zero, the series $\mathbf{\Phi}$ remains smaller than this sum, multiplied by the largest value of

$$\int_b^c B_{\mu \pm n} \lambda''(x) dx, \tag{60}$$

which tends to zero.

On the other hand, if the quantity $c$ is strictly greater than one, the sum (58), summed over the indices $n$ in the same ranges as above in (60),

$$\sum_\mu \frac{\mu^2}{(\mu - n)^2 x^2} = \frac{1}{\mu} \sum_n \frac{\frac{1}{\mu}}{(1 - \frac{n}{\mu})^2 (\frac{n}{\mu})^2}, \tag{61}$$

is bounded by

$$\frac{1}{\mu} \int \frac{dx}{(1, -x)^2 x^2}, \tag{62}$$

integrated in over the following intervals:

$$-\infty < x < -\frac{c' - 1}{\mu}, \quad \frac{c'' - 1}{\mu} < x < 1 - \frac{c''' - 1}{\mu}, \quad 1 + \frac{c^{IV} - 1}{\mu} < x < \infty; \tag{63}$$

for if one divides the real line into intervals of length $\frac{1}{\mu}$ symmetrically about the origin, and if in each such interval one takes the smallest value of the function, one bounds all terms of the series since the function has no local maximum.

If one carries out the integration, one obtains the expression

$$\frac{1}{\mu} \int \frac{dx}{x^2(1-x)^2} = \frac{1}{\mu} \left( -\frac{1}{x} + \frac{1}{1-x} + 2\log x - 2\log(1-x) \right) + \text{const.}, \tag{64}$$

and consequently one gets a value between the limits just mentioned which does not go to $\infty$ as $\mu$ tends to infinity.

By means of these results one can make the following remarks about the representability of a function by a trigonometric series, whose individual terms tend to zero:

I. If a periodic function $f(x)$ of period $2\pi$ is to be representable by a trigonometric series, whose individual terms tend to zero, there must exist a continuous function $F(x)$, depending on $f(x)$, so that

$$\frac{F(x+\alpha+\beta)-F(x+\alpha-\beta)-F(x-\alpha+\beta)-F(x-\alpha-\beta)}{4\alpha\beta} \tag{65}$$

tends to $f(x)$ when $\alpha$ and $\beta$ go to zero, provided their ration remains bounded from above and away from zero. In addition the integral

$$\mu^2\int_b^c F(x)\cos\bigl(\mu(x-a)\bigr)\lambda(x)dx \tag{66}$$

must go to zero as $\mu$ grows, provided $\lambda(x)$ and $\lambda'(x)$ vanish at the limits of the integral and are continuous in between, and provided $\lambda''(x)$ does not have indefinitely many local maxima and minima.

II. If conversely these two conditions are satisfied, there exists a trigonometric series, whose coefficients tend to zero, a series that represents the function $f(x)$ at all points where it converges. For if one determines the constants $C'$, $A_0$ so that

$$F(x)-C'x-A_0\frac{x^2}{2} \tag{67}$$

is a periodic function of period $2\pi$, and if one expresses this function according to Fourier's method as a trigonometric series

$$C-\frac{A_1}{1}-\frac{A_2}{4}-\frac{A_3}{9}-\cdots, \tag{68}$$

with coefficients

$$\begin{aligned} C &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(F(x)-C't-A_0\frac{t^2}{2}\right)dt,\\ -\frac{A_n}{n^2} &= \int_{-\pi}^{\pi}\left(F(x)-C't-A_0\frac{t^2}{2}\right)\cos\bigl(n(x-t)\bigr)dt, \end{aligned} \tag{69}$$

then, as explained in section V. of [5], the quantity

$$A_n=-\frac{n^2}{\pi}\int_{-\pi}^{\pi}\left(F(x)-C't-A_0\frac{t^2}{2}\right)\cos\bigl(n(x-t)\bigr)dt \tag{70}$$

must go to zero as $n\to\infty$; that, as mentioned in the text after (21), implies that the series

$$A_0+A_1+A_2+\cdots \tag{71}$$

converges to $f(x)$ at all places where it does converge.

III. Suppose $b<x<c$, and that $\rho(t)$ is a function with the following properties: both $\rho(t)$ and $\rho'(t)$ vanish at $t=b$ and $t=c$ and are continuous on the open interval

$(b,c)$; $\rho''(t)$ does not have infinitely many local maxima and minima; and at $t=x$ the function $\rho$ takes the value 1; both $\rho'$ and $\rho''$ vanish at $x=t$; $\rho'''$ and $\rho^{IV}$ are have finite values an are continuous. In that case the difference between

$$A_0+A_1+\cdots+A_n \tag{72}$$

and the integral

$$\frac{1}{2\pi}\int_b^c F(t)\frac{d^2\frac{\sin((2n+1)(x-t))/2)}{\sin((x-t)/2)}}{dt^2}\rho(t)dt \tag{73}$$

tends to zero as $n\to\infty$. The series

$$A_0+A_1+A_2+\cdots \tag{74}$$

will therefore converge, where it does converge, to $f(x)$.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)\frac{t^2}{2}\frac{d^2\frac{\sin((2n+1)(x-t))/2)}{\sin((x-t)/2)}}{dt^2}\lambda(t)dt \tag{75}$$

tends to a finite limit. It is not difficult to see, using integration by parts, that

$$\frac{1}{2\pi}\int_b^c \left(C't+A_0\frac{t^2}{2}\right)\frac{d^2\frac{\sin((2n+1)(x-t))/2)}{\sin((x-t)/2)}}{dt^2}\rho(t)dt \tag{76}$$

converges to $A_0$. Indeed,

$$A_1+A_2+\cdots+A_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(F(t)-C't-A_0\frac{t^2}{2}\right)\sum_{1\le m\le n}(-m^2)\cos\big(m(x-t)\big)dt \tag{77}$$

or equivalently

$$\begin{aligned}2\sum_{1\le m\le n}(-m^2)\cos\big(m(x-t)\big)&=2\sum_{1\le m\le n}\frac{d^2\cos(m(x-t))}{dt^2}=\frac{d^2\frac{\sin((2n+1)(x-t))/2)}{\sin((x-t)/2)}}{dt^2}\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(F(t)-C't-A_0\frac{t^2}{2}\right)=\frac{d^2\frac{\sin((2n+1)(x-t))/2)}{\sin((x-t)/2)}}{dt^2}dt.\end{aligned} \tag{78}$$

As follows from the argument around the formula (73), this quantity must tend to zero as $n\to\infty$, provided the following conditions are satisfied: both $\lambda$ and $\lambda'$ are continuous; $\lambda''$ does not have infinitely many local maxima and minima, $\lambda(t)$, $\lambda'(t)$ and $\lambda''(t)$ vanish at $t=x$; and both $\lambda'''(t)$ and $\lambda''''(t)$ are finite and continuous.

If one requires $\lambda(t)$ to take the value 1 outside the interval $(b,c)$, and inside that interval the value $1-\rho(t)$, conditions that evidently can be imposed, the difference between the expression $A_1+\cdots+A_n$ and the integral (78)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(F(t)-C't-A_0\frac{t^2}{2}\right)\frac{d^2\frac{\sin((2n+1)(x-t))/2)}{\sin((x-t)/2)}}{dt^2}\rho(t)dt \tag{79}$$

tends to zero as $n \to +\infty$, which implies by partial integration that

(80)
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(C't + A_0\frac{t^2}{2}\right)\frac{d^2\frac{\sin((2n+1)(x-t))/2)}{\sin((x-t)/2)}}{dt^2}\rho(t)dt$$

converges to $A_0$ as $n \to +\infty$.

These investigations have shown that the convergence at $x$ of the series $\Omega$, as defined in (21), depends only on the behavior of the function $f(x)$ near $x$, provided the coefficients of the series tend to zero. Whether the coefficients do tend to zero cannot be decided by its expressions in terms of definite integrals in many cases, but must be established by other means. But one case, in which one can decide this based on the nature of the function directly, needs to be mentioned separately: the case of a function $f(x)$ that is bounded and integrable throughout.

Indeed, if one divides the interval $[-\pi,\pi]$ into subintervals of length $\delta_1, \delta_2, \delta_3, \ldots,$ and if one denotes the greatest variation of $f(x)$ in the first interval by $D_1$, in the second interval by $D_2$, the expression

(81)
$$\delta_1 D_1 + \delta_2 D_2 + \delta_3 D_3 + \cdots$$

can be made arbitrarily small, provided all the $\delta_j$ are small enough.

If one divides the integral $\int_{-\pi}^{\pi} f(x)\sin(n(x-a))dx$, which apart from the factor $1/\pi$ expresses the coefficients of the series, or equivalently the integral $\int_a^{a+2\pi} f(x)\sin(n(x-a))dx$, into integrals over subintervals of length $2\pi/n$, and each of these subintervals contributes at most an amount $2\pi/n$, multiplied by the greatest absolute variation of $f(x)$ in that interval. The sum of these contributions must therefore tend to zero as $n \to +\infty$. Indeed, these contributions can be expressed as

(82)
$$\int_{a+2\pi s/n}^{a+2\pi(s+1)/n} f(x)\sin\big(n(x-a)\big)dx$$

The sine function is positive in the first half of the subinterval, negative in the second. If one denotes the greatest value of $f(x)$ in this subinterval by $M$ and the smallest value by $m$, then it is clear that the integral becomes larger if one replaces $f(x)$ in the first half by $M$ and in the second half by $m$; on the other hand, the integral becomes smaller if one replaces $f(x)$ in the first half by $m$ and in the second half by $M$. In the first case one obtains the value $\frac{2}{n}(M-n)$, and in the second case the value $\frac{2}{n}(m-M)$. Disregarding signs, the integral is smaller than $\frac{2}{n}(M-n)$, and the integral

(83)
$$\int_{a+2\pi}^{a} f(x)\sin\big(n(x-a)\big)dx$$

smaller than

(84)
$$\frac{2}{n}(M_1-m_1)+\frac{2}{n}(M_2-m_2)+\frac{2}{n}(M_3-m_3)+\cdots$$

if one denotes the largest value of $f(x)$ in the $i$-th interval by $M_i$ and the smallest value by $m_i$; this sum must tend to zero as $n \to +\infty$, provided the function $f(x)$ is integrable, since then the length of the subintervals tends to zero. In this case, then, the coefficients of the series tend to zero.

One case remains to be examined, the case when the terms of the series $\Omega$ at $x$ tend to zero as $n \to \infty$, even if this does not happen for all values of the argument. This case can be deduced from the previous one. For if one adds the terms of the series at $x+t$ and $x-t$, one obtains the series

$$2A_0 + 2A_1 \cos(t) + 2A_2 \cos(2t) + \cdots, \tag{85}$$

whose terms go to zero for every value of $t$, and which can therefore be reduced to the previous case. For this purpose, if one denotes the value of the infinite series

$$C + C'x + A_0 \frac{x^2}{2} + A_0 \frac{t^2}{2} - A_1 \frac{\cos(t)}{1} - A_2 \frac{\cos(2t)}{4} - A_3 \frac{\cos(3t)}{9} - \cdots \tag{86}$$

by $G(t)$, so that $\frac{F(x+t)+F(x-t)}{2}$ equals $G(t)$ at all points where the series for $F(x+t)$ and for $F(x-t)$ converge, one obtains the following conclusions:

I. If the terms of the series $\Omega$ – as in (21) – at the point $x$ tend to zero as $n$ tends to infinity, the integral

$$\mu^2 \int_b^c G(t) \cos\big(\mu(t-a)\big) \lambda(t) dt \tag{87}$$

must tend to zero as $\mu$ tends to infinity; here $\lambda(t)$ denotes a function which tends to zero as $\mu$ goes to infinity.

II. Suppose that the terms of the series $\Omega$ – as in example (21) – tend to zero at the point $x$. Then it depends only on the behavior of the function $G(t)$ near zero whether or not the series converges; the difference between the sum

$$A_0 + A_1 + \cdots + A_n \tag{88}$$

and the integral

$$\frac{1}{\pi} \int_0^b G(t) \frac{d^2 \frac{\sin((2n+1)t/2)}{\sin(t/2)}}{dt^2} \rho(t) dt \tag{89}$$

goes to zero as $n \to \infty$, provided $b$ is a constant between 0 and $\pi$, and $\rho(t)$ is a function with the following properties: both $\rho(t)$ and $\rho'(t)$ are continuous and vanish at $t = b$; $\rho''(t)$ does not have infinitely many local maxima and minima; and at $t = 0$, $\rho$ takes the value 1, both $\rho'$ and $\rho''$ vanish, with both $\rho'''(t)$ and $\rho''''(t)$ continuous and taking finite values.

The conditions for the representability of a function by a trigonometric series can be reduced somewhat, and thereby our investigations about the representability of a trigonometric series can go further without making any special assumptions about the nature of the function. For example, the assumption that $\rho''(0) = 0$ can be omitted if in the integral (89) one replaces $G(t)$ by $G(t)-G(0)$. But that does not make a significant difference.

As we turn to the consideration of special cases, we shall first try to complete the investigation of functions, which do not have infinitely local maxima an minima, an investigation that is still possible beyond the efforts of Dirichlet.

As has been remarked before, such a function can be integrated where it does not tend to infinity, and it is evident that this can happen only at a finite number of values of its argument. Dirichlet's argument to the effect that the integral

$$\frac{1}{\pi}\int_x^{x+b} f(t)\frac{\sin((x-t)(n+\frac{1}{2}))}{\sin((x-t)/2)}dt, \quad \text{with } 0<b<\pi, \tag{90}$$

converges to $\pi f(\lim_{x>0,x\to 0} f(x))$ as $n$ tends to $\infty$ is beyond reproach, even if one omits the unnecessary assumption that the function $f(t)$ is continuous. It only remains to be determined under what conditions in these integrals the contributions of places where the function tends to infinity tends to zero as $n$ goes to infinity. This investigation has not been undertaken; only Dirichlet has shown occasionally that this happens under the assumption that the function in question is integrable, an assumption that is unnecessary.

We have seen above that if the terms of the series $\Omega$ tend to zero for every value of $x$, the function $F(x)$, whose second derivative is $f(x)$, must be continuous with finite values, and that

$$\frac{F(x+\alpha)-2F(x)+F(x-\alpha)}{\alpha} \tag{91}$$

has to tend to zero as $\alpha\to 0$. As $t$ goes to zero, if the expression $F'(x+t)-F'(x-t)$ do not have infinitely many local maxima and minima, this expression must tend to a finite limit $L$ as $t$ goes to zero, or it must tend to infinity; it is then evident that

$$\frac{1}{\alpha}\int_0^{\alpha}\big(F'(x+\alpha)-F'(x-\alpha)\big)=\frac{F(x+\alpha)-2F(x)+F(x-\alpha)}{\alpha} \tag{92}$$

must also converge to $L$ or to $\infty$; this quantity can therefore tend to zero only if $F'(x+t)-\mathbb{F}'(x-t)$ converges to zero. Consequently, if $f(x)$ tends to infinity at $x=a$, at least $f(a+t)+f(a-t)$ can be integrated down to $t=0$. That suffices to ensure that the expression

$$\int_b^{a-\epsilon} f(x)\cos\big(n(x-a)\big)dx+\int_{a+\epsilon}^{c} f(x)\cos\big(n(x-a)\big)dx \tag{93}$$

converges as $\epsilon$ tends to zero; this quantity can be made arbitrarily small by choosing n large enough. Since $F(x)$ is continuous and takes finite values, the function $F'(x)$ is integrable down to $x=a$, and since $(x-a)F'(x)$ tends to zero as $x\to a$, provided this function does not have infinitely many local maxima and minima;

this implies that

$$\frac{d}{dx}\big((x-a)F'(x)\big) = (x-a)f(x) + F'(x) \tag{94}$$

can be integrated down to $x = a$. Therefore also the integral $\int f(x)\sin(n(x-a))dx$ can be performed all the way to $x-a$; to force the coefficients of the series to tend to zero, it is evidently only necessary that the integral

$$\int_b^c f(x)\sin\big(n(x-a)\big)dx, \quad \text{with } b < a < c, \tag{95}$$

tends to zero as $n \to \infty$. If one defines

$$\phi(x) = f(x)\sin\big(n(x-a)\big), \tag{96}$$

then, as Dirichlet has shown,

$$\begin{aligned}\int_b^c & f(x)\sin\big(n(x-a)\big)dx \\ &= \int_b^c \frac{\phi(x)}{x-a}\sin\big(n(x-a)\big)dx = \frac{\lim_{x>a, x\to a}\phi(x) + \lim_{x<a, x\to a}\phi(x)}{2}.\end{aligned} \tag{97}$$

Consequently

$$\phi(a+t) + \phi(a-t) = tf(x+t) - tf(x-t) \tag{98}$$

must tend to zero as $t \to 0$. And since $f(a+t) + f(a-t)$ can be integrated down to $t = 0$, the expression

$$tf(a+t) + tf(t-t) \tag{99}$$

must tend to zero as $t \to 0$, and consequently both $tf(a+t)$ and $tf(a-t)$ must tend to zero as $t \to 0$. Disregarding functions that have infinitely many local maxima and minima, to represent a function $f(x)$ by a trigonometric series – a series whose $n$-th coefficient tends to zero as $n \to \infty$ – it is both necessary and sufficient that at points $x = a$ where the function tends to $\pm\infty$, both $tf(a+t)$ and $tf(a-t)$ tend to zero as $t \to 0$ and that also both $tf(a+t)$ and $tf(a-t)$ go to zero as $t \to 0$, and that $f(a+t) + f(a-t)$ can be integrated down to zero.

A function $f(x)$, which does not have infinitely many local maxima and minima, can be represented by a trigonometric series whose coefficients do not tend to zero only at a finite number of values of $x$, since the integral

$$\mu^2 \int_b^c F(x)\cos\big(\mu(x-a)\big)\lambda(x)dx \tag{100}$$

tends to zero as $\mu$ goes to infinity only at a finite number of values of $x$, but we do not need to dwell on this further.

Concerning functions which do have infinitely local maxima and minima, it should be noted that such a function $f(x)$ can be integrable, yet not be representable by a Fourier series. As an example, consider the function $f(x)$ which between 0 and $2\pi$ is given by the formula

$$\frac{d(x^\nu \cos(\frac{1}{x}))}{dx}, \quad \text{and} \quad 0 < \nu < \frac{1}{2}. \tag{101}$$

To see this, consider the integral

$$\int_0^{2\pi} f(x) \cos\big(n(x-a)\big) dx. \tag{102}$$

To speak in general terms, as $n$ tends to infinity, the contribution of a neighborhood of $\sqrt{\frac{1}{n}}$ becomes so large, that the ratio of the integral (102) to the quantity

$$\frac{1}{2} \sin\left(2\sqrt{n} - na + \frac{\pi}{4}\right) \sqrt{\pi} n^{\frac{1-2\nu}{4}} \tag{103}$$

converges to 1, as will be argued below. In order to generalize this example, and to clarify the nature of this matter, let us define

$$\int f(x) dx = \phi(x) \cos\big(\psi(x)\big), \tag{104}$$

with $\phi(x)$ tending to zero and $\psi(x)$ to infinity as $x \to 0$. Furthermore these functions should have a continuous second derivative and not have infinitely many local maxima near the origin. Then

$$f(x) = \phi'(x) \cos\big(\psi(x)\big) - \phi(x)\psi'(x) \sin\big(\psi(x)\big). \tag{105}$$

That makes the integral $\int f(x) \cos(\psi(x)) dx$ equal to the sum of the four integrals

$$\frac{1}{2} \int \phi'(x) \cos\big(\psi(x) \pm n(x-a)\big) dx, \quad -\frac{1}{2} \int \phi(x)\psi'(x) \sin\big(\psi(x) \pm n(x-a)\big) dx. \tag{106}$$

Now, with $\psi(x)$ assumed to be positive-valued, let us consider the term

$$-\frac{1}{2} \int \phi(x)\psi'(x) \sin\big(\psi(x) \pm n(x-a)\big) dx. \tag{107}$$

In this integral, let us examine the value of $x$ at which the sine function changes signs most slowly.

If one defines $y = \psi(x) + n(x-a)$, that happens where $\frac{dy}{dx}$ vanishes, and therefore with $\alpha$ substituted for $x$, where

$$\psi'(\alpha) + n = 0. \tag{108}$$

Let us examine the behavior of the integral

$$-\frac{1}{2} \int_{\alpha-\epsilon}^{\alpha+\epsilon} \phi(x)\psi'(x) \sin(y) dx, \tag{109}$$

in case that $\epsilon$ tends to zero as $n \to \infty$, and let us introduce a new variable $y$. If one defines

$$\beta = \psi(\alpha) + n(\alpha - a), \tag{110}$$

then for sufficiently small values of $\epsilon$,

$$y = \beta + \psi''(\alpha)\frac{(x-\alpha)^2}{2} + \cdots. \tag{111}$$

Here $\psi''(\alpha)$ is positive, since $\psi(x)$ tends to $+\infty$ when $x$ goes to zero from above; furthermore

$$\frac{dy}{dx} = \psi''(\alpha)(x-a) = \pm\sqrt{2\psi''(\alpha)(y-\beta)}, \tag{112}$$

with the sign depending on whether $x-\alpha$ is positive or negative; also

(113)

$$\begin{aligned}
-\frac{1}{2}\int_{\alpha-\epsilon}^{\alpha+\epsilon}\phi(x)\psi'(x)\sin(y)dx &= \frac{1}{2}\int_{\beta+\psi''(\alpha)+\epsilon^2/2}^{\beta}\frac{\phi(\alpha)\psi'(\alpha)\epsilon^2/2}{\sqrt{2\psi''(\alpha)}}\left(\sin(y)\frac{dy}{\sqrt{y-\beta}}\right)\\
&\quad -\frac{1}{2}\int_{\beta}^{\beta+\psi''(\alpha)+\epsilon^2/2}\frac{\phi(\alpha)\psi'(\alpha)\epsilon^2/2}{\sqrt{2\psi''(\alpha)}}\left(\sin(y)\frac{dy}{\sqrt{y-\beta}}\right)\\
&= -\int_0^{\psi''(\alpha)\frac{\epsilon^2}{2}}\sin(y+\beta)\frac{\phi(\alpha)\psi'(\alpha)}{\sqrt{2\psi''(\alpha)}}\frac{dy}{\sqrt{2\psi''(\alpha)}}.
\end{aligned}$$

If one lets $\epsilon$ decrease as $n$ tends to infinity, at a rate such that $\psi''(\alpha)\epsilon^2$ tends to infinity, then

$$-\int_{a-\epsilon}^{a+\epsilon}\phi(x)\psi'(x)\sin\big(\psi(x)+n(x-a)\big)dx = -\sin\left(\beta+\frac{\pi}{4}\right)\frac{\sqrt{\pi}\phi(\alpha)\psi'(\alpha)}{2\sqrt{2\psi''(\alpha)}}, \tag{114}$$

modulo terms of lower order, provided ratio of

$$\int_0^{2\pi}\cos\big(n(y-a)\big)dx \tag{115}$$

to the quantity (114) converges to 1, since its other components tend to one as $n \to \infty$ as $n$ goes to infinity.

If one assumes that $\phi(x)$ and $\psi'(x)$ can be expanded in fractional powers of $x$ near the origin, starting with $x^{\nu}$ in the case of $\phi$ and with $x^{-\mu-1}$ in the case of $\psi'(x)$ – which forces $\nu > 0$ and $\mu \geq$ – then the expansion of

$$\frac{\phi(\alpha)\psi'(\alpha)}{\sqrt{2\psi'(''(\alpha))}} \tag{116}$$

in fractional powers of $x$ starts with $\alpha^{\nu-\frac{\mu}{2}}$, and therefore does not tend to zero near the origin if $\mu \geq 2\nu$. More generally, if $x\psi'(x)$ tends to $+\infty$ as $x \to 0$ – or equivalently, if this is the case for $\frac{\psi(x)}{\log x}$ – then $\phi(x)$ can be chosen so that $\lim_{x\to 0}\phi(x)=0$, but so that

$$\phi(x)\frac{\psi'(x)}{\sqrt{2\psi''(x)}}=\frac{\phi(x)}{\sqrt{-2\frac{d}{dx}\frac{1}{\psi'(x)}}}=\frac{\phi(x)}{\sqrt{-2\lim\frac{1}{x\psi'(x)}}} \tag{117}$$

tends to infinity as $x$ goes to zero. Consequently $f(x)$ is integrable down to zero. On the other hand, the integral

$$\int_0^{2\pi} f(x)\cos\big(n(x-a)\big)\,dx \tag{118}$$

does not tend to zero as $n\to\infty$. As one can see, in the integral $\int_0 f(x)dx$ the contributions near the origin tend to cancel each other, even though their ratio relative to the change in the variable $x$ grows rapidly; however, the factor $\cos(n(x-a))$ has the effect of making these contributions add up.

Even though an integrable periodic function $f(x)$ may have a divergent Fourier series, and even though its $n$-th term may become large as $n$ grows, it is possible for the Fourier series to converge on a dense subset of its domain. An example one might consider the function

$$\sum_{1\le n<\infty}\frac{(nx)}{n}; \tag{119}$$

here $(x)$ is Riemann's notation for the integer closest to $x$. It can be represented by the Fourier series

$$\sum_{1\le n<\infty}\frac{\sum^{\theta}-(-1)^{\theta}}{n\pi}\sin(2n\pi x); \tag{120}$$

with $(x)$ denoting the distance between the number $x$ and the integer closest to $x$ and the variable $\theta$ runs over all the proper divisors of $n$. This function is unbounded in any non-empty open interval, and therefore fails to be integrable in the sense of the Riemann integral. As another example, consider the two series

$$\sum_{0\le n<\infty} c_n\cos\big(n^2x\big),\qquad \sum_{1\le n<\infty} c_n\sin\big(n^2x\big), \tag{121}$$

with positive coefficients $c_n$ that are monotonely decreasing to zero, but such that the finite series $\sum_{1\le s<n} c_s$ does not have a finite limit as $n\to\infty$, whereas $\sum_{1\le s\le n} c_s$ tends to infinity as $n\to\infty$. For if the ratio of $x$ to $2\pi$ is rational, and expressed as a fraction of relatively prime integers has denominator $m$, then these series will converge or diverge with infinite limit, depending on whether the quantities

$$\sum_{0\le n\le m-1}\cos\big(n^2x\big),\qquad \sum_{1\le n\le m-1}\cos\big(n^2x\big) \tag{122}$$

equal zero or are non-zero. Both cases do occur infinitely often in any open interval, according to a well known theorem about "Kreisteinung" – i.e., ruler and compass constructions of regular $n$-gons.

To the same extent a Fourier series $\Omega$ can converge, even though the value of the series obtained by integration,

$$C' + A_0 x - \sum \frac{\frac{dA_n}{dx}}{n^2}, \tag{123}$$

which one obtains by integrating the series $\Omega$ term-by term, fails to be integrable[2] over any non-empty open interval. If, for example, the expression

$$\sum_{1 \le n < \infty} \frac{1}{n^3} (1 - q^n) \log\left(\frac{-\log(1-q^n)}{q^n}\right), \tag{124}$$

with the logarithms chosen so that they vanish at $q = 0$, is developed in increasing powers of $q$, and if one substitutes $e^{ix}$ for $q$, then the imaginary part becomes a trigonometric series; when this series if differentiated twice, it converges for infinitely many values of $x$ in any open interval, whereas its first derivative has an infinite limit infinitely often in any non-empty open interval.

To the same extent – i.e., infinitely often between any two unequal values of $x$ – a trigonometric series can converge even if its $n$-th coefficient does not tend to zero as $n \to \infty$. Here is a simple example of such a series:

$$\sum_{1 \le n < \infty} \sin(n! x\pi), \quad \text{where } n! = 1 \times 2 \times 3 \times \cdots \times n, \tag{125}$$

using customary notation. This series converges not only for rational values of $n$ – in which case the series (125) reduces to a finite series – but also for an infinite number of irrational values of $x$, the simplest of which are the numbers $\sin(1)$, $\cos(1)$, $\frac{2}{e}$ and their integral multiples, odd multiples of $e$, $\frac{e-e^{-1}}{4}$, and so forth.

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