Derived Class Field Theory

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1. Introduction

Our article is in memory of John Coates, in memory of his energy, his generosity of thought, his appreciation of ideas.

(*Barry M.:*)

> He was an inspiration to me from the earliest days that I knew him — when — beginning in 1969 — he was a Benjamin Pierce Assistant Professor at

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Harvard, to the later years, when he was based in Orsay, France and I was at the IHES and when the two of us would jog together as he would explain his latest mathematical thought.

(*Michael H.:*)

I had the privilege of meeting John as a new Ph.D. My thesis had been largely inspired by the Coates–Wiles paper, which had appeared just one year before. John's encouragement and support were precious at this stage of my career. Although I failed to find interesting applications of my thesis work and shifted my attention to other questions, I remained in close contact with John, especially after John moved to France. Always elegant, always diplomatic, always with just the slightest trace of a smile on his lips, during his brief stay at Orsay and the École Normale Supérieure John left an influence on number theory in France that is still felt today. After he settled in Cambridge he did the same for Europe as a whole. John was uniquely effective in helping to build European number theory, and this left a deep impression on me when he moved to France, showing me that it was possible to use the modest powers of a European academic creatively as well as constructively. I never made any decisions that might significantly affect our mathematical community without first consulting John. Inevitably, John's influence on me was primarily mathematical, through his own work and through that of his mathematical descendants. No other number theorist of his generation had such a vast mathematical family as John — I have published papers with eight of them, with more on the way. Inspired by Barry's work on Selmer groups and by unpublished work of Ralph Greenberg, John, together with his student Bernadette Perrin-Riou, reformulated and reinvigorated (classical) Iwasawa theory by extending it to motives. This perspective shaped my return to Iwasawa theory, starting with a joint paper with Jacques Tilouine — the last of John's students in France. And, though this may not be so immediately apparent, it also shapes my thoughts about the project on which this paper is a report.

(*Tony F.:*)

Unfortunately I never had the pleasure of meeting John Coates in person, but I have had many encounters with his mathematics, which was and continues to be an inspiration for me.

2. Beyond Class Field Theory

Let F be a number field and K be an open subgroup of the restricted product $\prod'_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^{\times}$, where the product runs over the set of places $\mathfrak{p}$ of F. Let K_∞ denote the maximal compact subgroup of $(F \otimes_{\mathbb{Q}} \mathbf{R})^{\times}$; thus $K_\infty \xrightarrow{\sim} (\pm 1)^{r_1} \times (S^1)^{r_2}$, where S^1 is the unit circle; here as usual, r_1 is the number of real embeddings and r_2 is the number of pairs of complex embeddings. Define the *idèle class groupoid* as the quotient stack

$$\mathbf{I}_K = \mathbf{I}_{F,K} := [F^{\times} \backslash \mathbf{A}_F^{\times} / K \cdot K_\infty].$$

Here the brackets mean that we take the quotient in the sense of groupoids, or in other words we form the *homotopy quotient*. The "idèle class group" traditionally considered in class field theory is the quotient group $\mathrm{I}_K = \mathbf{F}^\times \backslash \mathbf{A}_F^\times / K \cdot K_\infty$, which can be thought of as the group $\pi_0(\mathbf{I}_K)$ of connected components of $\mathbf{I}_K$. However, the groupoid $\mathbf{I}_K$ has an interesting homotopy type that we will also want to consider.

Traditional class field theory describes $\pi_0(\mathbf{I}_K)$, as K varies, as the abelianizations of certain Galois groups of the maximal field extension of F with ramification dictated by K. However, we have good reason to want to describe the entirety of $\mathbf{I}_K$, and not just its component group, in terms of Galois theory. In other words, we would like to enlarge class field theory to account for the entire idèle class groupoid and not just its group $\pi_0(\mathbf{I}_K)$ of connected components. The purpose of the present article is to explain that this is possible: what we shall see is that within this new framework, $\mathbf{I}_K$ accounts completely for what we call the *derived abelianization* of the absolute Galois group of F.

Why might we want this? The space $\mathbf{I}_K$ is the locally symmetric space associated to the reductive group $G = \mathrm{GL}(1)$ over F. We expect — thanks to ideas of Galatius-Venkatesh [6] — that there is an analogous but perhaps subtler relation between the topology of locally symmetric spaces attached to more general reductive groups and their corresponding Galois representations. See [4] for an introduction to this circle of ideas. The general conjectures seem intractable at present, but they encompass the case $G = \mathrm{GL}(1)$, so we might as well try to solve that (easiest) case first. This will be done in forthcoming joint work of the authors and Arpon Raksit [5]. Although the case of $\mathrm{GL}(1)$ is relatively simple, it arises as a useful technical tool in studying other G (for example, one might want to "twist by characters", or "fix determinants", etc.), and so nailing down this case should help in the more interesting cases as well. The present survey explains a component of [5] that we call "derived class field theory".

In this survey we aim to give an informal and intuitive explanation, therefore omitting technicalities on higher category theory and homotopical algebra, as well as focusing on simplifying special cases. Precise and complete details will appear in the article (joint with Raksit) [5]. See also the slides of Barry M.'s talk at the Coates Memorial Conference at https://bpb-us-e1.wpmucdn.cteveryyounwhaom/sites.harvard.edu/dist/a/189/files/2023/08/Beamer.Coates.2023.07.20.pdf.

We thank the referees at ICCM for their careful reading of the manuscript.

3. A Derived Langlands Correspondence for GL(1) – The Galois Side

3.1 Derived Abelianization

Let G be a discrete group. The abelianization of G is an abelian group G^{ab} for which the projection $G \longrightarrow G^{\mathrm{ab}}$ is the universal solution to the problem of morphisms from G to any abelian group. That is, for an abelian group A,

$$Hom_{\mathrm{gps}}(G,A) = Hom_{\mathrm{ab.gps}}(G^{\mathrm{ab}},A).$$

Explicitly, we have

$$G^{\mathrm{ab}} = G/[G,G] = \mathrm{H}_1(G,\mathbf{Z}). \tag{3.1}$$

Just as G^{ab} gives us one-dimensional homology of G (as in Equation (3.1) above) the *derived abelianization* of G, denoted $G^{\mathrm{ab},\bullet}$, is represented by a simplicial abelian group that is constructed canonically in the appropriate category and captures all of $\mathrm{H}_*(G,\mathbf{Z})$. Specifically, there is a canonical isomorphism

$$\pi_i(G^{\mathrm{ab},\bullet}) \simeq \mathrm{H}_{i+1}(G,\mathbf{Z}) \tag{3.2}$$

for $i \geq 0$.

Intuitively speaking, the derived abelianization should be a kind of "derived functor of abelianization". However, the process of "deriving" the abelianization functor cannot be approached as in classical homological algebra, since the category of groups is far from being the sort of abelian category to which the classical theory of derived functors applies. What one uses instead is Quillen's theory of *homotopical algebra* [11].

Recall that homological algebra is implemented using the notion of chain complex, which however is very specific to abelian categories. In contrast, Quillen's homotopical algebra uses the notion of *simplicial objects*, which applies to totally general categories. A simplicial object of a category C is a collection of objects $C_n \in \mathsf{C}$ for $n \geq 0$, together with maps $C_m \to C_n$ with combinatorics modeled on the maps of standard simplices

$$\Delta_n := \{(t_0,\dots,t_n) \in \mathbb{R}_+^{n+1} \colon \textstyle\sum t_i = 1\}.$$

More precisely, the simplex Δ_n is spanned by the $n+1$ vertices, with the vertex labeled i satisfying $t_i = 1$. There are maps $\Delta_n \to \Delta_m$ induced by non-decreasing maps $\{0,\dots,n\} \to \{0,\dots,m\}$. The category of finite non-empty sets is called the *simplex category* Δ, and a simplicial object of C is a functor $\Delta^{\mathrm{op}} \to \mathsf{C}$. The corresponding functor category $\mathsf{Fun}(\Delta^{\mathrm{op}},\mathsf{C})$ is abbreviated sC.

There is a so-called "Quillen equivalence" between simplicial sets and CW complexes (a class of "nice" topological spaces), which informally says that one can think of simplicial sets and CW complexes as being interchangeable up to homotopy. For this reason, one often refers to simplicial sets as "spaces", and thinks of the adjective "simplicial" as synonymous to "topological". A *simplicial group* (resp. *simplicial abelian group*) is a simplicial object in the category of groups (resp. abelian groups).

The formalism of derived functors makes use, not only of simplicial objects, but also of appropriate generalizations of "quasi-isomorphisms" and "projective resolutions". Such notions are provided by Quillen's theory of *model categories*, which is a specification of distinguished families of morphisms in sC satisfying suitable properties. The existence of a model category structure on sC is not guaranteed, but much work has gone into producing such structures on categories of interest. Often one starts with a standard model structure on the category sSet of simplicial sets, and then bootstraps from this to sets with finitary algebraic structure such as groups, abelian groups, rings, etc. We will not discuss these details; see instead [1]. Once a model category structure is in place, one constructs derived functors by a procedure analogous to the traditional calculus in derived categories, using "projective resolutions". (Actually in [5] this will all be used in a different way, using the framework of ∞-categories [9].)

What does this have to do with homological algebra? Recall that a chain complex is called *connective* if it is supported in non-negative degrees. If C is an abelian category, then we denote by $\mathsf{Ch}_+(\mathsf{C})$ the category of connective chain complexes of objects in C. The *Dold-Kan correspondence* [7, §III.2] gives an equivalence between sC and $\mathsf{Ch}_+(\mathsf{C})$, demonstrating that in the case of abelian categories the "simplicial" theory of homotopical algebra recovers the older "chain complex" theory of homological algebra.

Circling back to abelian groups: we use the formalism of simplicial abelian groups as the context for derived functors involving abelian groups. A simplicial abelian group $G^\bullet$ has homotopy groups $\pi_i(G^\bullet)$; these coincide with the homotopy groups of the topological space corresponding to the underlying simplicial set of $G^\bullet$. The "singular simplices" functor from topological spaces to simplicial sets promotes to a functor from topological abelian groups to simplicial abelian groups. With these preparatory remarks in place, we return to the subject of derived abelianization.

Construction 3.1 (Derived abelianization). The paper [5] gives several explicit descriptions of the derived abelianization $\Gamma^{\mathrm{ab},\bullet}$, which require a bit more language to explain. Instead, we will give a more down-to-earth model for its homotopy type. Let Γ be a simplicial abelian group. Let $(B\Gamma, e)$ be the bar construction on Γ, viewed as a pointed space (see [10, Chapter 16, §5] for an explanation of the bar construction).

Given any pointed space (X,x), there is the infinite symmetric product [8, p.282]

$$\mathrm{Sym}(X,x) = \varinjlim_n \mathrm{Sym}^n(X)$$

where $\mathrm{Sym}^n(X) = X^n/S_n$ and the transition maps append the basepoint x. It is a topological abelian monoid, under concatenation.

Then the homotopy type of the derived abelianization of Γ is represented by the topological abelian group $\Omega\,\mathrm{Sym}(B\Gamma, e)$ where Ω is the (based) loop space functor.

As a sanity check, note that if Γ is discrete, then

$$\pi_0(\Omega\,\mathrm{Sym}(B\Gamma,e)) \cong \pi_1(\mathrm{Sym}(B\Gamma,e)) \cong \mathrm{H}_1(\Gamma;\mathbf{Z}) \cong \Gamma^{\mathrm{ab}}.$$

This affirms the intuition that (for discrete Γ) $\Gamma^{\mathrm{ab},\bullet}$ should be a space whose underlying group of connected components is Γ^{ab}. In fact, the *Dold-Thom theorem* [2] implies that — as signaled in (3.1) above — for all $i \geq 0$ we have:

$$\pi_i(\Omega\,\mathrm{Sym}(B\Gamma,e)) \cong \pi_{i+1}(\mathrm{Sym}(B\Gamma,e)) \cong \mathrm{H}_{i+1}(\Gamma;\mathbf{Z}), \tag{3.3}$$

giving us some understanding of the higher homotopy groups of $\Gamma^{\mathrm{ab},\bullet}$ as well.

Under the Quillen equivalence between CW complexes and simplicial abelian groups, $\Omega\,\mathrm{Sym}(B\Gamma,e)$ may be viewed as a simplicial abelian group. Then under the Dold-Kan equivalence, it corresponds to a connective chain complex. This turns out to be a familiar object: as a chain complex, $\Gamma^{\mathrm{ab},\bullet}$ is quasi-isomorphic to the (reduced, shifted) homology chains $\overline{\mathrm{C}_*(\Gamma,\mathbf{Z})}[-1]$; this is a refinement of (3.3).

Remark 3.2 (Derived abelianization of profinite groups). Since we are interested in Galois groups, we will want to take the derived abelianization of profinite groups. In this case it is natural to modify the derived abelianization construction to produce a profinite abelian group. If Γ is a profinite group, then we denote by $\Gamma^{\mathrm{ab},\bullet}$ its profinite derived abelianization (the universal profinite simplicial abelian group to which it maps). This can be described explicitly as follows: if $\Gamma = \varprojlim_\alpha \Gamma_\alpha$ is a profinite presentation of Γ, then $\Gamma^{\mathrm{ab},\bullet} \cong \varprojlim_\alpha \Gamma_\alpha^{\mathrm{ab},\bullet}$.

3.2 Derived Abelianization of Galois Groups

Let $\Gamma_S = \pi_1(\mathrm{Spec}\,(\mathcal{O}_F[1/S]))$ be the Galois group of the maximal extension F_S/F unramified outside a finite set S of prime ideals, equipped with its natural profinite construction. We let $\Gamma_S^{\mathrm{ab},\bullet}$ be as in Remark 3.2. Then under a profinite version of the Dold-Kan correspondence, we have

$$\Gamma_S^{\mathrm{ab},\bullet} \xrightarrow{\sim} \overline{\mathrm{C}_*}(\Gamma_S,\widehat{\mathbf{Z}})[-1]. \tag{3.4}$$

In particular,

$$\pi_0(\Gamma_S^{\mathrm{ab},\bullet}) = \mathrm{H}_1(\Gamma_S,\widehat{\mathbf{Z}}) \cong \Gamma_S^{\mathrm{ab}} \tag{3.5}$$

is the classical profinite abelianization.

4. A Derived Langlands Correspondence for GL(1) – The Automorphic Side

Class field theory identifies the classical abelianization of $\pi_1(\mathrm{Spec}\,(\mathcal{O}_F[1/S]))$ with the class group of $\mathcal{O}_F[1/S]$. We will now describe an enhancement of this story. To simplify the exposition we will focus on the case where S is empty and F is totally imaginary. Otherwise, there are subleties coming from the interaction of real places with the prime 2. These subtleties are treated in detail in [5].

4.1 The Picard Groupoid of $\mathrm{Spec}\,(\mathcal{O}_F)$

Recall that the class group of $\mathcal{O}_F$ may be defined as the group of equivalence classes of line bundles over $\mathrm{Spec}\,(\mathcal{O}_F)$. A more refined structure is the *Picard groupoid* of $\mathrm{Spec}\,(\mathcal{O}_F)$, which is the category whose objects are line bundles on $\mathrm{Spec}\,(\mathcal{O}_F)$ and morphisms are isomorphisms of line bundles. This construction makes sense more generally on any scheme X, and in fact, the Picard groupoid of X can be naturally promoted to a simplicial abelian group (the underlying simplicial can be taken to be the nerve of the groupoid, and the group operation comes from tensor product of line bundles), which we denote $\mathbf{Pic}(X)$. We abbreviate $\mathbf{Pic}(\mathcal{O}_F) := \mathbf{Pic}(\mathrm{Spec}\,(\mathcal{O}_F))$. In these terms the class group may be described as

$$\mathrm{Cl}(\mathcal{O}_F) = \pi_0\,\mathbf{Pic}(\mathcal{O}_F). \tag{4.1}$$

This is a reflection of the fact that for any scheme X, there is a natural isomorphism between $\mathrm{H}^1(X, \mathbf{G}_m)$ and equivalence classes of line bundles on X. More generally, there is a cohomological description of the Picard groupoid. We may view $\mathrm{H}^1_{\mathrm{fppf}}(X, \mathbf{G}_m)$ as the 0th cohomology group of the cohomology complex $\mathbf{C}^\bullet_{\mathrm{fppf}}(X, \mathbf{G}_m[1])$, which is well-defined in the homotopy category of complexes. There is a truncation functor $\tau^{\leq 0}$ on the category of chain complexes, which extracts the connective cover of a complex, and it is a general fact that the Picard groupoid of X is naturally isomorphic to $\tau^{\leq 0}\mathbf{C}^\bullet_{\mathrm{fppf}}(X, \mathbf{G}_m[1])$, the connective cover of the fppf cohomology[1] complex $\mathbf{C}^\bullet_{\mathrm{fppf}}(X, \mathbf{G}_m[1])$. Taking the 0th cohomology group of this isomorphism recovers (4.1).

4.2 The Picard Groupoid and the Idèle Class Groupoid

If F is the function field of a curve X over a finite field, then Weil's construction identifies the idèle class groupoid $\mathrm{I}_{F,K_{max}}$, where K_{max} is the product over all places v of F of the maximal compact subgroups of $F_v^\times$, with the groupoid of line bundles on X. A similar construction can be applied when F is a number field, which involves metrics at archimedean places. However, these can be ignored under our simplifying assumption that F is totally imaginary. So we can get away (under this assumption) with just considering the Picard groupoid $\mathbf{Pic}(\mathcal{O}_F)$; we then get a natural homotopy equivalence

$$\mathbf{Pic}(\mathcal{O}_F) \simeq \mathbf{I}_{F,K_{max}}.$$

Here the left hand side is the simplicial abelian group defined above, while the right hand side is the topological abelian group $\mathrm{I}_{F,K_{max}}$, viewed as the geometric realization of the left hand side.

We let $K = K_{max}$ in what follows. We abbreviate by $C^*(\mathcal{O}_F, \bullet)$ the cohomology complex of $\mathrm{Spec}\,(\mathcal{O}_F)$ with coefficients in the fppf sheaf $\bullet$. We then have a cohomological description of the idèle class groupoid, as

$$\tau^{\leq 0}\mathrm{C}^\bullet(\mathcal{O}_F, \mathbf{G}_m[1]) \simeq \mathbf{Pic}(\mathcal{O}_F) \simeq \mathbf{I}_{F,K}. \tag{4.2}$$

[1] We could have equivalently taken Zariski or étale cohomology here, but in later situations we will really need to use fppf cohomology.

4.3 The Profinite Completion of the Picard Groupoid and Flat Cohomology

In applications to deformation rings, we need the profinite completion of the derived abelianization of $\pi_1(\operatorname{Spec}\mathcal{O}_F)$. We will compare this with the profinite completion (in the derived sense) of the Picard groupoid. We consider the Kummer exact sequence

$$1 \to \mu_n \to \mathbf{G}_m \xrightarrow{n} \mathbf{G}_m \to 1 \tag{4.3}$$

which we can write as an isomorphism in the derived category of sheaves on the fppf topology of $\operatorname{Spec}(\mathcal{O}_F)$:

$$[\mathbf{G}_m \xrightarrow{n} \mathbf{G}_m] \simeq \mu_n[1]; \quad (\mathbf{G}_m)\hat{} := \varprojlim_n \mathbf{G}_m/n \simeq \varprojlim_n \mu_n[1]. \tag{4.4}$$

The profinite completion of $\mathrm{C}^*(\mathcal{O}_F, \mathbf{G}_m)$ is therefore

$$\mathrm{C}^*(\mathcal{O}_F, \mathbf{G}_m)\hat{} := \varprojlim_n \mathrm{C}^*(\mathcal{O}_F, \mathbf{G}_m)/n \xrightarrow{\sim} \mathrm{C}^*(\mathcal{O}_F, \varprojlim_n \mathbf{G}_m/n) \cong \mathrm{C}^*(\mathcal{O}_F, \mu[1]) \tag{4.5}$$

where $\mu := \varprojlim_n \mu_n$ is the Tate module of roots of unity.

Combining (4.5) with (4.4), we identify the profinite completion of $\mathrm{C}^*(\mathcal{O}_F, \mathbf{G}_m)$ with the flat cohomology complex:

$$\mathrm{C}^*(\mathcal{O}_F, \mathbf{G}_m)\hat{} \quad \simeq \quad \mathrm{C}^*(\mathcal{O}_F, \mu[1]). \tag{4.6}$$

There is a generalization of profinite completion to simplicial sets and simplicial abelian groups. Combining (4.2) and (4.6) yields

$$\mathbf{Pic}(\mathcal{O}_F)\hat{} \xrightarrow{\sim} \tau^{\leq 0}\mathrm{C}^*(\mathcal{O}_F, \mu[2]). \tag{4.7}$$

5. Derived Class Field Theory via Poitou-Tate Duality

5.1 Derived Poitou-Tate Duality

Recall that an oriented manifold M enjoys Poincaré duality, which can be formulated as an isomorphism

$$\mathrm{H}_i(M, \mathbf{Z}) \cong \mathrm{H}_c^{n-i}(M, \mathbf{Z})$$

where $n = \dim M$. In fact, this can be promoted to an isomorphism of complexes (in a suitable localization of the category of complexes)

$$\mathrm{C}_i(M, \mathbf{Z}) \cong \mathrm{C}_c^{n-i}(M, \mathbf{Z}).$$

There is an analogy between number fields and 3-manifolds, under which Poincaré duality is analogous to the so-called Poitou-Tate duality. The latter is a bit subtle, but the upshot is that for a finite fppf sheaf $\mathcal{F}$ with Cartier dual $\mathcal{F}^D := \mathcal{H}om(\mathcal{F}, \mathbf{G}_m)$, there are isomorphisms

$$\mathrm{H}_i(\mathcal{O}_F, \mathcal{F}) \cong \mathrm{H}_c^{3-i}(\mathcal{O}_F, \mathcal{F}^D).$$

Here the compactly supported cohomology groups $\mathrm{H}_c^{3-i}(\mathcal{O}_F, \mathcal{F}^D)$ are a bit more involved to define in general – they are defined formally as cones of restriction maps to Archimedean places, where one also has to replace cohomology by Tate cohomology. However, under our simplifying assumptions we will have $\mathrm{H}_c^{3-i}(\mathcal{O}_F, \mathcal{F}^D) = \mathrm{H}^{3-i}(\mathcal{O}_F, \mathcal{F}^D)$, and we can ignore the compact support condition entirely. It turns out that in this case, similarly to the case of manifolds, one can promote Poitou-Tate duality to an isomorphism

$$\mathrm{C}_*(\mathcal{O}_F, \mathcal{F}) \cong \mathrm{C}^*(\mathcal{O}_F, \mathcal{F}^D[3]).$$

This promotion is quite formal but we prefer to leave the details to [5]. Applying this with $\mathcal{F} = \mu_n$ and taking limits in n, the upshot is a natural isomorphism

$$\mathrm{C}_*(\mathcal{O}_F, \widehat{\mathbf{Z}}) \cong \mathrm{C}^*(\mathcal{O}_F, \mu[3]).$$

Combining this with (4.7) gives the following:

$$\mathbf{Pic}(\mathcal{O}_F)^{\wedge} \xrightarrow{\sim} \tau^{\leq 0}\mathrm{C}^*(\mathcal{O}_F, \mu[2]) \xrightarrow{\sim} \tau^{\leq 0}\mathrm{C}_*(\pi_1(\mathrm{Spec}\,(\mathcal{O}_F)), \widehat{\mathbf{Z}}[-1]). \tag{5.1}$$

5.2 Derived Class Field Theory

Now we put together the isomorphism (5.1) with (3.4) and (4.6) to find the following dual description of the derived abelianization of Galois groups:

Theorem 5.1. *Suppose that F is a totally imaginary number field. Then there is a natural isomorphism of simplicial abelian groups*

$$\left(\pi_1(\mathrm{Spec}\,(\mathcal{O}_F))^{\mathrm{ab},\bullet}\right)^{\wedge} \simeq (\mathbf{I}_{F,K})^{\wedge}$$

where $K = \prod_v \mathcal{O}_{F,v}^{\times}$ is the maximal compact subgroup of the finite idèles, such that applying $\pi_0(-)$ recovers the usual isomorphism of class field theory

$$\pi_1(\mathrm{Spec}\,(\mathcal{O}_F))^{\mathrm{ab}} \cong \pi_0(\mathbf{I}_{F,K}).$$

A complete proof of this Theorem, in the more precise language of ∞-categories and animated sets, will appear in [5].

5.3 Allowing for Ramification

The version of derived class field theory explained above recovers the idèle quotient $\pi_0(\mathbf{I}_{F,K})$ when K is the product of the unit groups $\mathcal{O}_v^{\times}$ at all primes v. For more general K, one can interpret $\pi_0(\mathbf{I}_{F,K})$ as the group of equivalence classes of invertible $\mathcal{O}_F$-modules M with a given *level K structure*. This is defined as follows.

If $K \subset \prod_{v \in S} \mathcal{O}_v$ is the principal congruence subgroup K_J of level J, where $J \subset \mathcal{O}_F$ is an ideal supported at S, then a level K structure is a trivialization

$$\iota : \mathcal{O}_F/J \xrightarrow{\sim} M/JM.$$

In general, any open subgroup $K \subset \prod_{v \in S} \mathcal{O}_v$ contains some K_J, and a level K structure is an equivalence class of level K_J structures for the action of K/K_J. There is a (relatively formal) way to enhance Theorem 5.1 to encompass such level structures, using *relative derived abelianization*, which will be explained in [5].

6. Functoriality

6.1 Behavior Under Finite Field Extensions

Let E/F be a finite extension of degree n. Fix a finite set S of non-archimedean places of F and let $\Gamma_{F,S} = \pi_1(\mathcal{O}_{F,S})$, $\Gamma_{E,S} = \pi_1(\mathcal{O}_{E,S})$. Let $K_F^S = \prod_{v \notin S} \mathcal{O}_v \subset \mathbf{A}_F^\times$, $K_E^S \subset \mathbf{A}_E^\times$, be the corresponding subgroups. The following commutative diagrams are the derived versions of the familiar functorialities of class field theory; the profinite completions have been omitted.

(6.1)
$$\begin{tikzcd} \mathbf{I}_{K_E^S} \arrow[r] \arrow[d, "N_{E/F}"] & \Gamma_{E,S}^{\mathrm{ab},\bullet} \arrow[d, "\iota"] \\ \mathbf{I}_{K_F^S} \arrow[r] & \Gamma_{F,S}^{\mathrm{ab},\bullet} \end{tikzcd}$$

(6.2)
$$\begin{tikzcd} \mathbf{I}_{K_E^S} \arrow[r] & \Gamma_{E,S}^{\mathrm{ab},\bullet} \\ \mathbf{I}_{K_F^S} \arrow[r] \arrow[u, "\iota"] & \Gamma_{F,S}^{\mathrm{ab},\bullet} \arrow[u, "tr"'] \end{tikzcd}$$

The horizontal maps are the derived class field theory equivalences discussed above.

The vertical maps labelled ι are induced by the natural inclusions $F \hookrightarrow E$ or $\Gamma_{E,S} \hookrightarrow \Gamma_{F,S}$, which go in opposite directions for the idèle classes and Galois groups (inclusion followed by abelianization, in the latter case). The map $N_{E/F}$ is the norm map. As usual, the only map that requires explanation is the right-hand vertical map in (6.2). This is the transfer, which on homotopy groups is given by the corestriction on group cohomology.

6.2 Iwasawa Theory

Now suppose $F = F_0 \subset F_1 \subset \cdots \subset F_n \subset \ldots$ is a tower of extensions of F, with $\mathrm{Gal}(F_n/F) \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$, so that $F_\infty = \cup_n F_n$ is a $\mathbb{Z}_p$-*extension* of F. This is the setting of *Iwasawa theory* to which so much of John Coates's work was devoted. Write $\Gamma = \mathrm{Gal}(F_\infty/F)$.

In classical Iwasawa theory, one considers the limit and colimit of the classical abelianizations $(\Gamma^{\mathrm{ab}}_{F_n,S})^{\wedge}_p$ in this tower

$$(\Gamma_{F_\infty,S})^{\wedge}_p := \varprojlim_n (\Gamma^{\mathrm{ab}}_{F_n,S})^{\wedge}_p; \quad (A^{\mathrm{ab}}_{F_\infty,S})^{\wedge}_p = \varinjlim_n (\Gamma^{\mathrm{ab}}_{F_n,S})^{\wedge}_p. \tag{6.3}$$

Here $(-)^{\wedge}_p$ denotes p-adic completion. One views the resulting objects in (6.3) as compact and discrete modules, respectively, over the Iwasawa algebra $\mathbb{Z}_p[[\Gamma]]$, where the double brackets denote the completion of the group algebra with respect to the inverse limit topology. It may be interesting to investigate a derived enhancement of this story.

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