

AUXILIARY MONGE-AMPÈRE EQUATIONS IN GEOMETRIC ANALYSIS

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ABSTRACT. This is an introduction to a particular class of auxiliary complex Monge-Ampère equations which have been instrumental in L^∞ estimates for fully non-linear equations and various questions in complex geometry. The essential comparison inequalities are reviewed and shown to apply in many contexts. Adapted to symplectic geometry, with the auxiliary equation given now by a real Monge-Ampère equation, the method gives an improvement of an earlier theorem of Tosatti-Weinkove-Yau, reducing Donaldson's conjecture on the Calabi-Yau equation with a taming symplectic form from an exponential bound to an L^1 bound.

1. INTRODUCTION

It is well-known that comparisons with an auxiliary equation can be of powerful assistance in the study of partial differential equations. Comparisons with harmonic functions were used early on by De Giorgi [16], and have been applied since in many contexts, including e.g. in the book of L. Simon [58], and in X.J. Wang's method for Schauder estimates for the Poisson equation [70]. Since then, comparisons with many other equations have proved to be effective. Complex Monge-Ampère equations are particularly suitable as auxiliary equations, since the existence and smoothness of their solutions for given right hand sides have been established by Yau [72] in the case of compact Kähler manifolds, and by Caffarelli, Kohn, Nirenberg, and Spruck [6] in the case of the Dirichlet problem. Notable successes of Monge-Ampère equations as auxiliary equation include the estimates of Dinew and Kolodziej [19] relating volume and capacity, the bounds of Song and Tian [60] for the Kähler-Ricci flow, and the entropy estimates of Chen and Cheng [8] for the constant scalar curvature equation.

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Very recently, a specific class of auxiliary Monge-Ampère equations has been instrumental in many significant advances in complex geometry. This class was first introduced by the authors in joint work with F. Tong in [31], and already led to a pure PDE proof of the L^∞ estimates of Kolodziej [45], a goal which had eluded researchers in the field for close to a quarter of a century. As a PDE proof, the method immediately extends Kolodziej's estimates to a general class of non-linear equations satisfying a structural condition. Remarkably, this class has been shown by Harvey and Lawson [43] to be quite large, and include in particular all invariant Garding-Dirichlet operators. In fact the method turns out to be even more flexible and powerful than naively anticipated, and it has had since many unexpected applications. These include stability estimates for Monge-Ampère and Hessian equations [33]; L^∞ estimates for Monge-Ampère equations on nef classes rather than just Kähler classes [34]; sharp modulus of continuity for non-Hölder solutions [35]; extensions to parabolic equations [9]; extensions to equations on Hermitian manifolds [37]; extensions to form-type equations [37]; lower bounds for the Green's function [36]; uniform entropy estimates [38]; and diameter estimates and convergence theorems in Kähler geometry not requiring bounds on the Ricci curvature [40, 39].

The main purpose of this paper is to provide a survey of these developments. We shall describe in some detail the essential features of the particular class of auxiliary complex Monge-Ampère equations of interest. The main applications are sketched, with explanations of how the auxiliary Monge-Ampère equations are used. The precise statements of the results obtained are then given. In all but one case, the full treatment is left to references to the original papers in the literature. The one exceptional case is the application to taming symplectic forms on almost-Kähler manifolds. It is a conjecture of Donaldson [21], motivated by symplectic geometry, that on a compact 4-manifold equipped with an almost-complex structure J and a taming symplectic form Ω , the Calabi-Yau equation would admit a priori bounds to all orders. This conjecture had been reduced by Weinkove [71] to an L^∞ bound for the potential, and subsequently by Tosatti, Weinkove, and Yau [68] to a single exponential estimate. Using an auxiliary real Monge-Ampère equation, we can reduce it further to a single L^1 estimate. This result is treated in detail because it is new and does not appear anywhere else. It may also be noteworthy as evidence that the methods here can extend to the real or symplectic context as well.

2. THE AUXILIARY MONGE-AMPÈRE EQUATION

We begin by describing the class of Monge-Ampère equations which will serve later as comparison equations. Some key inequalities needed for the maximum principle are broadly described, which can be adapted later for different applications.

Let (X, ω_X) be a compact n -dimensional Hermitian manifold. For any Hermitian form ω on X and any smooth function φ with $\sup_X \varphi = 0$, set $\omega_\varphi = \omega + i\partial\bar{\partial}\varphi$, and

consider the relative endomorphism

$$(2.1) \quad h_\varphi = \omega_X^{-1} \omega_\varphi.$$

We denote by $\lambda[h_\varphi]$ the (un)-ordered vector of its eigenvalues. Let $f(\lambda)$ be a function on a convex cone $\Gamma \subset \mathbf{R}^n$ invariant under permutations, and consider the family of equations parametrized by ω ,

$$(2.2) \quad f(\lambda[h_\varphi]) = k_\omega(z), \quad \lambda[h_\varphi] \in \Gamma,$$

where $k_\omega(z)$ is a given positive function. For some estimates, it is convenient to introduce the constant $c_\omega > 0$ defined by the following normalization

$$(2.3) \quad k_\omega(z) = c_\omega e^{F\omega}, \quad \int_X e^{nF\omega} \omega_X^n = \int_X \omega_X^n.$$

We remark that for certain equations such as complex Monge-Ampère equations, it is necessary to add the normalizing constant $c_\omega > 0$ in view of the constraint for $e^{nF\omega}$ in (2.3).

As in [31], we require that $f : \Gamma \rightarrow \mathbf{R}_+$ satisfies the structural conditions:

- (1) $\Gamma \subset \mathbf{R}^n$ is a symmetric cone with

$$(2.4) \quad \Gamma_n \subset \Gamma \subset \Gamma_1.$$

Here Γ_k is the cone of vectors λ with $\sigma_j(\lambda) > 0$ for $1 \leq j \leq k$, where $\sigma_j(\lambda)$ is the j -th symmetric polynomial in λ . In particular, Γ_1 is the half-space defined by $\lambda_1 + \dots + \lambda_n > 0$, and Γ_n is the first octant, defined by $\lambda_j > 0$ for $1 \leq j \leq n$.

- (2) $f(\lambda)$ is symmetric in $\lambda = (\lambda_1, \dots, \lambda_n) \in \Gamma$ and it is homogeneous of degree one;
 (3) $\frac{\partial f}{\partial \lambda_j} > 0$ for each $j = 1, \dots, n$ and $\lambda \in \Gamma$;
 (4) There is a $\gamma > 0$ such that

$$(2.5) \quad \prod_{j=1}^n \frac{\partial f}{\partial \lambda_j} \geq \gamma \quad \forall \lambda \in \Gamma.$$

It is well-known that equations such as the Monge-Ampère equation, with $f(\lambda) = (\prod_{j=1}^n \lambda_j)^{\frac{1}{n}}$, or the Hessian equation with $f(\lambda) = \sigma_k(\lambda)^{\frac{1}{k}}$ where σ_k is the k -th order symmetric polynomial, or the p -Monge-Ampère equation of Harvey and Lawson [41, 42] with

$$f(\lambda) = \left(\prod_I \lambda_I \right)^{\frac{n!}{(n-p)!p!}}$$

where I runs over all distinct multi-indices $1 \leq i_1 < \dots < i_p \leq n$, $\lambda_I = \lambda_{i_1} + \dots + \lambda_{i_p}$, and Γ is the cone defined by $\lambda_I > 0$ for all p -indices I , all satisfy the structural condition (4). A remarkable recent result of Harvey and Lawson [43] is that the

condition (4) actually holds for very large classes of non-linear operators, including all invariant Garding-Dirichlet operators. As noted in [43], the condition (4) also arose independently in [1] in the study of $W^{2,p}$ interior regularity.

We would like to compare the solution φ of the equation (2.2) with the solution ψ of the following complex Monge-Ampère equation

$$(2.6) \quad (\omega + i\partial\bar{\partial}\psi)^n = \frac{\tau(-\varphi + q(z) - s)}{A} k_\omega^n(z) \omega_X^n, \quad \sup_X \psi = 0$$

where $s \geq 0$ is a nonnegative constant, $\tau(t)$ is a smooth strictly positive function, $q(z)$ is a given function, and A is a normalizing constant so that the integrals of both sides of (2.6) coincide. Explicitly, A can be written as in (3.5) below.

Let $G^{j\bar{k}}$ be the linearized operator of $\log f(\lambda)$, defined by

$$(2.7) \quad G^{j\bar{k}} = \frac{\partial}{\partial h_{\bar{k}j}} \log f(\lambda[h]).$$

Here we denote by $\lambda[h]$ the unordered n -tuple of eigenvalues of the endomorphism h . Fix a point $z_0 \in X$. Since ω_X is positive definite, and h_φ is a self-adjoint endomorphism with respect to ω_X , we can choose a holomorphic coordinate system centered at z_0 where $(\omega_X)_{\bar{j}k} = \delta_{jk}$ and $h_\varphi(z_0)$ is diagonal, $(h_\varphi)_{\bar{j}k}(z_0) = \lambda_j \delta_{jk}$. In particular $(\omega_\varphi)_{\bar{k}j}(z_0) = \lambda_j \delta_{jk}$.

Lemma 1. *The linearized endomorphism $G^{j\bar{k}}$ then satisfies the following: for $\lambda \in \Gamma$*

- (a) $G^{j\bar{k}}(z_0) = \frac{1}{f(\lambda)} \frac{\partial f}{\partial \lambda_j} \delta_{jk}$, and $G^{j\bar{k}}$ is positive definite;
- (b) $G^{j\bar{k}}(\omega_\varphi)_{\bar{k}j}(z_0) = 1$ and $G^{j\bar{k}}\omega_{\bar{k}j}(z_0) \geq 0$,
- (c) and, taking into account the equations satisfied by φ and ψ ,

$$(2.8) \quad \frac{1}{n} G^{j\bar{k}}(\omega_\psi)_{\bar{k}j} \geq \left(\frac{\gamma\tau(-\varphi + q - s)}{A} \right)^{\frac{1}{n}}.$$

Proof. The formula in (a) is a classical formula for the linearization of a fully non-linear operator $\log f(\lambda[h])$ at a diagonal matrix h from the theory of non-linear equations [61]. The positive-definiteness of $G^{j\bar{k}}$ is an immediate consequence of the ellipticity condition $\partial_j f(\lambda) > 0$ of the function f . Next, we can write at z_0 ,

$$(2.9) \quad G^{j\bar{k}}(\omega_\varphi)_{\bar{k}j} = \sum_{j=1}^n \frac{1}{f(\lambda)} \frac{\partial f}{\partial \lambda_j} \lambda_j = 1$$

by Euler's relation for homogeneous functions $f(\lambda)$ of degree 1. This proves the first equation in (b). To establish the second equation in (b), we observe that since $(\omega_X)_{\bar{k}j} = \delta_{jk}$ at z_0 , ω can be identified with the relative endomorphism $h_0 = (\omega_X)^{-1}\omega$, which is hermitian and positive with respect to ω_X . The expression $G^{j\bar{k}}\omega_{\bar{k}j}$ can then be identified with the trace $\text{Tr}(Gh_0)$, which is then the inner

product of two positive matrices. As such, it is positive, as can be seen for example by writing it in a basis where both corresponding quadratic forms are diagonal.

Finally, to establish (c), we apply the arithmetic-geometric inequality to write

$$(2.10) \quad \begin{aligned} \frac{1}{n} G^{j\bar{k}}(\omega_\psi)_{\bar{k}j} &\geq (\det(G^{j\bar{k}}(\omega_\psi)_{\bar{k}m}))^{\frac{1}{n}} = (\det G^{j\bar{k}} \cdot \det(\omega_\psi)_{\bar{k}m})^{\frac{1}{n}} \\ &= \left(\frac{1}{f(\lambda)^n} \left(\prod_{j=1}^n \frac{\partial f}{\partial \lambda_j} \right) \cdot \frac{\tau(-\varphi + q - s)}{A} k_\omega(z)^n \det(\omega_X) \right)^{\frac{1}{n}}. \end{aligned}$$

Applying the equation for φ , this inequality simplifies to

$$(2.11) \quad \frac{1}{n} G^{j\bar{k}}(\omega_\psi)_{\bar{k}j} \geq \left(\prod_{j=1}^n \frac{\partial f}{\partial \lambda_j} \cdot \frac{\tau(-\varphi + q - s)}{A} \det(\omega_X) \right)^{\frac{1}{n}}$$

Applying now the structural condition on $f(\lambda)$, we obtain the desired inequality (c). \square

Next, we need a ‘‘comparison function Φ ’’, relating the solution φ of the equation (2.2) to the solution ψ of the auxiliary Monge-Ampère equation (2.6). For the applications considered in this survey, the comparison function Φ is usually of the form

$$(2.12) \quad \Phi = -\varepsilon(-\psi + q(z) + \Lambda)^b - \varphi + \tilde{q}(z) - s$$

with $0 < b < 1$ a fixed constant, ε, Λ non-negative constants to be chosen later, and $\tilde{q}(z)$ a smooth function. We obtain the key inequality relating φ and ψ if we can choose all the data in Φ so as to guarantee that $\Phi \leq 0$ everywhere. To do so, we typically apply the maximum principle, and make use of the following general calculations:

Lemma 2. *Fix a point z_0 and a holomorphic coordinate system centered at z_0 as above. Then we have the following inequality at z_0*

$$(2.13) \quad \begin{aligned} G^{j\bar{k}}\Phi_{\bar{k}j} &\geq \varepsilon n b (-\psi + q(z) + \Lambda)^{b-1} \left(\frac{\gamma \tau(-\varphi + q - s)}{A} \right)^{\frac{1}{n}} - 1 \\ &\quad + G^{j\bar{k}} \{ \omega_{\bar{k}j} + (1 - \varepsilon b (-\psi + q + \Lambda)^{b-1}) (\omega_q)_{\bar{k}j} + \tilde{q}_{\bar{k}j} \}. \end{aligned}$$

Proof. A direct calculation gives

$$(2.14) \quad \begin{aligned} \Phi_{\bar{k}j} &= \varepsilon b (-\psi + q(z) + \Lambda)^{b-1} (\psi - q)_{\bar{k}j} - \varphi_{\bar{k}j} + \tilde{q}_{\bar{k}j} \\ &\quad - \varepsilon b (b - 1) (-\psi + q(z) + \Lambda)^{b-2} (\psi - q)_j (\psi - q)_{\bar{k}} \end{aligned}$$

Rewriting this expression using $(\psi - q)_{\bar{k}j} = (\omega_\psi)_{\bar{k}j} - (\omega_q)_{\bar{k}j}$ and $\varphi_{\bar{k}j} = (\omega_\varphi)_{\bar{k}j} - \omega_{\bar{k}j}$, we obtain

$$G^{j\bar{k}}\Phi_{\bar{k}j} = \varepsilon b (-\psi + q + \Lambda)^{b-1} G^{j\bar{k}}(\omega_\psi)_{\bar{k}j} - \varepsilon b (-\psi + q + \Lambda)^{b-1} G^{j\bar{k}}(\omega_q)_{\bar{k}j}$$

$$(2.15) \quad \begin{aligned} & -G^{j\bar{k}}(\omega_\varphi)_{\bar{k}j} + G^{j\bar{k}}\omega_{\bar{k}j} + G^{j\bar{k}}\tilde{q}_{\bar{k}j} \\ & -\varepsilon b(b-1)(-\psi + q + \Lambda)^{b-2}G^{j\bar{k}}(\psi - q)_j(\psi - q)_{\bar{k}} \end{aligned}$$

Since $G^{j\bar{k}} > 0$ and $0 < b < 1$, we can drop the expression involving $G^{j\bar{k}}(\psi - q)_j(\psi - q)_{\bar{k}}$ from the right hand side. The desired inequality then follows from those in Lemma 1. \square

The lemma is particularly useful when the last term on the right hand side of (2.13) happens to be positive and can be dropped. We obtain then an upper bound for the solution φ of the given equation (2.2) in terms of the solution ψ of the auxiliary Monge-Ampère equation. The following is the simplest illustration, which will be shown later to apply to L^∞ estimates on Kähler manifolds:

Lemma 3. *Let φ and ψ satisfy the equations (2.2) and (2.6), under the preceding hypotheses on the operator $f(\lambda)$. Assume that the function $\tau(t)$ satisfies the condition*

$$(2.16) \quad \tau(t) \geq t^a, \quad t \in [0, \infty)$$

for some fixed power $a > 0$. Then for any $s \geq 0$, we have

$$(2.17) \quad -\varphi - s \leq c_{n,a,\gamma} A^{\frac{1}{a+n}} (-\psi + \Lambda)^{\frac{n}{n+a}}$$

for all $z \in X$ and all $s \geq 0$, if the constants b , ε and Λ are chosen to be

$$(2.18) \quad b = \frac{n}{n+a}, \quad \varepsilon = (nb\gamma^{\frac{1}{n}})^{-\frac{n}{a+n}} A^{\frac{1}{a+n}}, \quad \Lambda^{1-b} = \varepsilon b.$$

Here $c_{n,a,\gamma}$ is a constant depending only on n, a, γ .

Proof. Let Φ be defined as in (2.12), with $q(z) = \tilde{q}(z) = 0$. We apply Lemma 2. Since $q = \tilde{q} = 0$, we have $\omega_q = \omega$ and the last expression on the right hand side of (2.13) reduces to

$$(2.19) \quad \begin{aligned} & G^{j\bar{k}}\{\omega_{\bar{k}j} + (1 - \varepsilon b(-\psi + q + \Lambda^{b-1}))(\omega_q)_{\bar{k}j} + \tilde{q}_{\bar{k}j}\} \\ & = G^{j\bar{k}}\{1 - \varepsilon b(-\psi + \Lambda)^{b-1}\}\omega_{\bar{k}j} \geq (1 - \varepsilon b\Lambda^{b-1})G^{j\bar{k}}\omega_{\bar{k}j} \geq 0 \end{aligned}$$

since $-\psi \geq 0$, $0 < b < 1$, and $G^{j\bar{k}}\omega_{\bar{k}j} \geq 0$ by (b) of Lemma 1.

Let z_0 be a point where Φ attains its maximum on X . We shall show that $\Phi(z_0) \leq 0$. If $-\varphi(z_0) - s \leq 0$, the function Φ is manifestly ≤ 0 at its maximum z_0 , and we are done. Otherwise, we note that $0 \geq G^{j\bar{k}}\Phi_{\bar{k}j}(z_0)$ and apply Lemma 2. As just noted, we can drop the last term on the right hand side of (2.13), and bound $\tau(-\varphi - s)$ from below by $(-\varphi - s)^a$. We find

$$(2.20) \quad 1 \geq n\varepsilon b(-\psi + \Lambda)^{b-1} \left(\frac{\gamma(-\varphi - s)^a}{A} \right)^{\frac{1}{n}}$$

at z_0 , which can be rewritten as

$$(2.21) \quad -(nb\gamma^n)^{-\frac{1}{a+n}} A^{\frac{1}{a+n}} (-\psi + \Lambda)^{\frac{n}{n+a}} - \varphi - s \leq 0.$$

With the choice of b , ε , and Λ indicated in the lemma, we can recognize the left hand side as $\Phi(z_0)$. Since z_0 is a maximum for Φ , it follows that $\Phi(z) \leq 0$ for any $z \in X$. This last statement can be recast as in the form stated in the Lemma. \square

3. APPLICATION TO THE COMPACT KÄHLER CASE

The first application that we discuss is the one where the above class of auxiliary equations was originally introduced [31], in order to provide a PDE proof of the L^∞ estimates for the complex Monge-Ampère equation originally established by Kolodziej [45]. We discuss this case in some detail, since it also serves as a template for other subsequent applications.

Assume in this section that X is a compact Kähler manifold, and both ω_X and ω are Kähler forms. We would like to derive L^∞ estimates for the solutions φ of the equation (2.2). Our goal is to find in the present case of non-linear equations an adaptation of the strategy going back to De Giorgi for L^∞ bounds for linear equations in divergence form. In this strategy, a lower bound for φ is obtained by showing that the set

$$(3.1) \quad \Omega_s = \{\varphi < -s\}$$

is empty starting from some S_0 which can be estimated. We shall deduce this from suitable growth conditions on the function

$$(3.2) \quad \phi(s) = \frac{1}{V_\omega} \int_{\Omega_s} k_\omega^n(z) \omega_X^n, \quad V_\omega = \int_X \omega^n,$$

which will follow themselves from a reverse Hölder inequality for the key function A_s defined for $s > 0$ by

$$(3.3) \quad A_s = \frac{1}{V_\omega} \int_{\Omega_s} (-\varphi - s) k_\omega^n(z) \omega_X^n.$$

For this we need an auxiliary Monge-Ampère equation. Let $\tau_\ell(t)$ be a sequence of smooth strictly positive functions on \mathbf{R} which decreases monotonically to the function $\mathbf{R} \ni t \rightarrow t\chi_{\mathbf{R}_+}(t)$ as $\ell \rightarrow \infty$, and which are uniformly bounded from above by $1 + t\chi_{\mathbf{R}_+}(t)$. Here $\chi_{\mathbf{R}_+}$ is the characteristic function of \mathbf{R}_+ . For each $s \in \mathbf{R}$, let $\psi_\ell(z)$ be the solution of the following auxiliary Monge-Ampère equation

$$(3.4) \quad (\omega + i\partial\bar{\partial}\psi_\ell)^n = \frac{\tau_\ell(-\varphi - s)}{A_{\ell,s}} k_\omega^n(z) \omega_X^n,$$

where the constant $A_{\ell,s}$ is defined by

$$(3.5) \quad A_{\ell,s} = \frac{1}{V_\omega} \int_X \tau_\ell(-\varphi - s) k_\omega^n(z) \omega_X^n.$$

By Yau's theorem [72], the above equation admits a unique smooth and ω -pluri-subharmonic (PSH) solution ψ_ℓ normalized by $\sup_X \psi_\ell = 0$.

We can now apply Lemma 3, with $a = 1$, $b = n/(n+1)$, $q \equiv 0$. Since $\psi_\ell \leq 0$, the condition $-\psi_\ell + q \geq 0$ is satisfied, and we obtain an estimate of the form

$$(3.6) \quad \frac{-\varphi - s}{A_{\ell,s}^{1/(n+1)}} \leq c_{n,\gamma}(-\psi_\ell + c_{n,\gamma}A_{\ell,s})^{\frac{n}{n+1}}$$

where $c_{n,\gamma}$ denote generically positive constants depending only on n and γ . Restricting to the set $\Omega_s = \{-\varphi - s > 0\}$, we can take the $n/(n+1)$ -root of both sides, multiply the resulting inequality by a constant β_0 , take the exponential, and integrate over Ω_s . We obtain

$$(3.7) \quad \int_{\Omega_s} \exp \beta_0 \left(\frac{-\varphi - s}{A_{\ell,s}^{1/(n+1)}} \right)^{\frac{n+1}{n}} \omega_X^n \leq \exp(c_{n,\gamma}\beta_0 A_{\ell,s}) \int_{\Omega_s} \exp(-c_{n,\gamma}\beta_0 \psi_\ell) \omega_X^n.$$

We can now invoke the well-known inequality for α -invariants, which states that for all ω Kähler forms with $\omega \leq \kappa \omega_X$ for some fixed $\kappa > 0$, there exists a constant α so that,

$$(3.8) \quad \int_X \exp(-\alpha_0 \psi) \omega_X^n \leq C(\alpha_0, n, \omega_X, \kappa)$$

for any $\alpha_0 < \alpha$ and any ω -plurisubharmonic function ψ with $\sup_X \psi = 0$ [44, 64]. Thus we have

$$(3.9) \quad \int_{\Omega_s} \exp \left\{ \beta_0 \left(\frac{-\varphi - s}{A_{\ell,s}^{1/(n+1)}} \right)^{\frac{n+1}{n}} \right\} \omega_X^n \leq c_{\alpha_0, n, \omega_X, \kappa, \gamma} \exp(c_{n,\gamma}\beta_0 A_{\ell,s}).$$

We can now let $\ell \rightarrow \infty$ and obtain

$$(3.10) \quad \int_{\Omega_s} \exp \left\{ \beta_0 \left(\frac{-\varphi - s}{A_s^{1/(n+1)}} \right)^{\frac{n+1}{n}} \right\} \omega_X^n \leq c_{\alpha_0, n, \omega_X, \kappa, \gamma} \exp(c_{n,\gamma}\beta_0 A_s),$$

where $A_s = \lim_{\ell \rightarrow \infty} A_{\ell,s}$ with A_s defined as in (3.3).

Let E be the energy of φ defined by

$$(3.11) \quad E = \frac{1}{V_\omega} \int_X (-\varphi) k_\omega^n(z) \omega_X^n.$$

Then $A_s \leq E$ for any $s \geq 0$, and the preceding inequality implies the following,

$$(3.12) \quad \int_{\Omega_s} \exp \left\{ \beta_0 \left(\frac{-\varphi - s}{A_s^{1/(n+1)}} \right)^{\frac{n+1}{n}} \right\} \omega_X^n \leq c \exp(cE)$$

which suffices for our purposes.

The inequality (3.10) is the key inequality in our method. It is not difficult to show that it implies a reverse Hölder inequality. For this, we apply the Young's inequality in the following form

$$(3.13) \quad UV \leq U\eta(U) + V\eta^{-1}(V)$$

for any monotone strictly increasing function $\eta : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ with $\lim_{u \rightarrow 0} \eta(u) = 0$. Here η^{-1} is the inverse of the function η . We make the choice $\eta(u) = (\log(1+u))^p$, $\eta^{-1}(v) = \exp(V^{\frac{1}{p}}) - 1$, and $U = e^{nF_\omega}$, $V = v(z)^p$, where we have rewritten $k_\omega(z)$ as $k_\omega(z) = c_\omega e^{F_\omega}$ as in (2.3). This gives

$$\begin{aligned} e^{nF_\omega} v(z)^p &\leq e^{nF_\omega} \log^p(1 + e^{nF_\omega}) + v(z)^p (e^{v(z)} - 1) \\ &\leq c_p \{e^{nF_\omega} (1 + |nF_\omega|^p) + e^{2v(z)}\}. \end{aligned}$$

Next, take

$$(3.14) \quad v(z) = \frac{1}{2} \beta_0 \left(\frac{-\varphi - s}{A_s^{1/(n+1)}} \right)^{\frac{n+1}{n}}$$

and integrate both sides over Ω_s . In view of (3.10), we find

Lemma 4. *The following inequality holds*

$$(3.15) \quad \frac{1}{V_\omega} \int_{\Omega_s} (-\varphi - s)^{p \frac{n+1}{n}} k_\omega^n(z) \omega_X^n \leq c \frac{c_\omega^n}{V_\omega} A_s^{\frac{p}{n}} (\|e^{nF_\omega}\|_{L^1(\log L)^p} + ce^{cE})$$

where $c = C(X, \omega_X, \beta_0, p)$ is a constant, and $\|e^{nF_\omega}\|_{L^1(\log L)^p} = \int_X |F_\omega|^p e^{nF_\omega} \omega_X^n$.

We observe that A_s is essentially the L^1 norm of $-\varphi - s$, so the inequality we just obtained can be interpreted as a reverse Hölder inequality. It implies immediately the following growth rate for the function $\phi(s)$ as in (3.2),

Lemma 5. *Fix $p > n$. Then we have the inequalities*

- (a) $A_s \leq B_0 \phi(s)^{1+\delta_0}$, for $\delta_0 = (p-n)/np > 0$,
- (b) For any $r > 0$, $A_s \geq r\phi(r+s)$,
- (c) $r\phi(s+r) \leq B_0 \phi(s)^{1+\delta_0}$, where the constant B_0 is given by $[c \frac{c_\omega^n}{V_\omega} (\|e^{nF_\omega}\|_{L^1(\log L)^p} + ce^{cE})]^{\frac{1}{p}}$.

Proof. We begin with the proof of (a). By Hölder's inequality, we have

$$(3.16) \quad A_s \leq \left\{ \frac{1}{V_\omega} \int_{\Omega_s} (-\varphi - s)^{p \frac{n+1}{n}} k_\omega^n(z) \omega_X^n \right\}^{\frac{n}{(n+1)p}} \left(\frac{1}{V_\omega} \int_{\Omega_s} k_\omega^n(z) \omega_X^n \right)^{\frac{1}{q}}$$

where q is defined by $\frac{n}{p(n+1)} + \frac{1}{q} = 1$. The first factor on the right hand side has been shown to be bounded by a multiple of $A_s^{1/(n+1)}$. Putting this factor on the

left hand side yields

$$(3.17) \quad A_s \leq B_0 \left(\frac{1}{V_\omega} \int_{\Omega_s} e^{nF_\omega} \omega_X^n \right)^{\frac{1+n}{qn}} = B_0 \phi(s)^{\frac{1+n}{qn}}.$$

The exponent $1 + n/(qn)$ is readily worked out to be $1 + \delta_0$, establishing (a). To see (b), we observe that, trivially, $\Omega_{r+s} = \{\varphi < -s - r\} \subset \Omega_s$, and hence

$$(3.18) \quad A_s = \frac{1}{V_\omega} \int_{\Omega_s} (-\varphi - s) e^{nF_\omega} \omega_X^n \geq r \frac{1}{V_\omega} \int_{\Omega_{s+r}} e^{nF_\omega} \omega_X^n = r \phi(s+r)$$

establishing (b). The last statement (c) is a trivial consequence of (a) and (b). \square

We can now invoke a classic lemma of De Giorgi, which says that positive monotone decreasing functions $\phi(s)$ which tend to 0 as $s \rightarrow \infty$, and satisfy the growth rate condition stated as (c) in Lemma 5, must vanish for some $s \geq S_0$, where $S_0 > 0$ can be estimated in terms of B_0 and δ_0 . But this implies that Ω_s must be empty for $s \geq S_0$, and hence $\varphi \geq -S_0$. Thus we have obtained the following bound for φ [31]:

Theorem 1. *Let φ be a solution to the equation where the operator $f(\lambda)$ satisfies the structural conditions listed in (1–4). Assume that the Kähler form ω satisfies the condition $\omega \leq \kappa \omega_X$ for some positive constant κ . Fix $p > n$. Then we have the L^∞ bound*

$$(3.19) \quad \varphi \geq -C$$

where C is a constant depending only on $X, \omega_X, n, p, \gamma, \kappa$ and the following three quantities

$$(3.20) \quad \frac{c_\omega^n}{V_\omega}, \quad E, \quad \text{Ent}_p(\omega) = \|e^{nF_\omega}\|_{L^1(\log L)^p}.$$

In the case of the Monge-Ampère equation $f(\lambda) = (\prod_{j=1}^n \lambda_j)^{\frac{1}{n}}$, it is easy to see that $\frac{c_\omega^n}{V_\omega} = \frac{1}{[\omega_X^n]}$ and that the energy E can be bounded by a constant depending only on $X, \omega_X, n, \gamma, \kappa$ and $\text{Ent}_{\{p=1\}}(\omega)$. Thus Theorem 1 gives L^∞ estimates for φ depending only on the entropy $\text{Ent}_p(\omega)$ for $p > n$, recovering in this way the L^∞ estimates of Kolodziej [45], even in the more general version allowing degenerations of the background metric established by Demailly-Pali [17], and Eyssidieux-Guedj-Zeriahi [22]. We stress however that it holds for the general class of operators $f(\lambda)$ satisfying the structural conditions (1–4), which is quite large, as shown by Harvey and Lawson [43].

4. APPLICATION TO ENERGY ESTIMATES FROM ENTROPY

The previous Theorem 1 had, for a fixed background metric ω , reduced L^∞ bounds for general non-linear equations of the form (2.2) to the three quantities

$c_\omega^n/[\omega^n]$, E , and $\text{Ent}_p(\omega)$ for $p > n$. Actually, it has been known for some time that bounds for the energy E can be derived from bounds for the entropy [2, 8]. This was even one of the key steps in the work of X.X. Chen and J.R. Cheng [8] on the equation for Kähler metrics of constant scalar curvature. As shown in [31], the arguments of [8] can be adapted to provide bounds for the energy, assuming that the entropy is bounded. However, all of these bounds depend on the background metric ω , and are only useful for fixed ω .

This problem of bounding uniformly the energy by the entropy is addressed by the following theorem [37]:

Theorem 2. *Let (X, ω_X) be a compact n -dimensional Kähler manifold without boundary. Let ω be any Kähler form on X with*

$$(4.1) \quad \omega \leq \kappa \omega_X$$

for some constant $\kappa > 0$. Consider the equation (2.2) with the operator $f(\lambda)$ satisfying the conditions (1-4). Then for any $p > 0$, any C^2 solution φ to (2.2) satisfies the following

(i) *Trudinger-like inequalities*

$$(4.2) \quad \int_X e^{\alpha(-\varphi)^q} \omega_X^n \leq C_T,$$

(ii) *and energy-like estimates*

$$(4.3) \quad \int_X (-\varphi)^{pq} e^{nF\omega} \omega_X^n \leq C_e.$$

Here the exponent q is given by $q = \frac{n}{n-p}$ if $p < n$, and can be any strictly positive exponent if $p \geq n$. The constants C_T and C_e are computable constants depending only on $n, p, q, \omega_X, \kappa, \gamma$, and upper bounds for the following two quantities

$$(4.4) \quad \frac{c_\omega^n}{V_\omega}, \quad \text{Ent}_p(\omega) = \int_X e^{nF\omega} |F_\omega|^p \omega_X^n,$$

and the term $\alpha > 0$ is a constant that depends only on $n, p, \gamma, \frac{c_\omega^n}{V_\omega}$ and κ .

We observe that, in the case of a fixed background Kähler metric ω , this theorem was proved as Theorem 3 in [31]. As stressed above, the point of the new theorem is to have uniform estimates, even when the background metric ω may degenerate to the boundary of the Kähler cone. For the same reason, we consider the case $p > n$. When $p > n$ and the background Kähler form ω is fixed, it follows from [31], Theorem 1, that the solution φ of the equation is actually bounded, and the above Trudinger-like and energy-like estimates follow at once. But here again, the existing results do not give the estimates uniform in ω that we seek.

We only describe the version of the auxiliary complex Monge-Ampère equation that we need, together with the comparison inequality, referring to [38] for full details.

Thus let $a = pq = \frac{np}{n-p}$ (a is any positive number if $p = n$). We solve the following complex Monge-Ampère equation

$$(4.5) \quad (\omega + i\partial\bar{\partial}\psi_{s,\ell})^n = \frac{\tau_\ell(-\varphi - s)^a}{A_{s,\ell}} c_\omega^n e^{nF_\omega} \omega_X^n, \quad \sup_X \psi_{s,\ell} = 0.$$

Here the constant $A_{s,\ell}$ is defined by

$$(4.6) \quad A_{s,\ell} = \frac{c_\omega^n}{V_\omega} \int_X \tau_\ell(-\varphi - s)^a e^{nF_\omega} \omega_X^n$$

to make the equation (4.5) compatible. By assumption $[\omega]$ is a Kähler class, so by Yau's theorem [72], (4.5) admits a unique smooth solution $\psi_{s,\ell}$. We observe that as $\ell \rightarrow \infty$

$$(4.7) \quad A_{s,\ell} \rightarrow A_s := \frac{c_\omega^n}{V_\omega} \int_{\Omega_s} (-\varphi - s)^a e^{nF_\omega} \omega_X^n.$$

Next we consider comparison functions of the form

$$\Phi := -\varepsilon(-\psi_{s,\ell} + \Lambda)^b - \varphi - s,$$

where ε and Λ are constants, and b is a suitable power. Using the identities in Section 2 and the maximum principle, we then show that

$$(4.8) \quad \Phi \leq 0$$

if the constants are given by

$$(4.9) \quad b = \frac{n}{n+a} \in (0, 1), \quad \text{and} \quad \varepsilon = \frac{1}{\gamma^{1/(n+a)} (nb)^{n/(n+a)}} A_{s,\ell}^{\frac{1}{n+a}},$$

and Λ is chosen so that $\varepsilon b \Lambda^{-(1-b)} = 1$, that is,

$$(4.10) \quad \Lambda = \frac{b^{1/(1-b)}}{(\gamma^{1/(n+a)} (nb)^{n/(n+a)})^{1/(1-b)}} A_{s,\ell}^{\frac{1}{a}}.$$

The inequality $\Phi \leq 0$ on X implies that $-\varphi - s$ can be controlled by $c_{a,n,\gamma} A_{s,\ell}^{\frac{1}{n+a}} (-\psi_{s,\ell} + A_{s,\ell}^{\frac{1}{a}})^b$, after which the theorem can be established following the strategy of Section 3.

5. APPLICATION TO STABILITY ESTIMATES

It is not difficult to adapt the same method of L^∞ estimates to stability estimates. As in Section 3, let (X, ω_X) be a compact Kähler manifold, and ω be a Kähler metric such that $\omega \leq \kappa \omega_X$ for some $\kappa > 0$. We consider the following complex Monge-Ampère equations

$$(5.1) \quad (\omega + i\partial\bar{\partial}u)^n = c_\omega e^f \omega_X^n, \quad \text{and} \quad (\omega + i\partial\bar{\partial}v)^n = c_\omega e^h \omega_X^n,$$

with the constant $c_\omega = \int_X \omega^n$ and $\int_X e^f \omega_X^n = \int_X e^h \omega_X^n = 1$. The functions u, v are normalized such that

$$\max_X(u - v) = \max_X(v - u).$$

Assume that for some $K > 0$ and $p > n$,

$$\max(\|e^f\|_{L^1(\log L)^p(X, \omega_X)}, \|e^h\|_{L^1(\log L)^p(X, \omega_X)}) \leq K.$$

Theorem 3. *Under these assumptions, there is a constant $C > 0$ depending on n, p, ω_X, κ , and K such that*

$$\sup_X |u - v| \leq C \|e^f - e^h\|_{L^1}^\beta,$$

where $\beta = \beta(n, p) = (n + 3 + \frac{p-n}{pn})^{-1} > 0$.

We outline the main idea of the proof, and refer to [33] for the details. Fix a small $r > 0$. First we may assume $\int_{\{v \leq u\}} (e^f + e^h) \omega_X^n \leq 1$. We then consider the auxiliary equation

$$(5.2) \quad (\omega + i\partial\bar{\partial}\psi)^n = c_\omega \frac{\tau_\ell(-\varphi + q(z) - s)}{A_{\ell,s}} e^h \omega_X^n, \quad \sup_X \psi = 0,$$

with $\varphi = v - (1 - r)u$, $q(z) = -3\beta_0 r$, where β_0 is an upper bound of $\|u\|_{L^\infty}$ and $\|v\|_{L^\infty}$. β_0 exists from Section 3. Take the comparison function

$$(5.3) \quad \Phi = -\varepsilon(-\psi + \Lambda)^{\frac{n}{n+1}} - \varphi + q(z) - s$$

with $\varepsilon = c_1(n)A_{\ell,s}^{1/(n+1)}$ and $\Lambda = c_2(n)(\varepsilon/r)^{n+1}$ for appropriate constants $c_1(n)$ and $c_2(n)$. Arguing as in Section 3, we arrive at $v - u \geq -Cr$ for a constant $C > 0$ depending only on n, p, κ, ω_X and $K > 0$.

We remark that analogous stability estimates as in Theorem 3 hold for complex Hessian equations as well [33, 20].

6. APPLICATION TO THE NEF CASE

So far, we have considered equations whose background form ω is Kähler (although it may degenerate to the boundary of the Kähler cone). In this section, following [34], we show how it can be adapted to nef classes, which are those in the closure of the Kähler cone.

As above, (X, ω_X) is a Kähler manifold. We assume only the class $[\omega]$ is Kähler and $\omega \leq \kappa\omega_X$, but the $(1, 1)$ -form ω may not be positive.

We consider both the Monge-Ampère and Hessian equations. While L^∞ estimates do not hold in the usual form, the following estimates can be established [34]:

Theorem 4. *Assume that (X, ω_X) is an n -dimensional compact Kähler manifold, and let ω be a closed $(1,1)$ -form which may not be positive, but $[\omega]$ is a Kähler class with $\omega \leq \kappa \omega_X$.*

(a) *Let $f(\lambda) = (\prod_{j=1}^n \lambda_j)^{\frac{1}{n}}$, corresponding to the Monge-Ampère equation. Recall that the envelope associated to ω is the ω -PSH function defined by*

$$(6.1) \quad \mathcal{V}_\omega = \sup\{u \mid u \in \text{PSH}(X, \omega), u \leq 0\}.$$

Then for any fixed $p > n$, there is a constant C depending only on $\omega_X, n, p, \kappa, \text{Ent}_p(\omega)$, so that the solution φ of the equation (2.2) satisfies

$$(6.2) \quad 0 \leq -\varphi + \mathcal{V}_\omega \leq C.$$

(b) *Let $f(\lambda) = \sigma_k(\lambda)^{\frac{1}{k}}$, corresponding to the k -th Hessian equation. Define the envelope $\mathcal{V}_{\omega,k}$ corresponding to the Γ_k cone by*

$$(6.3) \quad \mathcal{V}_{\omega,k} = \sup\{v \leq 0\}$$

where v runs over non-negative C^2 functions with the vector of eigenvalues of the relative endomorphism $\omega_X^{-1}(\omega + i\partial\bar{\partial}v)$ lying in the Γ_k cone. Define the energy E_k by

$$(6.4) \quad E_k = \int_X (-\varphi + \mathcal{V}_{\omega,k}) e^{kF\omega} \omega_X^n.$$

Recall also the constant c_ω as defined for the equation (2.2). Then for any $p > n$, there is a constant C depending only on $\omega_X, n, o, \kappa, E_k$, and $\|e^{kF\omega}\|_{L^1(\log L)^p}, \frac{c_\omega^n}{[\omega]^k [\omega_X]^{n-k}}$ so that

$$(6.5) \quad 0 \leq -\varphi + \mathcal{V}_{\omega,k} \leq C.$$

We note that Part (a) on the Monge-Ampère equation had been proved with different methods by Boucksom et al. [5] and Fu-Guo-Song [23]. Part (b) on Hessian equations is new and due to [34].

As in earlier applications, we indicate only the auxiliary Monge-Ampère equation to be used, and the comparison function Φ . We only discuss the proof of Part (a), the one of Part (b) being similar.

A first technical difficulty is that \mathcal{V}_ω is only $C^{1,1}$. However, it has been shown by Berman [2] that there exists a sequence of smooth and strictly ω -PSH functions $\{u_\beta\}_{\beta=1}^\infty$ which converge uniformly to \mathcal{V}_ω . In the following we can use u_β in place of \mathcal{V}_ω , and then take limit $\beta \rightarrow \infty$.

The auxiliary Monge-Ampère equation is then

$$(6.6) \quad (\omega + i\partial\bar{\partial}\psi)^n = \frac{\tau_\ell(-\varphi + u_\beta - s)}{A_{\ell,\beta}} k_\omega^n(z) \omega_X^n, \quad \sup_X \psi = -1$$

and the comparison function Φ is defined by

$$(6.7) \quad \Phi = -\varepsilon(-\psi + u_\beta + 1 + \Lambda)^{\frac{n}{n+1}} - \varphi + u_\beta - s.$$

In the general setting of Section 2, this corresponds to our general ansatz with $q = \tilde{q} = u_\beta$ and the simple shift in the normalization of ψ , with $\sup_X \psi = -1$. We note that an analogue of (2.19) still holds by the fact that $\omega_{u_\beta} > 0$ although ω may not be positive. We can then show that $\Phi \leq 0$, and establish the desired bounds following our general template.

7. APPLICATION TO THE MODULUS OF CONTINUITY

It has been shown by Kolodziej [45] that the solution of the complex Monge-Ampère equation on a compact Kähler manifold is Hölder continuous if the right hand side is of class L^q for some $q > 1$. He has also shown that if the right-hand side is in some Orlicz space (see (2.1.3) in [45]), then it must be continuous. But even so, his arguments are not direct, and we don't have any information on the modulus of continuity of the solution. The modulus of continuity is typically a delicate question. However, it turns out that it can also be addressed using the class of auxiliary Monge-Ampère equations discussed in the present paper.

More specifically, let (X, ω_X) be again a compact Kähler manifold of complex dimension n . We consider the complex Monge-Ampère equation with $\int_X e^F \omega_X^n = \int_X \omega_X^n$

$$(7.1) \quad (\omega_X + i\partial\bar{\partial}\varphi)^n = e^F \omega_X^n, \quad \omega_\varphi = \omega_X + i\partial\bar{\partial}\varphi > 0.$$

Then the following estimate is established in [35]:

Theorem 5. *Fix $p > n$. Then we have*

$$|\varphi(x) - \varphi(y)| \leq \frac{C}{|\log d(x,y)|^\alpha}, \quad \forall x, y \in X$$

for some constant $C > 0$ depending on n, p, ω_X and $\|e^F\|_{L^1(\log L)^p}$. Here $d(x, y)$ denotes the geodesic distance of x, y in the Riemannian manifold (X, ω_X) , and $\alpha = \min\{\frac{p}{n+1}, \frac{p-n}{n}\} > 0$.

The proof of Theorem 5 relies on the following auxiliary complex Monge-Ampère equation:

$$(7.2) \quad (\omega_X + i\partial\bar{\partial}\psi)^n = \frac{\tau_\ell(-\varphi + q(z) - s)}{A_{\ell,s}} e^F \omega_X^n$$

with $q = (1 - |\log \delta|^{-p/(n+1)})\varphi_\delta - 2\delta$ and φ_δ is the (rescaled) Kiselman-Legendre transform of φ at level $\delta > 0$. With the comparison function

$$(7.3) \quad \Phi = -\varepsilon(-\psi + \Lambda)^{\frac{n}{n+1}} - \varphi + q(z) - s,$$

we can argue as in Section 3 to conclude that $\varphi_\delta - \varphi \leq 2\delta + |\log \delta|^{-p/(n+1)}\varphi_\delta + S_\infty$ for $S_\infty = C|\log \delta|^{-(p-n)/n}$. The proof of Theorem 5 then follows from the fact that $\varphi_\delta(z)$ is equal to $\max \varphi$ over the ball $B(z, \delta)$ up to a controlled error term.

8. APPLICATION TO THE HERMITIAN CASE

A striking feature of the above method of auxiliary Monge-Ampère equations is the ease with which it can be adapted to the case of Hermitian manifolds. Thus we let (X, ω_X) be a compact Hermitian manifold with ω_X a fixed Hermitian metric, $\omega_\varphi = \omega_X + i\partial\bar{\partial}\varphi$, $h_\varphi = \omega_X^{-1}\omega_\varphi$, and consider the equation

$$(8.1) \quad f(\lambda[h_\varphi]) = e^F, \quad \lambda[h_\varphi] \in \Gamma, \quad \sup_X \varphi = 0,$$

where the operator $f(\lambda)$ defined on a cone Γ satisfies the structural conditions (1–4) spelled out in Section 2. We have the following result from [37]

Theorem 6. *For any $p > n$, a C^2 solution φ of the equation (8.1) must satisfy the L^∞ bound*

$$(8.2) \quad \sup_X |\varphi| \leq C$$

where C is a constant depending only on $X, \omega_X, n, p, \gamma$ and $\|e^{nF}\|_{L^1(\log L)^p}$.

In the case of the Monge-Ampère equation, L^∞ estimates were first obtained in the Hermitian setting by Cherrier [13] and Tosatti and Weinkove [65], assuming a pointwise bound for e^F . The sharper version with entropy bounds $\|e^{nF}\|_{L^1(\log L)^p}$ was obtained by Dinew and Kolodziej [19], and required a highly non-trivial extension of pluripotential theory to the Hermitian setting. An approach based on envelopes has been recently proposed by Guedj and Lu [30]. Our theorem applies to much more general classes of equations, and its proof is arguably the simplest, as it bypasses the complicated integration by parts with torsion terms which arise in Hermitian geometry. We provide a brief sketch. The original and complete argument can be found in [37].

A first observation is that the previous auxiliary complex Monge-Ampère equation cannot be applied as it is. The reason is that, on a compact Hermitian manifold, unlike on Kähler manifolds, solutions exist only up to an undetermined constant. Because of this, we shall use instead as auxiliary equation the Dirichlet problem for a complex Monge-Ampère equation on a Euclidean ball, which has been shown by Caffarelli, Kohn, Nirenberg, and Spruck [6] to always admit a smooth solution.

Thus, fix r_0 small enough, but depending only on (X, ω_X) so that, for any $z_0 \in X$, there is a coordinate system z centered at z_0 so that

$$(8.3) \quad \frac{1}{2}i\partial\bar{\partial}|z|^2 \leq \omega_X \leq 2i\partial\bar{\partial}|z|^2 \text{ in } B(z_0, 2r_0) = \{|z| < 2r_0\}.$$

Let $x_0 \in X$ be a point where φ attains its minimum. We shall show that

$$(8.4) \quad \varphi(x_0) \geq -C$$

where C is a constant depending only on $X, \omega_X, p, \gamma, \|e^{nF}\|_{L^1(\log L)^p}$, and $\|\varphi\|_{L^1(X, \omega_X)}$. It is not difficult to show by a separate argument that $\|\varphi\|_{L^1(X, \omega_X)}$ is bounded by

a constant depending only n, ω_X , for all functions with $\sup_X \varphi = 0$ with

$$\lambda[h_\varphi] \in \Gamma \subset \Gamma_1 = \{\lambda; \lambda_1 + \dots + \lambda_n > 0\}.$$

The desired theorem (i.e. Theorem 6) would follow.

We introduce now the auxiliary equation. Let $\Omega = B(x_0, 2r_0)$, and for each s with $0 < s < 2r_0^2$, set

$$(8.5) \quad u_s(z) = \varphi(z) - \varphi(x_0) + \frac{1}{2}|z|^2 - s$$

and $\Omega_s = \{z \in \Omega; u_s(z) < 0\}$. Let $\psi_{s,\ell}$ be the solution of the following Dirichlet problem,

$$(8.6) \quad (i\partial\bar{\partial}\psi_{s,\ell})^n = \frac{\tau_\ell(u_s)}{A_{s,\ell}} e^{nF(z)} \omega_X^n \quad \text{on } \Omega, \quad \psi_{s,\ell} = 0 \quad \text{on } \partial\Omega,$$

where the coefficients $A_{s,\ell}$ are defined by

$$(8.7) \quad A_{s,\ell} = \int_{\Omega} \tau_\ell(-u_s) e^{nF} \omega_X^n \rightarrow A_s = \int_{\Omega_s} (-u_s) e^{nF} \omega_X^n, \text{ as } \ell \rightarrow \infty.$$

By [6], the solution $\psi_{s,\ell}$ of this Dirichlet problem exists and is unique.

We can now state the key comparison inequality between u_s and $\psi_{s,\ell}$

$$(8.8) \quad -u_s \leq C(n, \gamma) A_{s,\ell}^{\frac{1}{n+1}} (-\psi_{s,\ell})^{\frac{n}{n+1}}$$

on $\bar{\Omega}$, where $C(n, \gamma)$ depends only on n and γ . Note that the auxiliary Monge-Ampère equation is of the general form considered in (2.6), with $\omega = 0$, $\bar{q}(z) = 0$, $\Lambda = 0$, and $q(z) = \varphi(z_0) - \frac{1}{2}|z|^2$, (and $s \rightarrow -s$). In particular $q_{\bar{k}j} = -\frac{1}{2}\delta_{kj}$. It is then easy to apply the maximum principle to the function

$$(8.9) \quad \Phi = -\varepsilon(-\psi_{s,\ell})^{\frac{n}{n+1}} - u_s$$

with the preliminary computations as in Lemma 2, and the desired inequality follows with $\varepsilon^{n+1} = A_{s,\ell} \gamma^{-1} (n+1)^n n^{-2n}$.

We can now follow the template provided by the compact Kähler case. First we establish the inequality

$$(8.10) \quad \int_{\Omega_s} \exp\left\{\beta \frac{(-u_s)^{\frac{n+1}{n}}}{A_s^{1/n}}\right\} \omega_X^n \leq C$$

where β and C are strictly positive constants depending only on n, γ, r_0 . This is the analogue in the Hermitian case of the inequality (3.12). For this, we apply Young's inequality as in the compact Kähler case, with the α -invariant replaced by the following inequality of Kolodziej [45] for plurisubharmonic functions ψ

$$(8.11) \quad \int_D e^{-\alpha\psi} dV \leq C$$

on bounded pseudoconvex domains $D \subset \mathbf{C}^n$ with $\psi = 0$ on ∂D , and Monge-Ampère measure $\int_D (i\partial\bar{\partial}\psi)^n = 1$. Here α is a strictly positive constant¹.

Next, we can apply Young's inequality to the exponential inequality (8.10) and obtain for any $p > n$ the following reverse Hölder inequality

$$(8.12) \quad A_s \leq C_0 \left(\int_{\Omega_s} e^{nF} \omega_X^n \right)^{1+\delta_0}$$

with $\delta_0 = \frac{1}{n} - \frac{1}{p}$, and C is a constant depending only on ω_X, n, p, γ and $\|e^{nF}\|_{L^1(\log L)^p}$. Setting

$$(8.13) \quad \phi(s) = \int_{\Omega_s} e^{nF} \omega_X^n$$

we can rewrite the reverse inequality as $A_s \leq C_0(\phi(s))^{1+\delta_0}$. Since it is easily seen that $A_s \geq t \int_{\Omega_{s-t}} e^{nF} \omega_X^n$, we conclude that for any $t \in (0, s)$

$$(8.14) \quad t\phi(s-t) \leq C_0(\phi(s))^{1+\delta_0}.$$

We can now apply a version of De Giorgi's lemma for monotone increasing and positive functions $\phi(s)$ on $(0, s_0)$ satisfying $\lim_{s \rightarrow 0^+} \phi(s) = 0$ and the growth rate (8.14), which says that under these conditions, there must exist a constant $c_0 > 0$ depending only on s_0, C_0 and δ_0 so that

$$(8.15) \quad \phi(s_0) \geq c_0.$$

Finally, we note the elementary inequality

$$(8.16) \quad \phi(s_0) \log(-\varphi(x_0) - s_0)^{\frac{1}{2}} \leq \int_{\Omega_{s_0}} \log \frac{-\varphi - 2^{-1}|z|^2}{(-\varphi(x_0) - s_0)^{\frac{1}{2}}} e^{nF} \omega_X^n$$

which results itself from the elementary inequality $-\varphi - \frac{1}{2}|z|^2 > -\varphi(x_0) - s_0$ on Ω_0 since we may assume without loss of generality that $s_0 < \frac{1}{2}$ and $\varphi(x_0) < -2$. Applying the Young's inequality to the integrand on the right hand side, we find

$$\phi(s_0) \log(-\varphi(x_0) - s_0)^{\frac{1}{2}} \leq \frac{1}{2} \max \left\{ \|e^{nF}\|_{L^1(\log L)^p} + C'_p C_1(\omega_X), C''_p \frac{\|\varphi\|_{L^1} + C_2(\omega)}{(-\varphi(x_0) - s_0)^{\frac{1}{2}}} \right\}.$$

Since $\phi(s_0)$ is bounded from below by a constant, it follows that $-\varphi(x_0) - s_0$ must be bounded from above. The theorem is proved.

9. APPLICATION TO $(n - 1)$ -FORM MONGE-AMPÈRE EQUATIONS

A new generation of problems in complex geometry, notably from non-Kähler geometry and mathematical physics, has led to non-linear equations of the form

¹ Kolodziej's inequality is a generalization to all dimensions of a classic inequality in one complex dimension due to Brezis and Merle [4]. Kolodziej's original proof used pluripotential theory. A PDE proof has been provided recently by Wang, Wang, and Zhou [69].

$f(\lambda) = e^F$, but where λ are not the eigenvalues of a hessian matrix $h_\varphi = \omega_X^{-1}(\omega + i\partial\bar{\partial}\varphi)$, but of a more general matrix involving the derivatives of φ to second order. One basic example of such an equation is the so-called $(n-1)$ -form Monge-Ampère equation, solved by G. Székelyhidi, V. Tosatti, and B. Weinkove [63] in their solution of the Gauduchon conjecture. Central to the study of this equation is the L^∞ estimate. In this section, we show how our method of auxiliary Monge-Ampère equations can be used to establish this estimate, with in fact slightly weaker and more general hypotheses than in [63].

Let (X, ω_X) be a compact Hermitian manifold. If ω is any other Hermitian metric, we consider the smooth functions φ for which the form

$$(9.1) \quad \tilde{\omega}_\varphi = \omega + \frac{1}{n-1}(\Delta_{\omega_X} \varphi \omega_X - i\partial\bar{\partial}\varphi)$$

is positive, where $\Delta_{\omega_X} \varphi = n \frac{i\partial\bar{\partial}\varphi \wedge \omega_X^{n-1}}{\omega_X^n}$ is the rough Laplacian with respect to the metric ω_X . Then

Theorem 7. *Let $\tilde{h}_\varphi = \omega_X^{-1} \tilde{\omega}_\varphi$, and $\lambda[\tilde{h}_\varphi]$ be its eigenvalues. Fix any $p > n$. Then any smooth solution φ of the equation*

$$(9.2) \quad \prod_{j=1}^n (\tilde{\lambda}_\varphi)_j = e^{nF}$$

with $\tilde{\omega}_\varphi > 0$ and $\sup_X \varphi = 0$ satisfies

$$(9.3) \quad \sup_X |\varphi| \leq C$$

where C is a constant depending only on ω_X, n, p, ω and $\|e^{nF}\|_{L^1(\log L)^p(X, \omega_X)}$.

We observe that this result is more precise than in [67, 66] in the sense that only the norm $\|e^{nF}\|_{L^1(\log L)^p(X, \omega_X)}$ is needed, and not pointwise norms. In [62], G. Székelyhidi obtains an estimate which depends on $\|e^{nF}\|_{L^q}$ for $q > 2$, using Blocki’s approach [2] based on the Alexandrov-Bakelman-Pucci (ABP) maximum principle.

We sketch the proof. A first observation is that the tensor $\Theta^{i\bar{j}}$, as defined in [67] by

$$(9.4) \quad \Theta^{k\bar{j}} = \frac{1}{n-1}((\text{Tr}_{\tilde{\omega}_\varphi} \omega_X) \omega_X^{k\bar{j}} - \tilde{\omega}_\varphi^{k\bar{j}})$$

is positive definite. In fact, in coordinates where $(\omega_X)_{j\bar{k}} = \delta_{jk}$, $(\tilde{\omega}_\varphi)_{j\bar{k}} = \lambda_j \delta_{jk}$ at a given point, then $\Theta^{k\bar{j}} = \frac{1}{n-1}(\sum_{\ell \neq j} \frac{1}{\lambda_\ell}) \delta_{jk}$.

Next, fix r_0 small enough so that, at any point $z_0 \in X$, there is a holomorphic coordinate system z with

$$(9.5) \quad \frac{1}{2}i\partial\bar{\partial}|z|^2 \leq \omega_X \leq 2i\partial\bar{\partial}|z|^2, \quad z \in B(z_0, 2r_0) = \{|z| < 2r_0\}.$$

Let x_0 be a minimum point for φ , $\Omega = B(x_0, 2r_0)$, and fix a small constant $\varepsilon' > 0$ depending only on X, ω_X, ω so that

$$(9.6) \quad \omega \geq \frac{2\varepsilon'}{n-1} (\text{Tr}_{\omega_X} \omega) \omega_X.$$

Define now for $s \in (0, s_0)$, $s_0 = 4\varepsilon' r_0^2$,

$$(9.7) \quad u_s(z) = \varphi(z) - \varphi(x_0) + \varepsilon' |z|^2 - s, \quad z \in \Omega.$$

Then $u_s(z) > 0$ for $z \in \partial\Omega$, and hence the sublevel set $\Omega_s = \{z | u_s(z) < 0\} \cap \Omega$ is relatively compact in Ω and also an open set. We can now consider the solution $\psi_{s,\ell}$ of the following auxiliary Dirichlet problem on Ω for the complex Monge-Ampère equation,

$$(9.8) \quad (i\partial\bar{\partial}\psi_{s,\ell})^n = \frac{\tau_\ell(-u_s)}{A_{s,\ell}} e^{nF} \omega_X^n \quad \text{on } \Omega, \quad \psi_{s,\ell} = 0 \quad \text{on } \partial\Omega$$

with $A_{s,\ell}$ defined by

$$(9.9) \quad A_{s,\ell} = \int_{\Omega} \tau_\ell(-u_s) e^{nF} \omega_X^n \rightarrow A_s = \int_{\Omega_s} (-u_s) e^{nF} \omega_X^n.$$

By [6], this Dirichlet problem admits a unique solution $\psi_{s,\ell}$. From the very equation, the solution $\psi_{s,\ell}$ satisfies $\int_{\Omega} (i\partial\bar{\partial}\psi_{s,\ell})^n = 1$.

The key comparison estimate is now

$$(9.10) \quad -u_s \leq \left(\frac{n+1}{n} \right)^{\frac{n}{n+1}} A_{s,\ell}^{\frac{1}{n+1}} (-\psi_{s,\ell})^{\frac{n}{n+1}}$$

which follows from the non-positivity on Ω of the test function

$$(9.11) \quad \Phi = -\varepsilon (-\psi_{s,\ell})^{\frac{n}{n+1}} - u_s, \quad \varepsilon = A_{s,\ell}^{\frac{1}{n+1}} \left(\frac{n+1}{n} \right)^{\frac{n}{n+1}}.$$

To show that $\Phi \leq 0$, we apply the maximum principle at a maximum point x_0 for Ψ , but with the operator $Lv = \Theta^{j\bar{k}} v_{\bar{k}j}$, so that $L\Phi(x_0) \leq 0$. The desired inequality follows using the arithmetic-geometric inequality, along the same lines as Lemmas 1–3.

Once we have the key comparison estimate (9.10), the proof follows the same template as in the compact Kähler case, or more precisely the compact Hermitian case. Thus we apply the exponential estimate of Kolodziej to arrive at a reverse Hölder inequality, which allows an application of De Giorgi's lemma, and the proof is completed using a general L^1 bound for $-\varphi$.

Finally, we note that there has been considerable interest recently in more general equations involving $f(\lambda[\tilde{h}_\varphi])$, motivated in part by non-Kähler geometry, mirror symmetry, and equations from string theories (see e.g. [26, 15, 56, 50]). In particular, subsolutions and general C^2 estimates have been obtained in [28, 62]. A frequent feature of these new equations is the appearance of gradient terms. For related developments, see [25, 29, 14, 51, 52, 53] and references therein.

10. APPLICATION TO GREEN'S FUNCTIONS

Perhaps surprisingly, the auxiliary Monge-Ampère equations which we have been discussing turn out to very effective in many seemingly unrelated problems in geometric analysis. In this section, we show how they can be applied to derive lower bounds for the Green's function. The Green's function is the solution of a linear partial differential equation, so this would be a case of a linear problem solved by a non-linear method. Such methods had been instrumental in the study of Schrödinger equations [12, 57, 49, 59], and in the celebrated work of M. Kuranishi [46, 47, 48] on the embeddability of strongly pseudo-convex manifolds, as pointed out by C. Fefferman. However, it does not appear that comparisons with the Monge-Ampère equation had been used before in linear problems. In our case, this non-linear method is needed because it gives bounds which are uniform in the underlying geometry.

In this section, we describe some of the results obtained in [36]. The setting is then a compact Kähler manifold (X, ω_X) of dimension n , and a closed nonnegative $(1, 1)$ -form χ whose cohomology class $[\chi]$ is nef and big. That is, $[\chi]$ lies in the closure of the Kähler cone and $\int_X \chi^n > 0$. Then $[\chi + t\omega_X]$ is a Kähler class for any $t \in (0, 1]$. For any $\varepsilon > 0$, $N > 0$, $\gamma \geq 1$, and any fixed $t \in (0, 1]$, we introduce the following class of Kähler metrics,

$$(10.1) \quad \mathcal{M}'_t(N, \varepsilon, \gamma) = \left\{ \omega \in [\chi + t\omega_X]; \frac{1}{V} \int_X e^{(1+\varepsilon)F_\omega} \omega_X^n \leq N, \sup_X e^{-F_\omega} \leq \gamma^{-1} \right\},$$

where F_ω denotes the relative volume form of ω ,

$$(10.2) \quad F_\omega = \log \left(\frac{\omega^n / [\omega^n]}{\omega_X^n / [\omega_X^n]} \right).$$

We are interested in bounds which hold uniformly for $\omega \in \mathcal{M}'_t(N, \varepsilon, \gamma)$. The following is a basic example of an estimate which would be well-known and readily established for a fixed Kähler form ω , but requires a non-linear method in order to have uniform bounds with respect to $\omega \in \mathcal{M}'_t(N, \varepsilon, \gamma)$:

Theorem 8. *Let ω be any Kähler form in the class $\mathcal{M}'_t(N, \varepsilon, \gamma)$.*

(a) *Let $v \in L^1(X, \omega^n)$ be any function satisfying $\int_X v \omega^n = 0$. Let $\Omega_s = \{v > s\}$ be the super-level sets of v . Assume that*

$$(10.3) \quad v \in C^2(\bar{\Omega}_0), \quad \Delta_{\omega} v \geq -a \quad \text{in } \Omega_0$$

for some $a > 0$. Then

$$(10.4) \quad \sup_X v \leq C(a + \|v\|_{L^1(X, \omega^n)})$$

where C is a constant depending only on $n, \omega_X, \chi, \varepsilon, N$, and γ .

(b) Assume now that $v \in C^2(X)$ and that

$$(10.5) \quad |\Delta_\omega v| \leq 1, \quad \text{and} \quad \int_X v \omega^n = 0.$$

Then there is a constant C depending only on $n, \omega_X, \chi, \varepsilon, N, \gamma$ such that

$$\|v\|_{L^1(X, \omega^n)} \leq C.$$

The proof relies again on an auxiliary complex Monge-Ampère equation. Replacing v by v/a we may assume $a = 1$. Thus let as before $u \rightarrow \tau_\ell(u)$ be again a sequence of smooth strictly positive functions which decrease to the function $u \rightarrow u \chi_{\mathbf{R}_+}(u)$ as $\ell \rightarrow \infty$, and consider the solution $\psi_{s,\ell}$ of the following equation

$$(10.6) \quad (\omega + i\partial\bar{\partial}\psi_{s,\ell})^n = \frac{\tau_\ell(v-s)}{A_{s,\ell}} \omega^n, \quad \sup_X \psi_{s,\ell} = 0,$$

where $A_{s,\ell}$ is again a normalizing constant

$$(10.7) \quad A_{s,\ell} = \frac{1}{[\omega^n]} \int_X \tau_\ell(v-s) \omega^n \rightarrow \frac{1}{[\omega^n]} \int_{\Omega_s} (v-s) \omega^n = A_s$$

as $\ell \rightarrow \infty$. Yau's theorem insures then the existence of a unique solution $\psi_{s,\ell}$.

We need to compare v to $\psi_{s,\ell}$. For this, we express ω by the $\partial\bar{\partial}$ -lemma as

$$(10.8) \quad \omega = \chi + t\omega_X + i\partial\bar{\partial}\varphi, \quad \sup_X \varphi = 0$$

for a unique function φ . Then the key estimate is the following: there exists a constant Λ depending only on $n, p, \chi, \omega_X, \text{Ent}_p(\omega)$ so that

$$(10.9) \quad -\psi + \varphi + \Lambda \geq 1, \quad \text{and} \quad \Phi \leq 0$$

where the test function Φ is defined by

$$(10.10) \quad \Phi = -\varepsilon(-\psi + \varphi + \Lambda)^{\frac{n+1}{n+1}} + v - s$$

with $\varepsilon^{n+1} = (\frac{n+1}{n^2})^n (a + \varepsilon n)^n A_{s,\ell}$. The proof of this is analogous to the ones we have used for the previous auxiliary complex Monge-Ampère equations.

Once we know that $\Phi \leq 0$, we can apply an α -invariant inequality, uniform for all Kähler classes bounded by a fixed multiple of ω_X to obtain an inequality of the form

$$(10.11) \quad \int_{\Omega_s} \exp\left(\alpha \frac{(v-s)^{\frac{n+1}{n}}}{A_{s,\ell}^{1/n}}\right) \omega_X^n \leq C$$

and hence, using Young's inequality, the following reverse Hölder inequality for $p > n$,

$$(10.12) \quad \int_{\Omega_s} (v-s)^{\frac{(n+1)p}{n}} e^F \omega_X^n \leq C A_{s,\ell}^{\frac{p}{n}} \rightarrow C A_s^{\frac{p}{n}}.$$

This readily implies $A_s \leq (\int_{\Omega_s} e^F \omega_X^n)^{\frac{1+n}{np'}}$, with p' the dual exponent of $p(n+1)/n$. An easy consequence is the growth inequality

$$(10.13) \quad r\phi(s+r) \leq C\phi(s)^{1+\delta_0}, \quad s \geq 0, r > 0$$

for the monotone decreasing function $\phi(s) = \int_{\Omega_s} e^F \omega_X^n$. Again by a De Giorgi lemma, it follows that $\phi(s)$ must vanish for $s > S_0$, where S_0 can be estimated by the constants in the growth condition. Thus $v \leq S_0$, and part (a) of Theorem 8 is proved.

Next, we sketch the proof of Part (b) of Theorem 8. We observe that the proof of Part (a) did not require a uniform lower bound γ for the volume form in the definition of the class $\mathcal{M}'_t(N, \varepsilon, \gamma)$, but the proof of Part (b) will.

We argue by contradiction. Thus assume that there exists a sequence of metrics $\omega_j \in \mathcal{M}'_{t_j}(N, \varepsilon, \gamma)$ with $\{t_j\}_j \subset (0, 1)$ and a sequence of functions $\hat{v}_j \in C^2(X)$ satisfying $\|\hat{v}_j\|_{L^1(X, \omega_j^n)} = 1$ and

$$(10.14) \quad \Delta_{\omega_j} \hat{v}_j = \hat{h}_j, \quad \int_X \hat{v}_j \omega_j^n = 0, \quad \sup_X |\hat{h}_j| \rightarrow 0$$

as $j \rightarrow \infty$. Multiplying the above equation by \hat{v}_j and integrating by parts gives

$$(10.15) \quad \int_X |\nabla \hat{v}_j|_{\omega_j}^2 \omega_j^n = \left| \int_X \hat{h}_j \hat{v}_j \omega_j^n \right| \leq V_{\omega_j}^{\frac{1}{2}} \sup_X |\hat{h}_j| \rightarrow 0.$$

On the other hand, we can write

$$(10.16) \quad \begin{aligned} \int_X |\nabla \hat{v}_j|_{\omega_X} \omega_X^n &\leq \int_X (|\nabla \hat{v}_j|_{\omega_j}^2 \text{tr}_{\omega_X} \omega_j)^{\frac{1}{2}} \omega_X^n \\ &\leq \left(\int_X |\nabla \hat{v}_j|_{\omega_j}^2 e^{F_j} \omega_X^n \right)^{\frac{1}{2}} \left(\int_X (\text{tr}_{\omega_X} \omega_j) e^{-F_j} \omega_X^n \right)^{\frac{1}{2}}. \end{aligned}$$

The first factor in the right hand side tends to 0 as $j \rightarrow \infty$, since

$$(10.17) \quad \int_X |\nabla \hat{v}_j|_{\omega_j}^2 e^{F_j} \omega_X^n = \frac{V_X}{V_{\omega_j}} \int_X |\nabla \hat{v}_j|_{\omega_j}^2 \omega_j^n \rightarrow 0$$

and $V_{\omega_j} \geq [\chi^n]$ for all ω_j . In view of the lower bound for the volume form of metrics $\omega_j \in \mathcal{M}'_{t_j}(N, \varepsilon, \gamma)$, we have

$$(10.18) \quad \int_X (\text{tr}_{\omega_X} \omega_j) e^{-F_j} \omega_X^n \leq \frac{1}{n\gamma} \int_X \omega_j \wedge \omega_X^{n-1} \leq C$$

where C is a constant independent of j . This implies that

$$(10.19) \quad \int_X |\nabla \hat{v}_j|_{\omega_X} \omega_X^n \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

From this, it is not difficult to deduce that \hat{v}_j is uniformly bounded in the Sobolev space $W^{1,1}(X, \omega_X)$, and that, after passing to subsequences, it must converge to a

constant \hat{v}_∞ in $L^1(X, \omega_X^n)$. This constant can on one hand be verified to be nonzero in view of the normalization $\|\hat{v}_j\|_{L^1(X, \omega_j^n)} = 1$, and on the other hand to be 0, in view of the condition $\int_X \hat{v}_j \omega_j^n = 0$ satisfied by \hat{v}_j . This is a contradiction and Theorem 8 is proved.

We can now establish uniform estimates for the Green's function. Recall now that the Green's function with respect to the Kähler metric ω is defined by the equations

$$(10.20) \quad \Delta_\omega G(x, y) = -\delta_x(y) + \frac{1}{V_\omega}, \quad \int_X G(x, y) \omega^n = 0.$$

Theorem 9. *Fix $\varepsilon > 0, N > 0$ and $\gamma \in (0, 1)$. Then for any Kähler metric $\omega \in \mathcal{M}'_t(\varepsilon, N, \gamma)$, the corresponding Green's function $G(x, y)$ satisfies the following estimates, with constants C which depend only on $n, \omega_X, \chi, \varepsilon, N$ and γ :*

- (a) $\int_X |G(x, \cdot)| \omega^n \leq C$;
- (b) $\inf_{y \in X} G(x, y) \geq -C$;
- (c) *For any $\delta > 0$, there is a constant C_δ depending additionally on δ so that, for all $x \in X$,*

$$(10.21) \quad \int_X |G(x, \cdot)|^{\frac{n}{n-1}-\delta} \omega^n + \int_X |\nabla G(x, \cdot)|^{\frac{2n}{2n-1}-\delta} \omega^n \leq C_\delta.$$

Parts (a) and (b) of Theorem 9 are direct consequences of Theorem 8. For example, for any fixed x , the function $v = -G(x, y)$ is in $C^\infty(X \setminus \{x\})$ and satisfies the conditions in Part (a) of Theorem 8 with $\Delta_\omega v(y) = -\frac{1}{V_\omega} \geq -\frac{1}{[\chi^n]}$ for $y \in \{v \geq 0\}$. Thus Theorem 8 applies, giving the lower bound

$$(10.22) \quad \inf_{y \in X} G(x, y) \geq -C(1 + \|G(x, \cdot)\|_{L^1(X, \omega^n)})$$

for some constant depending only on n, p, ω_X, χ, N . With a bit more work, we can deduce from Part (b) of Theorem 8 that $\|G(x, \cdot)\|_{L^1(X, \omega^n)} \leq C$. Combined with the preceding inequality, we obtain Parts (a) and (b) of Theorem 9.

The proof of Part (c) is harder and requires a new idea, involving comparisons with another auxiliary complex Monge-Ampère equation. The key inequality to be established is a uniform bound for the L^q norm of $G(x, \cdot)$,

$$(10.23) \quad \int_X |G(x, y)|^q \omega^n(y) \leq C_q,$$

first for $q \in 1 + \frac{1}{r_0}$ for some $r_0 > n$, and then iteratively for any $q < \frac{n}{n-1}$. By Part (b), we can add a uniform constant to $G(x, \cdot)$ to obtain a function $\mathcal{G}(x, \cdot) \geq 1$. Fix $r_0 > n$ and a large $k \gg 1$ and set

$$(10.24) \quad H_k(y) = \min\{\mathcal{G}(x, y), k\}$$

which we assume is smooth, by smoothing it out if necessary. The above integral is closely related to the following integral

$$(10.25) \quad \int_X \mathcal{G}(x, y) H_k(y)^{\frac{1}{r_0}} \omega^n(y)$$

which is itself closely related (via the Green's formula) to the solution u_k of the following Laplace equation

$$(10.26) \quad \Delta_\omega u_k = -H_k^{\frac{1}{r_0}} + \frac{1}{V_\omega} \int_X H_k^{\frac{1}{r_0}} \omega^n, \quad \frac{1}{V_\omega} \int_X u_k \omega^n = 0.$$

To estimate u_k , we introduce another auxiliary complex Monge-Ampère equation,

$$(10.27) \quad (\omega + i\partial\bar{\partial}\psi_k)^n = \frac{H_k^{\frac{n}{r_0}} + 1}{V_\omega^{-1} \int_X (H_k^{\frac{n}{r_0}} + 1) \omega^n} \omega^n, \quad \sup_X \psi_k = 0.$$

It can then be shown that

$$(10.28) \quad \sup_X |\psi_k| \leq C$$

and that

$$(10.29) \quad \varepsilon' u_k + (\psi_k - \varphi) - \frac{1}{V_\omega} \int_X (\psi_k - \varphi) \omega^n \leq C$$

where C and ε' are uniform constants, and φ is the potential for the Kähler metric $\omega \in [\chi + t\omega_X]$ introduced earlier in (10.8). From here, the desired inequality follows.

Finally, we establish integral bounds for $\nabla G(x, y)$. First we note the elementary inequality

$$(10.30) \quad \int_X \frac{|\nabla_y \mathcal{G}(x, y)|_{\omega(y)}^2}{\mathcal{G}(x, y)^{1+\beta}} \omega^n(y) \leq \frac{1}{\beta}$$

which holds for all $\beta > 0$, and follows from applying Green's formula to $u(y) = \mathcal{G}(x, y)^{-\beta}$. Next, setting

$$(10.31) \quad H_k(y) = \min \left\{ \frac{|\nabla_y \mathcal{G}(x, y)|_{\omega(y)}^2}{\mathcal{G}(x, y)^{1+\beta}}, k \right\}$$

and arguing as in the estimate of $\|G(x, \cdot)\|_{L^{1+\frac{1}{r_0}}(X, \omega^n)}$, we can show that

$$(10.32) \quad \int_X \mathcal{G}(x, y) H_k(y)^{\frac{1}{r_0}} \omega^n(y) \leq C.$$

The desired $L^s(X, \omega^n)$ bound for $\nabla G(x, \cdot)$ ultimately follows. □

It is instructive to compare the preceding theorem with the classic result of Cheng and Li [11] on lower bounds for the Green's function in Riemannian geometry. This result asserts the existence of a uniform lower bound depending on only the dimension, the diameter, the volume, and a lower bound on the Ricci curvature. In our Kähler setting, Theorem 9 is easily seen to imply the following theorem, where only a lower bound on the scalar curvature, combined with an integral estimate for the volume form, suffice to give lower bounds for the Green's function which are uniform in ω :

Theorem 10. *Let ω be any Kähler metric in $[\omega_X]$. Then if $\|e^{F_\omega}\|_{L^{1+\varepsilon}(X, \omega_X)} \leq N$ for some $\varepsilon > 0, N > 0$, and the scalar curvature $R(\omega)$ satisfies $R(\omega) \geq -\kappa$ for some $\kappa \geq 0$, then the Green's function of (X, ω) satisfies*

$$(10.33) \quad \inf_{y \in X} G(x, y) \geq -C, \quad x \in X,$$

for a constant C depending only on $n, \omega_X, \varepsilon, N$, and κ .

We note that the class $\mathcal{M}'_t(\varepsilon, N, \gamma)$ of Hermitian metrics which we have used so far is not the only class to which the methods of this section apply. Other classes are described in [36] as well, with a key difference being the replacement of a pointwise lower bound on e^{F_ω} by an integral bound. In fact, for applications to diameter bounds to be described in the next section, it will be important to relax further the constant lower bound γ to a non-negative function which may vanish along a closed set of Hausdorff dimension strictly less than $2n - 1$.

In the remaining part of this section, we describe some applications of the above bounds for the Green's function to a priori estimates for the complex Monge-Ampère equation. A priori estimates are often obtained by applying the maximum principle to elliptic differential inequalities satisfied by the quantity under consideration. Sharp lower bounds for the Green's function can provide a more effective tool, especially if we consider conditions involving integrals. The following is a sample of sharp estimates which can be obtained in this manner.

Recall that we have assumed that the $(1, 1)$ -form χ is non-negative, and the Kähler class $[\chi]$ is big. By Kodaira's lemma, there is an effective divisor D in X such that

$$(10.34) \quad \chi - \varepsilon_0 \text{Ric}(h_D) \geq \delta_0 \omega_X$$

for some positive constants ε_0, δ_0 , and a Hermitian metric h_D on the line bundle $[D]$ associated with D . Let s_D be a holomorphic section defining D with $\sup_X |s_D|_{h_D}^2 = 1$. Let ω be any metric in $[\chi + t\omega_X]$. Let φ be its potential, i.e. $\omega = \chi + t\omega_X + i\partial\bar{\partial}\varphi$ and $\sup_X \varphi = 0$. Thus, in the notation (10.2) for the relative volume form F_ω , φ satisfies the complex Monge-Ampère equation

$$(10.35) \quad (\chi + t\omega_X + i\partial\bar{\partial}\varphi)^n = ce^{F_\omega} \omega_X^n, \quad c = \frac{V_\omega}{V_X}, \quad \sup_X \varphi = 0.$$

Theorem 11. Fix $\varepsilon, N, \gamma \in (0, 1)$ and $p > n$. Then for any $t \in (0, 1]$ and $\omega \in \mathcal{M}'_t$, we have the estimate

$$(10.36) \quad |\nabla\varphi|_{\omega_X}^2 \leq \frac{C}{|s_D|_{h_D}^{2A}}$$

where C depends only on $n, \varepsilon, \chi, \omega_X, N, \gamma, p$, and $A > 0$ depends only on $n, \varepsilon, \chi, \omega_X, N, \gamma$.

Note that gradient bounds had been obtained before, but under pointwise assumptions on $|\nabla F|$ [3, 55]. An earlier result requiring an L^p bound for $|\nabla F|$ with $p \geq 2n$ is in [10, 32]. The range $p > n$ in the above theorem is sharp.

Theorem 12. Under the same assumptions as in Theorem 11, but with $p > 2n$, we have the estimate

$$(10.37) \quad |i\partial\bar{\partial}\varphi|_{\omega_X}^2 \leq \frac{C}{|s_D|_{h_D}^{2B}}$$

where C depends only on $n, \varepsilon, \chi, \omega_X, N, \gamma, p$ and $\int_X |\nabla F|_{\omega_X}^p e^{F\omega} \omega_X^n$, and $B > 0$ depends only on $n, \varepsilon, \chi, \omega_X, N, \gamma$.

We have formulated the above estimates for families of degenerating metrics. But even in the case of a fixed background Kähler form ω , the above estimates improve on the known ones. For example, we have

Theorem 13. Consider the complex Monge-Ampère equation

$$(10.38) \quad (\omega_X + i\partial\bar{\partial}\varphi)^n = e^F \omega_X^n, \quad \sup_X \varphi = 0$$

on an n -dimensional compact Kähler manifold (X, ω_X) . Assume that F satisfies the condition

$$(10.39) \quad \|e^F\|_{L^{1+\varepsilon}(X, \omega_X)} \leq N \\ \sup_X e^{-F} \leq \gamma^{-1}.$$

Then for any $p > 2n$, we have

- (a) $\sup_X |i\partial\bar{\partial}\varphi|_{\omega_X}^2 \leq C$, where $C > 0$ is a constant depending only on $n, p, \omega_X, \varepsilon, N, \gamma$ and $\int_X |\nabla F|_{\omega_X}^p e^F \omega_X^n$.
- (b) $\sup_X |\nabla i\partial\bar{\partial}\varphi|_{\omega_X}^2 \leq C$, where $C > 0$ is a constant depending only on $n, p, \omega_X, \varepsilon, N, \gamma$,

$$(10.40) \quad \gamma, \int_X |\nabla F|_{\omega_X}^p e^F \omega_X^n, \quad \int_X |i\partial\bar{\partial}F|_{\omega_X}^p e^F \omega_X^n,$$

and upper and lower bounds for the endomorphism $\omega_X^{-1}(\omega_X + i\partial\bar{\partial}\varphi)$.

For the estimate in (a) to hold in general, we do need $p \geq 2n$. We also note that previous C^3 bounds had required a C^3 bound for F , and that the proof of (b) also relied on the approach of [54], which relied on estimating the connection forms instead of the potentials.

11. APPLICATION TO DIAMETER BOUNDS

In general, estimates for the Green's function can imply estimates for the diameter of the underlying metric. This can be seen as follows [39].

Let x_0, y_0 be points with $d_\omega(x_0, y_0) = \text{diam}(X, \omega)$, and define the function d on X by $d(y) = d_\omega(x_0, y)$. Then d is a Lipschitz function with Lipschitz constant 1. The Green's formula applied to $d(y)$ gives

$$(11.1) \quad d(x) = \frac{1}{[\omega^n]} \int_X d(y) \omega(y)^n + \int_X \langle \nabla_y G(x, y), \nabla d(y) \rangle_{\omega(y)} \omega(y)^n.$$

The fact that $d(x_0) = 0$ gives

$$\begin{aligned} \frac{1}{[\omega^n]} \int_X d(y) \omega(y)^n &= - \int_X \langle \nabla_y G(x_0, y), \nabla d(y) \rangle_{\omega(y)} \omega(y)^n \\ &\leq \int_X |\nabla_y G(x_0, y)|_{\omega(y)} \omega(y)^n. \end{aligned}$$

We can then write

$$\begin{aligned} \text{diam}(X, \omega) = d(y_0) &= \frac{1}{[\omega^n]} \int_X d(y) \omega(y)^n + \int_X \langle \nabla_y G(y_0, y), \nabla d(y) \rangle_{\omega(y)} \omega(y)^n \\ &\leq \int_X |\nabla_y G(x_0, y)|_{\omega(y)} \omega(y)^n + \int_X |\nabla_y G(y_0, y)|_{\omega(y)} \omega(y)^n \end{aligned}$$

which shows that the diameter can be estimated by an integral bound for the gradient of the Green's function.

Thus the bounds obtained in the previous section already imply some diameter bounds. However, for many geometric applications, such as diameters in the Kähler-Ricci flow, it is important to relax the conditions on the lower bound γ for the volume form. It turns out that this is possible, albeit quite non-trivial. Thus the following theorems were established in [39]:

Theorem 14. *Let (X, ω_X) be an n -dimensional connected compact Kähler manifold. For given parameters $A > 0, K > 0, p > n$ and γ a continuous non-negative function on X , let $\mathcal{V}(X, \omega_X, n, A, p, K, \gamma)$ be the following space of Kähler metrics*

$$\mathcal{V}(X, \omega_X, n, A, p, K, \gamma) = \left\{ \omega; [\omega] \cdot [\omega_X]^{n-1} \leq A, \mathcal{N}_{X, \omega_X, p}(\omega) \leq K, \frac{\omega^n}{\omega_X^n} \geq \gamma \right\}$$

where $\mathcal{N}_{X, \omega_X, p}(\omega)$ is the p -Nash entropy, defined by

$$(11.2) \quad \mathcal{N}_{X, \omega_X, p}(\omega) = \int_X |F|^p e^F \omega_X^n = \|e^F\|_{L^1(\log L)^p(\omega_X)}, \quad F = \frac{1}{V_\omega} \frac{\omega^n}{\omega_X^n}.$$

Assume that

$$(11.3) \quad \dim_{\mathcal{H}}\{\gamma = 0\} < 2n - 1$$

where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension. Then for any $A, K > 0$ and $p > n$, there exist constants $C, c > 0$ depending only on $X, \omega_X, n, A, p, K, \gamma$ and α depending only on n and p so that

- (a) $\int_X |G(x, \cdot)| \omega^n + \int_X |\nabla G(x, \cdot)| \omega^n + (-\inf_{y \in X} G(x, y)) V_\omega \leq C$;
- (b) $\text{diam}(X, \omega) \leq C$;
- (c) $\frac{\text{Vol}_\omega(B_\omega(x, R))}{\text{Vol}_\omega(X)} \geq cR^\alpha$ for any $x \in X, R \in (0, 1)$.

We stress that the above theorem can give bounds on the diameter even when no lower bound on the Ricci curvature is available. This is of particular importance for the Kähler-Ricci flow. More generally, we obtain the following Kähler analogue of Gromov’s precompactness theorem for metric spaces:

Theorem 15. *Let (X, ω_X) be a connected n -dimensional Kähler manifold, and let γ be a non-negative function with*

$$(11.4) \quad \dim_{\mathcal{H}}\{\gamma = 0\} < 2n - 1.$$

Then for any $A, K > 0, p > n$, any sequence $\{\omega_j\}$ in $\mathcal{V}(X, \omega_X, n, A, p, K, \gamma)$ admits a subsequence converging in Gromov-Hausdorff topology to a compact metric space (X_∞, d_∞) .

Several applications of these theorems to the Kähler-Ricci flow and to the asymptotic behavior of fibrations near the singular fibre can be found in [39].

12. APPLICATION TO THE TAMING OF SYMPLECTIC FORMS

let (X, J) be a compact *almost complex* manifold with J the *almost complex structure*. Suppose $m = 2n$ is the real dimension of X . A Riemannian metric \tilde{g} on X is called *almost Kähler*, if \tilde{g} is J -compatible, i.e. $\tilde{g}(JY, JZ) = \tilde{g}(Y, Z)$ for any vector fields Y, Z and the associated 2-form $\omega_{\tilde{g}}$ defined by $\omega_{\tilde{g}}(Y, Z) = \tilde{g}(JY, Z)$ is *closed*.

Let Ω be a *taming symplectic form*, that is, $\Omega(Y, JY) > 0$ for $Y \neq 0$, and g be the associated *almost Hermitian metric* of Ω , i.e.

$$g(Y, Z) = \frac{1}{2}\Omega(Y, JZ) + \frac{1}{2}\Omega(Z, JY), \quad \forall \text{ vector fields } Y, Z.$$

Write dV_g for the volume form of the Riemannian metric g . For a smooth function $F \in C^\infty(X)$ normalized by $\int_X e^F dV_g = \int_X \Omega^n$, we consider the following Calabi-Yau equation on X

$$(12.1) \quad \det \tilde{g} = e^{2F} \det g,$$

where we require that $\omega_{\tilde{g}}$ is an *almost Kähler form* with $[\omega_{\tilde{g}}] = [\Omega]$. As shown by Donaldson [21], the existence of solutions to this equation would have important consequences in symplectic geometry (see also Conjecture 1.2 in [68]).

It has been proved by Tosatti, Weinkove, and Yau in [68] that the C^2 -*a priori* estimates of \tilde{g} satisfying the equation (12.1) can be derived by the L^∞ estimates of $\varphi \in C^\infty(X)$, which satisfies the linear equation

$$(12.2) \quad \Delta_{\tilde{g}}\varphi = 2n - 2n \frac{\omega_{\tilde{g}}^{n-1} \wedge \Omega}{\omega_{\tilde{g}}^n} = 2n - \text{tr}_{\tilde{g}} g, \quad \sup_X \varphi = 0.$$

Here $\Delta_{\tilde{g}}$ is the usual Riemannian Laplacian operator of the metric \tilde{g} . Note that the equation (12.2) admits a unique solution since the function on the right-hand-side has integral zero against the volume form $dV_{\tilde{g}}$.

We assume

$$\|e^F\|_{L^2(X, dV_g)}^2 = \int_X e^{2F} dV_g \leq K$$

for some constant $K > 0$. Our main result is the following L^∞ estimate of φ in (12.2):

Theorem 16. *Suppose \tilde{g} is an almost Kähler metric solving the equation (12.1) and φ solves (12.2). Then there exists a constant $C > 0$ depending on n, g and K such that*

$$\sup_X |\varphi| \leq C(1 + \|\varphi\|_{L^1(X, e^{2F} dV_g)}).$$

In the following, we denote $\{x^1, \dots, x^m\}$ local *real* coordinates on some open subset of X . Let \tilde{g} be an *almost Kähler* metric, and $\tilde{\omega}(Y, Z) = \tilde{g}(JY, Z)$ be the associated symplectic 2-form. Locally we have

$$(12.3) \quad \tilde{g} = \tilde{g}_{ij} dx^i \otimes dx^j, \quad \tilde{\omega} = \frac{1}{2} \tilde{\omega}_{ij} dx^i \wedge dx^j, \quad J = J_i^j dx^i \otimes \frac{\partial}{\partial x^j},$$

where the summations are taken over $i, j \in \{1, 2, \dots, m\}$. It follows from straightforward calculations that

$$(12.4) \quad \tilde{\omega}_{ij} = \tilde{g}_{ik} J_j^k, \quad \tilde{\omega}_{ij} = -\tilde{\omega}_{ji}.$$

The form $\tilde{\omega}$ being *almost Kähler* means

$$0 = d\tilde{\omega} = \frac{1}{2} \frac{\partial \tilde{\omega}_{ij}}{\partial x^l} dx^l \wedge dx^i \wedge dx^j = \frac{1}{6} \left(\frac{\partial \tilde{\omega}_{ij}}{\partial x^l} + \frac{\partial \tilde{\omega}_{li}}{\partial x^j} + \frac{\partial \tilde{\omega}_{jl}}{\partial x^i} \right) dx^l \wedge dx^i \wedge dx^j$$

in other words,

$$(12.5) \quad \frac{\partial \tilde{\omega}_{ij}}{\partial x^l} + \frac{\partial \tilde{\omega}_{li}}{\partial x^j} + \frac{\partial \tilde{\omega}_{jl}}{\partial x^i} = 0, \quad \forall i, j, l.$$

Multiplying both sides of (12.5) by $\tilde{\omega}^{ij} := \tilde{g}^{ik}J_k^j$ which is skew-symmetric in i, j , and taking summation over i, j , we get

$$(12.6) \quad \tilde{g}^{ik}J_k^j \frac{\partial \tilde{\omega}_{ij}}{\partial x^l} + 2\tilde{g}^{ik}J_k^j \frac{\partial \tilde{\omega}_{jl}}{\partial x^i} = 0.$$

Substituting (12.4) to (12.6), we obtain

$$\begin{aligned} 0 &= \tilde{g}^{ik}J_k^j \frac{\partial \tilde{g}_{ip}}{\partial x^l} J_j^p + \tilde{g}^{ik}J_k^j \tilde{g}_{ip} \frac{\partial J_j^p}{\partial x^l} + 2\tilde{g}^{ik}J_k^j \frac{\partial \tilde{g}_{jp}}{\partial x^i} J_l^p + 2\tilde{g}^{ik}J_k^j \tilde{g}_{jp} \frac{\partial J_l^p}{\partial x^i} \\ &= -\tilde{g}^{ik} \frac{\partial \tilde{g}_{ik}}{\partial x^l} + J_k^j \frac{\partial J_j^k}{\partial x^l} + 2\tilde{g}^{ik}J_k^j \tilde{g}_{jp} \frac{\partial J_l^p}{\partial x^i} + 2\tilde{g}^{ik} \frac{\partial \tilde{g}_{kl}}{\partial x^i} - 2\tilde{g}^{ik} \tilde{g}_{jp} J_l^p \frac{\partial J_k^j}{\partial x^i} - 2\tilde{g}^{ik} \tilde{g}_{jp} J_k^j \frac{\partial J_l^p}{\partial x^i} \\ &= -\tilde{g}^{ik} \frac{\partial \tilde{g}_{ik}}{\partial x^l} + 2\tilde{g}^{ik} \frac{\partial \tilde{g}_{kl}}{\partial x^i} + J_k^j \frac{\partial J_j^k}{\partial x^l} - 2J_p^j \frac{\partial J_l^p}{\partial x^i} + 2\tilde{g}^{ik} \tilde{g}_{lp} J_j^p \frac{\partial J_k^j}{\partial x^i} + 2J_p^j \frac{\partial J_l^p}{\partial x^i}, \\ &= -\tilde{g}^{ik} \frac{\partial \tilde{g}_{ik}}{\partial x^l} + 2\tilde{g}^{ik} \frac{\partial \tilde{g}_{kl}}{\partial x^i} + J_k^j \frac{\partial J_j^k}{\partial x^l} + 2\tilde{g}^{ik} \tilde{g}_{lp} J_j^p \frac{\partial J_k^j}{\partial x^i}, \end{aligned}$$

from which we derive that

$$(12.7) \quad \tilde{g}^{ik} \frac{\partial \tilde{g}_{kl}}{\partial x^i} - \frac{1}{2} \tilde{g}^{ik} \frac{\partial \tilde{g}_{ik}}{\partial x^l} = -\frac{1}{2} J_k^j \frac{\partial J_j^k}{\partial x^l} - \tilde{g}^{ik} \tilde{g}_{lp} J_j^p \frac{\partial J_k^j}{\partial x^i}.$$

It then follows that

$$(12.8) \quad \tilde{g}^{ik} \tilde{\Gamma}_{ik}^q = \tilde{g}^{ql} \left(\tilde{g}^{ik} \frac{\partial \tilde{g}_{kl}}{\partial x^i} - \frac{1}{2} \tilde{g}^{ik} \frac{\partial \tilde{g}_{ik}}{\partial x^l} \right) = -\frac{1}{2} \tilde{g}^{ql} J_k^j \frac{\partial J_j^k}{\partial x^l} - \tilde{g}^{ik} J_j^q \frac{\partial J_k^j}{\partial x^i}.$$

From (12.8), we see that the second term in the Laplacian of a function u , $\Delta_{\tilde{g}} u = \tilde{g}^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} - \tilde{g}^{ik} \tilde{\Gamma}_{ik}^q \frac{\partial u}{\partial x^q}$ is independent of the first order derivatives of the metric coefficients \tilde{g}_{ij} .

Let $x_0 \in X$ be a minimum point of φ , i.e. $\varphi(x_0) = \inf_X \varphi$. Choose a fixed number $r_0 > 0$ such that $2r_0 \leq$ the injectivity radius of the fixed Riemannian manifold (X, g) . Take the normal coordinates of (X, g) centered at x_0 , $(U, \{x^1, \dots, x^m\})$. Without loss of generality we may assume that on U the following holds

$$(12.9) \quad \frac{1}{2} \delta_{ij} \leq g_{ij} \leq 2\delta_{ij},$$

and the *Euclidean ball*

$$B(x_0, 2r_0) = \{x \in U : |x| < 2r_0\} \subset\subset U$$

where $|x| = \sqrt{\sum_{j=1}^m (x^j)^2}$ is the usual Euclidean norm of the coordinates $x = (x^1, \dots, x^m)$. We also have a constant $C'_J > 0$ depending on J, g such that

$$(12.10) \quad \sup_U \left(\left| \sum_{j,k} J_k^j \frac{\partial J_j^k}{\partial x^l} \right|_g + \left| \sum_j J_j^q \frac{\partial J_k^j}{\partial x^i} \right|_g \right) \leq C'_J.$$

From the equation (12.8), we have for *any* smooth function ψ on U , there exists a uniform constant $C_J > 0$ such that on U

$$(12.11) \quad |\tilde{g}^{ik} \tilde{\Gamma}_{ik}^q \psi_q| \leq C_J |\nabla \psi|_g \cdot \text{tr}_{\tilde{g}} g,$$

where $\psi_q = \frac{\partial \psi}{\partial x^q}$ and $|\nabla \psi|_g^2 = g^{ij} \psi_i \psi_j$ is the gradient of ψ with respect to the *fixed* metric g . We emphasize that the constant C_J in (12.11) depends *only* on g, J and can be made to be independent of the choice of coordinates, though the LHS of (12.11) is only locally defined. Indeed, we can see from (12.10) that C'_J depends on $|J|_g$ and $|\nabla_g J|_g$, both of which are globally defined.

Let $\eta \in (0, 1)$ be a small positive constant to be determined. For any $0 < s \leq s_0 = \eta r_0^2$, we consider the function defined on $B(x_0, 2r_0)$

$$(12.12) \quad u_s(x) := \varphi(x) - \varphi(x_0) + \eta |x|^2 - s.$$

By the choice of the point x_0 , it is clear that $u_s \geq -s$. Define the sublevel set of u_s by

$$(12.13) \quad \Omega_s := \{x \in B(x_0, 2r_0) \mid u_s(x) < 0\}.$$

Note that $x_0 \in \Omega_s$ so Ω_s is a *nonempty* open subset of $B(x_0, 2r_0)$. Moreover, by the choice of $s \leq \eta r_0^2$, we see that on $B(x_0, 2r_0) \setminus B(x_0, r_0)$

$$u_s \geq \eta r_0^2 - s \geq 0,$$

hence we have $\Omega_s \subset B(x_0, r_0)$. We solve the following *real* Monge-Ampère equation on the Euclidean ball $B(x_0, 2r_0)$

$$(12.14) \quad \det \left(\frac{\partial^2 \psi_{s,\ell}}{\partial x^i \partial x^j} \right) = \frac{\tau_\ell(-u_s)}{A_{s,\ell}} e^{2F} \det g, \quad \text{in } B(x_0, 2r_0),$$

and $\psi_{s,\ell} = 0$ on $\partial B(x_0, 2r_0)$. Here $\psi_{s,\ell}$ is strictly convex in the ball $B(x_0, 2r_0)$ and

$$A_{s,\ell} = \int_{B(x_0, 2r_0)} \tau_\ell(-u_s) e^{2F} (\det g) dx > 0$$

is chosen so that $\int_{B(x_0, 2r_0)} \det \left(\frac{\partial^2 \psi_{s,\ell}}{\partial x^i \partial x^j} \right) dx = 1$. Note that as $\ell \rightarrow \infty$

$$A_{s,\ell} \rightarrow A_s := \int_{\Omega_s} (-u_s) e^{2F} \det g$$

and $A_s \leq C(n, g)K \leq \frac{1}{2}C_1$, where $C_1 > 0$ depends only on n, g and K . Hence for all ℓ sufficiently large which we always assume, we have

$$(12.15) \quad A_{s,\ell} \leq C_1.$$

Let β_n be the volume of the unit ball in \mathbf{R}^{2n} .

Lemma 6. *There exist a constant $C_2 = C_2(n) > 0$ such that*

$$-\inf_{B(x_0, 2r_0)} \psi_{s, \ell} \leq C_2 r_0, \quad \sup_{B(x_0, r_0)} |\nabla \psi_{s, \ell}| \leq C_2.$$

Proof. Since by the definition of $\psi_{s, \ell}$, $\int_{B(x_0, 2r_0)} \det\left(\frac{\partial^2 \psi_{s, \ell}}{\partial x^i \partial x^j}\right) dx = 1$, so it follows from the standard ABP maximum principle (e.g. [24])

$$-\inf_{B(x_0, 2r_0)} \psi_{s, \ell} \leq -\inf_{\partial B(x_0, 2r_0)} \psi_{s, \ell} + \frac{4r_0}{\beta_n} \left[\int_{B(x_0, 2r_0)} \det\left(\frac{\partial^2 \psi_{s, \ell}}{\partial x^i \partial x^j}\right) dx \right]^{1/m} = \frac{4}{\beta_n} r_0.$$

To see the second inequality, for any fixed point $x \in B(x_0, r_0)$, denote $V = \frac{D\psi_{s, \ell}(x)}{|D\psi_{s, \ell}(x)|}$ to be the unit vector in the direction of $D\psi_{s, \ell}(x)$ (if $D\psi_{s, \ell}(x) = 0$, there is nothing to prove, so here we assume $D\psi_{s, \ell}(x) \neq 0$). Then clearly $|D\psi_{s, \ell}(x)| = D\psi_{s, \ell}(x) \cdot V$. Consider the half line $L: 0 \leq t \mapsto x + tV$ which intersects $\partial B(x_0, r_0)$ and $\partial B(x_0, 2r_0)$ at $L(t_1)$, $L(t_2)$, respectively. We have $t_2 - t_1 \geq r_0$ and $0 > \psi_{s, \ell}(L(t_1)) \geq -\frac{4}{\beta_n} r_0$ and $\psi_{s, \ell}(L(t_2)) = 0$. Then by the convexity of the function $t \mapsto \psi_{s, \ell}(L(t))$, we have

$$|D\psi_{s, \ell}(x)| = \psi_{s, \ell}(L(t))' \Big|_{t=0} \leq \frac{\psi_{s, \ell}(L(t_2)) - \psi_{s, \ell}(L(t_1))}{t_2 - t_1} \leq \frac{4}{\beta_n}.$$

Taking supremum over all $x \in B(x_0, r_0)$ finishes the proof of the lemma. □

Take positive constants

$$(12.16) \quad \Lambda = \frac{2n}{1+2n} (10C_J C_2)^{2n+1} A_{s, \ell}, \quad \varepsilon = \left(\frac{2n+1}{2n}\right)^{\frac{2n}{2n+1}} A_{s, \ell}^{\frac{1}{2n+1}}$$

where $C_2 > 0$ is the constant in Lemma 6 and $C_J > 0$ as in (12.11). We observe that by (12.15), it holds that Λ is bounded above by the uniform constant $\frac{2n}{1+2n} (10C_J C_2)^{2n+1} C_1$.

Define a function Φ on $B(x_0, 2r_0)$ by

$$(12.17) \quad \Phi(x) = -\varepsilon(-\psi_{s, \ell}(x) + \Lambda)^{\frac{2n}{2n+1}} - u_s(x), \quad \forall x \in B(x_0, 2r_0).$$

We claim that $\Phi \leq 0$ on this ball. As a continuous function, Φ achieves its maximum at some point $x_{\max} \in \overline{B(x_0, 2r_0)}$. If $x_{\max} \notin \Omega_s$, then by the definition of Ω_s , clearly we have $\Phi(x_{\max}) < 0$. So we assume $x_{\max} \in \Omega_s \subset B(x_0, r_0)$. By the maximum principle, it follows that $\frac{\partial \Phi}{\partial x^q} \Big|_{x_{\max}} = 0$ and $\frac{\partial^2 \Phi}{\partial x^i \partial x^j} \Big|_{x_{\max}} \leq 0$. Hence at x_{\max}

$$\Delta_{\tilde{g}} \Phi = \tilde{g}^{ij} \frac{\partial^2 \Phi}{\partial x^i \partial x^j} - \tilde{g}^{ij} \tilde{\Gamma}_{ij}^q \frac{\partial \Phi}{\partial x^q} = \tilde{g}^{ij} \frac{\partial^2 \Phi}{\partial x^i \partial x^j} \leq 0.$$

We then calculate at x_{\max} .

$$0 \geq \Delta_{\tilde{g}} \Phi = \frac{2n\varepsilon}{2n+1} (-\psi_{s, \ell} + \Lambda)^{-\frac{1}{2n+1}} \Delta_{\tilde{g}} \psi_{s, \ell} - \Delta_{\tilde{g}} \varphi - \eta \Delta_{\tilde{g}} |x|^2$$

$$\begin{aligned}
 & + \frac{2n\varepsilon}{(2n+1)^2} (-\psi_{s,\ell} + \Lambda)^{-\frac{2n+2}{2n+1}} |\nabla \psi_{s,\ell}|_{\tilde{g}}^2 \\
 (12.18) \quad & \geq \frac{2n\varepsilon}{2n+1} (-\psi_{s,\ell} + \Lambda)^{-\frac{1}{2n+1}} \Delta_{\tilde{g}} \psi_{s,\ell} - 2n + \text{tr}_{\tilde{g}} g - \eta \Delta_{\tilde{g}} |x|^2.
 \end{aligned}$$

We first look at the term $-\eta \Delta_{\tilde{g}} |x|^2$ in (12.18). It satisfies

$$(12.19) \quad -\eta \Delta_{\tilde{g}} |x|^2 = -2\eta \tilde{g}^{ij} \delta_{ij} + 2\eta \tilde{g}^{ij} \tilde{\Gamma}_{ij}^q x^q \geq -4\eta \text{tr}_{\tilde{g}} g - 2\eta C_J r_0 \text{tr}_{\tilde{g}} g \geq -\frac{1}{10} \text{tr}_{\tilde{g}} g,$$

where $C_J > 0$ is the uniform constant in (12.11) and we have chosen $\eta > 0$ such that

$$\eta(4 + 2C_J r_0) = 1/10$$

and this fixes the uniform constant η . We will denote $D_{ij}^2 \psi_{s,\ell} = \frac{\partial^2 \psi_{s,\ell}}{\partial x^i \partial x^j}$ to be (Euclidean) Hessian of the function $\psi_{s,\ell}$. For the first term in (12.18), we have

$$\begin{aligned}
 (12.20) \quad & \frac{2n\varepsilon}{2n+1} (-\psi_{s,\ell} + \Lambda)^{-\frac{1}{2n+1}} \Delta_{\tilde{g}} \psi_{s,\ell} \\
 & = \frac{2n\varepsilon}{2n+1} (-\psi_{s,\ell} + \Lambda)^{-\frac{1}{2n+1}} \tilde{g}^{ij} \left(\frac{\partial^2 \psi_{s,\ell}}{\partial x^i \partial x^j} \right) + \frac{2n\varepsilon}{2n+1} (-\psi_{s,\ell} + \Lambda)^{-\frac{1}{2n+1}} \tilde{g}^{ij} \tilde{\Gamma}_{ij}^q \frac{\partial \psi_{s,\ell}}{\partial x^q} \\
 & \geq \frac{4n^2\varepsilon}{2n+1} (-\psi_{s,\ell} + \Lambda)^{-\frac{1}{2n+1}} \left(\frac{\det D^2 \psi_{s,\ell}}{\det \tilde{g}} \right)^{1/2n} - \frac{2n\varepsilon \Lambda^{-\frac{1}{2n+1}}}{2n+1} C_J \left| \frac{\partial \psi_{s,\ell}}{\partial x^q} \right| \cdot \text{tr}_{\tilde{g}} g \\
 & \geq \frac{4n^2\varepsilon}{2n+1} (-\psi_{s,\ell} + \Lambda)^{-\frac{1}{2n+1}} \left(\frac{-u_s}{A_{s,\ell}} \right)^{1/2n} - \frac{2n\varepsilon \Lambda^{-\frac{1}{2n+1}}}{2n+1} C_J C_2 \cdot \text{tr}_{\tilde{g}} g \\
 & \geq \frac{4n^2\varepsilon^{1+\frac{1}{2n}}}{(2n+1)A_{s,\ell}^{1/2n}} \left(\frac{-u_s}{\varepsilon(-\psi_{s,\ell} + \Lambda)^{2n/(2n+1)}} \right)^{1/2n} - \frac{1}{10} \text{tr}_{\tilde{g}} g \\
 & = 2n \left(\frac{-u_s}{\varepsilon(-\psi_{s,\ell} + \Lambda)^{2n/(2n+1)}} \right)^{1/2n} - \frac{1}{10} \text{tr}_{\tilde{g}} g.
 \end{aligned}$$

Combining the equations (12.20), (12.19), (12.18), we see that at x_{\max}

$$0 \geq 2n \left(\frac{-u_s}{\varepsilon(-\psi_{s,\ell} + \Lambda)^{2n/(2n+1)}} \right)^{1/2n} - 2n + \frac{4}{5} \text{tr}_{\tilde{g}} g$$

from which we easily derive that $\frac{(-u_s)}{\varepsilon(-\psi_{s,\ell} + \Lambda)^{2n/(2n+1)}} < 1$, that is, $\Phi|_{x_{\max}} < 0$. Hence we finish the proof of the claim that $\Phi \leq 0$. In particular on Ω_s it holds that

$$(12.21) \quad -u_s \leq C(n) A_{s,\ell}^{\frac{1}{2n+1}} (-\psi_{s,\ell} + \Lambda)^{\frac{2n}{2n+1}} \leq C_3 A_{s,\ell}^{\frac{1}{2n+1}},$$

for some $C_3 > 0$ that depends on n, g, J, K . Here we have applied Lemma 6 to see that $|\psi_{s,\ell}| \leq C_2 r_0$ and the fact that Λ is a uniformly bounded constant. Letting

$\ell \rightarrow \infty$ we conclude from (12.21) that

$$(12.22) \quad -u_s \leq C_3 A_s^{\frac{1}{2n+1}}.$$

Integrating both sides of (12.22) against the measure $e^{2F}(\det g)dx$ over Ω_s , we get

$$(12.23) \quad A_s = \int_{\Omega_s} (-u_s) e^{2F}(\det g)dx \leq C_3 A_s^{\frac{1}{2n+1}} \int_{\Omega_s} e^{2F}(\det g)dx.$$

So we have

$$(12.24) \quad A_s \leq C_3^{\frac{2n+1}{2n}} \left(\int_{\Omega_s} e^{2F}(\det g)dx \right)^{1+\frac{1}{2n}} = C_4 \phi(s)^{1+\frac{1}{2n}},$$

where we denote $\phi(s) = (\int_{\Omega_s} e^{2F}(\det g)dx)^{1+\frac{1}{2n}}$. On the other hand, for any $0 < t < s$, on the open set Ω_{s-t} we have

$$u_s(x) = u_{s-t}(x) - t < -t, \text{ i.e. } -u_s(x) > t$$

It is elementary that

$$A_s \geq \int_{\Omega_{s-t}} (-u_s) e^{2F}(\det g)dx \geq t \phi(s-t).$$

Combining the above, we see that

$$(12.25) \quad t \phi(s-t) \leq C_4 \phi(s)^{1+\frac{1}{2n}}, \quad \forall 0 < t < s \leq s_0.$$

It is not hard to see that $\phi(s)$ is an *increasing* and *continuous* function in $s \in (0, s_0]$ and $\phi(s) > 0$ for any $s \in (0, s_0]$ and $\lim_{s \rightarrow 0^+} \phi(s) = 0$. We can apply a version of De Giorgi's lemma to show that $\frac{2C_4}{1-2^{-1/2n}} \phi(s_0)^{1/2n} \geq s_0$. Hence there is a uniform constant $c_0 > 0$ such that

$$(12.26) \quad \phi(s_0) \geq c_0 > 0.$$

Applying (12.24) with $s = s_0$, we obtain $A_{s_0} \leq C_5$ for a constant $0 < C_5 = C_4(2^{2n} \beta_n K)^{1+\frac{1}{2n}}$. From the definition of A_{s_0} , we derive that

$$(12.27) \quad (-\varphi(x_0)) \cdot \phi(s_0) \leq s_0 \phi(s_0) + \int_{\Omega_{s_0}} (-\varphi) e^{2F}(\det g)dx + C_5.$$

The equation (12.5) then implies that

$$(12.28) \quad -\inf_X \varphi = -\varphi(x_0) \leq C_6 + C_7 \int_X (-\varphi) e^{2F} dV_g,$$

for some constant $C_7 > 0$. So we have proved the inequality for the solution φ to the equation (12.2)

$$(12.29) \quad \sup_X |\varphi| \leq C_8(1 + \|\varphi\|_{L^1(X, e^{2F} dV_g)}).$$

This finishes the proof of Theorem 16. □

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