On Derivation Lie Algebras of Singularities and Torelli-Type Theorems

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Abstract. Since Brieskorn gave the connection between simple Lie algebras and ADE singularities in 1970, it has become an important problem to establish connections between singularities and solvable (nilpotent) Lie algebras. Recently, we have constructed some new natural maps between the set of complex analytic isolated singularities and the set of finite dimensional solvable (nilpotent) Lie algebras. Furthermore, we use these new Lie algebras to obtain Torelli-type theorems of simple elliptic singularities. The main purpose of this paper is to summarize the results that we have obtained on new Lie algebras arising from isolated hypersurface singularities.

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1. Introduction

In this paper, we announce the recent results, obtained in [8], [24], and [25] on new Lie algebras arising from isolated singularities and corresponding Torelli-type theorems. It is known that finite dimensional Lie algebras are semi-direct products of the semi-simple Lie algebras and solvable Lie algebras. Simple Lie algebras and semi-simple Lie algebras have been well understood, but not the solvable (nilpotent) Lie algebras. Brieskorn gave the connection between simple Lie algebras and simple singularities. It is important to establish connections between isolated singularities and solvable (nilpotent) Lie algebras. We first recall Brieskorn's theory which gives the way to construct ADE singularities from simple Lie algebras.

Let G be a semi-simple Lie group acting on its Lie algebra $\mathcal{G}$ by the adjoint action and let $\mathcal{G}/G$ be the variety corresponding to the G-invariant polynomials on $\mathcal{G}$. The quotient morphism $\gamma : \mathcal{G} \to \mathcal{G}/G$ was intensively studied by Kostant ([31], [32]). Further, let $\mathcal{H} \subset \mathcal{G}$ be a Cartan subalgebra of $\mathcal{G}$ and W be the corresponding Weyl group. Then,

(i) The space $\mathcal{G}/G$ may be identified with the set of semi-simple G classes in $\mathcal{G}$ such that γ maps an element $x \in \mathcal{G}$ to the class of its semi-simple part x_s. Thus $\gamma^{-1}(0) = N(\mathcal{G})$ is a nilpotent variety. Recall that an element $x \in N(\mathcal{G})$ is termed "regular" (resp., "subregular") if its centralizer has minimal dimension (resp., minimal dimension $+ 2$).

(ii) By a theorem of Chevalley, the space $\mathcal{G}/G$ is isomorphic to $\mathcal{H}/W$, an affine space of dimension $r = \mathrm{rank}(\mathcal{G})$. The isomorphism is given by the map of a semi-simple class to its intersection with $\mathcal{H}$ (a W orbit).

The following beautiful theorem of Brieskorn [6] conjectured by Grothendieck [16] establishes connec-

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tions between the ADE singularities and the simple Lie algebras.

Theorem 1.1 ([6]). *Let $\mathcal{G}$ be a simple Lie algebra over $\mathbb{C}$ of type A_r, D_r, E_r. Then*

(i) the intersection of the variety $N(\mathcal{G})$ of the nilpotent elements of $\mathcal{G}$ with a transverse slice S to the subregular orbit, which has codimension 2 in $N(\mathcal{G})$, is a surface $S \cap N(\mathcal{G})$ with an isolated rational double point of the type corresponding to the algebra $\mathcal{G}$.

(ii) the restriction of the quotient $\gamma : \mathcal{G} \to \mathcal{H}/W$ to the slice S is a realization of a semi-universal deformation of the singularity in $S \cap N(\mathcal{G})$.

The details of Brieskorn's theory can be found in Slodowy's papers ([42], [43]).

Historically, there is a marked difference is noted between the classification theory of semi-simple Lie algebras and the classification theories of solvable or nilpotent Lie algebras. While the semi-simple theory is beautiful, the others lack anything resembling elegance. For semi-simple Lie algebras over the complex numbers one has the Killing form, Dynkin diagrams, root space decompositions, the Serre presentation, the theory of highest weight representations, the Weyl character formula for finite-dimensional representations, and much more ([17], [27]). In the theory of solvable Lie algebras one has the theorems of Lie and Engel along with Malcev's reduction of the classification problem to the same problem for nilpotent algebras [34]. There does not seem to be any nice way to classify nilpotent Lie algebras (such as a graph or diagram for each algebra). Therefore, it is of great importance to establish a connection between isolated singularities and solvable (nilpotent) Lie algebras. In [8], [24], and [25], some natural maps between the set of complex analytic isolated singularities and the set of finite dimensional solvable (nilpotent) Lie algebras have been constructed. These connections help people to understand the solvable (nilpotent) Lie algebras from the geometric point of view.

In sections 2 and 3 we recall the definitions of Yau algebra and Local Hessian Lie algebra. We list Torelli-type theorems of simple elliptic singularities in section 4, as well as other results about the dimension of the new Lie algebra. In sections 5, the notation about fewnomial singularities is setted up. The computational results of new Lie algebras for fewnomial singularities was given in section 6. In section 7, we present the new results about k-th Yau algebra. Finally, the final section 8 is dedicated to the Lie algebras associated to isolated complete intersection singularities.

2. Yau Algebra

Let $\mathcal{O}_n$ denote the $\mathbb{C}$-algebra of germs of analytic functions defined at the origin of $\mathbb{C}^n$ and $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_n$. For any isolated hypersurface singularity $(V,0) \subset (\mathbb{C}^n,0)$ where $V = V(f) = \{f = 0\}$, the factor-algebra $A(V) = \mathcal{O}_n/(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$ is finite dimensional. This factor-algebra is called the moduli algebra of V and its dimension $\tau(V)$ is called Tyurina number. The order of the lowest nonvanishing term in the power series expansion of f at 0 is called the multiplicity (denoted by $mult(f)$) of the singularity $(V,0)$. It is well-known that a polynomial $f \in \mathbb{C}[x_1, \cdots, x_n]$ is said to be weighted homogeneous if there exist positive rational numbers $w_1, \cdots, w_n$ (weights of $x_1, \cdots, x_n$) such that, $\sum a_i w_i = 1$ for each monomial $\prod x_i^{a_i}$ appearing in f with nonzero coefficient. The weight type of f is denoted as $(w_1, \cdots, w_n)$. The Milnor number of the isolated hypersurface singularity is defined by $\mu = \dim \mathbb{C}[x_1, \cdots, x_n]/(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$. In [37], it was shown that the Milnor number of a weighted homogeneous hypersurface singularity of weight type $(w_1, \cdots, w_n)$ can be calculated by: $\mu = (\frac{1}{w_1} - 1)(\frac{1}{w_2} - 1) \cdots (\frac{1}{w_n} - 1)$. According to a beautiful theorem of Saito [38], if f is a weighted homogeneous polynomial, then after a biholomorphic change of coordinates the Milnor number coincides with the Tjurina number, i.e., $\mu = \tau$.

In [36], Mather and Yau proved that the complex structure of $(V,0)$ determines and is determined by its moduli algebra.

Theorem 2.1 ([36]). *The analytic isomorphism type of an isolated hypersurface singularity determine and is determined by the isomorphism class of its moduli algebra. i.e.,*

$$(V_1,0) \cong (V_2,0) \Longleftrightarrow A(V_1) \cong A(V_2).$$

Subsequently, motivated from the Mather-Yau theorem, Yau [48] introduced the Lie algebra to $(V,0)$ as follows:

Recall that a derivation of commutative associative algebra A is defined as a linear endomorphism D of A satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$. Thus for such an algebra A one can consider the Lie algebra of its derivations $\mathrm{Der}(A,A)$ with the bracket defined by the commutator of linear endomorphisms.

Let $V = \{f = 0\}$ be a germ of an isolated hypersurface singularity at the origin of $\mathbb{C}^n$ defined by $f \in \mathbb{C}[x_1, x_2, ..., x_n]$ and $A(V)$ be the moduli algebra. Then $L(V) := \mathrm{Der}(A(V), A(V))$. Yu [53] calls $L(V)$ the Yau algebra of V (cf. [29]). Its dimension $\lambda(V)$ is called the Yau number by Elashvili and Khimshiashvili [13].

The $L(V)$ is shown to be solvable by Yau [50]. Yau and his collaborators have systematically studied the Lie algebras of isolated hypersurface singularities since 1980s ([3], [4], [7], [9], [10], [44], [18]–[25], [47], [48]–[50], [51, 52], [53]).

3. Local k-th Hessian Lie Algebra

Let $(V,0)$ be a hypersurface with an isolated singularity at the origin defined by the holomorphic function $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$. In [26], we introduced a series of new derivation Lie algebras associated to an isolated hypersurface singularity $(V,0)$.

Definition 3.1. Let $Hess(f)$ be the Hessian matrix (f_{ij}) of the second order partial derivatives of f and $h(f)$ be the Hessian of f, i.e. the determinant of this matrix $Hess(f)$. More generally, for each k satisfying $0\le k\le n$ we denote by $h_k(f)$ the ideal in $\mathcal{O}_n$ generated by all $k\times k$-minors in the matrix $Hess(f)$. In particular, the ideal $h_n(f)=(h(f))$ is a principal ideal. For each k as above, consider the k-th Hessian algebra of the polynomial f defined by

$$H_k(f)=\mathcal{O}_n/(f+J(f)+h_k(f)).$$

where $J(f)=(\frac{\partial f}{\partial x_i})$. In particular, $H_0(f)$ is exactly the well-known Tyurina algebra $A(V)$. The dimension of $H_k(f)$ is denoted as τ_k and $\tau_0=\tau$.

It is known that the isomorphism class of the local k-th Hessian algebra $H_k(f)$ is a contact invariant of f, i.e. depends only on the isomorphism class of the germ $(V,0)$ [12]. It is natural to give the following definition.

Definition 3.2. Let $(V,0)$ be an isolated hypersurface singularity. The new derivation Lie algebra arising from the isolated hypersurface singularity $(V,0)$ is defined as $L_k(V):=\mathrm{Der}(H_k(f),H_k(f)), 0\le k\le n$. The $L_k(V)$ is finite dimensional Lie algebra and its dimension is denoted as $\lambda_k(V)$.

Remark 3.1. *Notice that $H_0(f)$ is exactly the Tyurina algebra $A(V)$, thus $L_k(V)$ is a generalization of Yau algebra $L(V)$ and $L_0(V)=L(V)$. These numbers $\lambda_k(V)$ are new numerical analytic invariants of an isolated hypersurface singularity.*

In [8], we firstly studied $L_n(V)$ (note that we use a different notation $L^*(V)$ and $\lambda^*(V)$ instead of $L_n(V)$ and $\lambda_n(V)$ there). We used the $L_n(V)$ to investigate the complex analytic structures of singularities. In the following, we recall the main results obtained in [8].

Dimca [11] obtains a beautiful characterization of zero-dimensional isolated complete intersection singularities. In fact, we first introduce $L_n(V)$ based on the Dimca's characterization theorem. Then we found that $L_n(V)$ can be generalized to $L_k(V), 0\le k\le n$.

Remark 3.1. Let $V=V(f)$ be an isolated weighted homogeneous hypersurface singularity. Assume that X defined by $(\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n})$ is a zero-dimensional isolated complete intersection singularities. Then $\mathrm{Sing}(X)$ is defined by $(f,\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n},\mathrm{Det}(\frac{\partial^2 f}{\partial x_i\partial x_j})_{i,j=1,\cdots,n})$.

Theorem 3.1 (Dimca [11]). *Two zero-dimensional isolated complete intersection singularities X and Y are isomorphic if and only if their singular subspaces $\mathrm{Sing}(X)$ and $\mathrm{Sing}(Y)$ are isomorphic.*

Theorem 3.1 implies that in order to study analytic isomorphism type of zero dimensional isolated complete intersection singularity X, we only need to consider the Artinian local algebra $A^*(V)$ which is the coordinate ring of $\mathrm{Sing}(X)$. Thus $A^*(V)$ is defined as the quotient

$$\mathcal{O}_n/(f,\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n},\mathrm{Det}(\frac{\partial^2 f}{\partial x_i\partial x_j})_{i,j=1,\cdots,n}).$$

Combining Theorem 3.1 with Mather-Yau theorem, we know that $A^*(V)$ is a complete invariant of quasi-homogeneous isolated hypersurface singularities (i.e., $A^*(V)$ determines and is determined by the analytic isomorphism type of the singularity). We call $A^*(V)$ the generalized moduli algebra of V. Based on this important observation, in [8], we introduce the following new invariants for isolated hypersurface singularities.

Definition 3.3. Let $V=\{f=0\}$ be a germ of isolated hypersurface singularity at the origin of $\mathbb{C}^n$ defined by $f\in\mathbb{C}[x_1,x_2,...,x_n]$. The new Lie algebra arising from the isolated hypersurface singularity V is defined as $L^*(V):=\mathrm{Der}(A^*(V),A^*(V))$ (or $\mathrm{Der}(A^*(V))$ for short). Its dimension is denoted as $\lambda^*(V)$.

It is natural to present the following question.

Question 3.1. For which type of singularities, is $L^*(V)$ a complete invariants? In other words, if V_1,V_2 are two singularities of such type, then $L^*(V_1)\cong L^*(V_1)$ if and only if $V_1\cong V_2$.

Recall that the simple (Kleinian, rational double point) singularities which play significant role in singularity theory [2], consist of two series $A_k:\{x_1^{k+1}+x_2^2=0\}\subset\mathbb{C}^2, k\ge 1$, $D_k:\{x_1^2x_2+x_2^{k-1}=0\}\subset\mathbb{C}^2, k\ge 4$, and three exceptional singularities E_6,E_7,E_8 defined in $\mathbb{C}^2$ by polynomials $x_1^3+x_2^4, {x_1}^3+x_1{x_2}^3, x_1^3+x_2^5$, respectively.

In [8], we gave an affirmative answer to Question 3.1 for simple singularities and simple elliptic singularities (for the definitions, see [2]).

Yau algebras are solvable. However, the new Lie algebra $L^*(V)$ is not solvable in general. An example is: x^3+y^3, and its new Lie algebra is spanned by $x\partial_x, y\partial_y, x\partial_y, y\partial_x$. Then it is easy to check that the derived series does not decay to zero. However, we prove that the new Lie algebra is solvable when the multiplicity of the singularity is at least 4. We first recall an important result obtained by Schulze.

Theorem 3.2 ([40]). *Let S be a zero-dimensional local $\mathbb{C}$-algebra of embedding dimension embdim(S) and order ord(S), and denote its first deviation by $\varepsilon_1(S)$.*

Then the Lie algebra $\mathrm{Der}_{\mathbb{C}}(S,S)$ *is solvable if* $\varepsilon_1(S)+1 < \mathrm{embdim}(S)+\mathrm{ord}(S)$.

Recall that, by definition, $\varepsilon_1(S) = \dim_{\mathbb{C}} H_1(S)$ where $H_\bullet(S)$ is the Koszul algebra of S. More explicitly, when $S = R/I$ in Theorem 3.2, where $R = \mathcal{O}_n$ and $I \subseteq R$ is a zero-dimensional ideal with $I \subseteq \mathrm{m}^m$, $\mathrm{m} = (x_1, \cdots, x_n)$ and $m \geq 2$ is chosen to be maximal. Then $n = \mathrm{embdim}(S), m = \mathrm{ord}(S)$, and $\varepsilon_1(S) = \dim_{\mathbb{C}}(I/\mathrm{m}I)$ is the minimal number of generators of I ([5], Thm. 2.3.2(b)).

This result applies in particular to the generalized moduli algebra $A^*(V)$. If f is not quasi-homogeneous, then $\mathrm{Der}(A^*(V),A^*(V))$ is the same as Yau algebra, thus it is solvable. Otherwise we have the following result.

Corollary 3.1 ([8]). *If f is quasi-homogeneous and mult$(f) \geq 4$, then the new Lie algebra* $\mathrm{Der}(A^*(V),A^*(V))$ *is solvable.*

4. Torelli-Type Theorems

Given a family of complex projective hypersurfaces in $\mathbb{C}P^n$, the Torelli problem studied by Griffiths and his school asks whether the period map is injective on that family, i.e., whether the family of complex hypersurfaces can be distinguished by means of their Hodge structures. A complex projective hypersurface in $\mathbb{C}P^n$ can be viewed as a complex hypersurface with isolated singularity in $\mathbb{C}^{n+1}$. Let $V = \{z \in \mathbb{C}^{n+1} : f(z) = O\}$ be a complex hypersurface with isolated singularity at the origin. Seeley and Yau investigated the family of isolated complex hypersurface singularities using Yau algebras and obtained two deep Torelli-type theorems for simple elliptic singularities $\tilde{E}_7$ and $\tilde{E}_8$ [44]. In recent years we generalize the results in [44] to other new introduced derivation Lie algebras. I.e., we answer the following natural question: whether the family of isolated complex hypersurface singularities can be distinguished by means of their other new Lie algebras. The family of hypersurface singularities here is not arbitrary. First of all, as in projective case, we are actually studying the complex structures of an isolated hypersurface singularity. In view of the theorem of Lê and Ramanujan [33], we require the Milnor number μ being constant along this family. Recall that the dimension of the moduli algebra (denoted by τ) is a complex analytic invariant, so it suffices to consider only a (μ,τ)-constant family of isolated complex hypersurface singularities [44]. As an example, the simple elliptic singularities give rise to such families. We shall prove two Torelli-type theorems for simple elliptic singularities $\tilde{E}_7$ and $\tilde{E}_8$ respectively. However, there is no Torelli-type result for $\tilde{E}_6$, since $L^*(V_t)$ is a trivial family. Our method for $\tilde{E}_7$ (i.e., using the cohomology of a Lie algebra) is completely new and can be used to prove Torelli-type theorems for more general singularities. There are several advantages of our approach. First of all, it works for general complex hypersurface singularities without homogeneity assumption. Second, it allows us to construct a continuous invariant explicitly. Third, it gives a general method to produce a continuous family of nilpotent Lie algebras.

For recent progress on the local k-th Hessian Lie algebra, see [35]. We proposed a new conjecture about the non-existence of negative weight derivations of the new moduli algebras of weighted homogeneous hypersurface singularities and verify this conjecture up to dimension three.

This paper is to summarize mainly the following results that we have obtained in [8]. The details and proofs can be found there. In the following theorems, we use V_t to denote one of the three one-parameter families of simple ellpitic singularities (see [44]).

Theorem A. *The Torelli-type theorem holds for simple elliptic singularities $\tilde{E}_8$. That is, $L^*(V_{t_1}) \cong L^*(V_{t_2})$ as Lie algebras, for $t_1 \neq t_2$ in $\mathbb{C} - \{t \in \mathbb{C} : 4t^3 + 27 = 0\}$, if and only if V_{t_1} and V_{t_2} are analytically isomorphic (i.e., $t_1^3 = t_2^3$). In particular, $\tilde{E}_8$ give rise to a non-trivial one-parameter family of solvable (resp. nilpotent) Lie algebras of dimension 12 (resp. 11).*

Theorem B. *The weak Torelli-type theorem holds for simple elliptic singularities $\tilde{E}_7$, i.e., $L^*(V_t)$ is a non-trivial one-parameter family. In particular, $\tilde{E}_7$ give rise to a non-trivial one-parameter family of solvable (resp. nilpotent) Lie algebras of dimension 11 (resp. 10).*

However the new Lie algebra can not distinguish $\widetilde{E_6}$. The $\widetilde{E_6}$ is a simple elliptic singularity defined by $\{(x,y,z) \in \mathbb{C}^3 \mid x^3 + y^3 + z^3 = 0\}$. Its (μ,τ)-constant family is given by

$$V_t = \{(x,y,z) \in \mathbb{C}^3 \mid f_t(x,y,z) = x^3 + y^3 + z^3 + txyz = 0\}$$

with $t^3 + 27 \neq 0$ (cf. [48]). The moduli algebra $A(V_t)$ of V_t is given by

$$\begin{aligned} A(V_t) &= \mathbb{C}\{x,y,z\}/(\frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y}, \frac{\partial f_t}{\partial z}) \\ &= <1,x,y,z,xy,yz,zx,xyz>, \end{aligned}$$

with multiplication rules

$$\begin{aligned} x^2 = -\frac{t}{3}yz,\ y^2 = -\frac{t}{3}zx,\ z^2 = -\frac{t}{3}xy, \\ x^2y = xy^2 = y^2z = yz^2 = x^2z = 0. \end{aligned}$$

Let $\mathrm{Hess}(f_t)$ be the Hessian matrix of f_t. Then the generalized moduli algebra is

$$\begin{aligned} A^*(V_t) &:= \mathcal{O}_n/(\frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y}, \frac{\partial f_t}{\partial z}, \mathrm{Det}(\mathrm{Hess}(f_t))) \\ &= A(V_t)/(x^2y^2) = <1,x,y,z,xy,yz,zx> \end{aligned}$$

with multiplication rules

$$x^2 = -\frac{t}{3}yz,\ y^2 = -\frac{t}{3}zx,\ z^2 = -\frac{t}{3}xy,$$

and

$$x^2y = xy^2 = y^2z = yz^2 = x^2z = xyz = 0.$$

By calculation, a basis for the new Lie algebra $L^*(V_t) = \mathrm{Der}(A^*(V_t), A^*(V_t))$ denoted as L_t^* for short is:

$$x\partial_x + y\partial_y + z\partial_z,\ yz\partial_x,\ yz\partial_y,\ yz\partial_z,\ xz\partial_x,\ xz\partial_y,\ xz\partial_z,\ xy\partial_x,\ xy\partial_y,\ xy\partial_z,$$

for $t \neq 0$ and $216 - \frac{t^6}{27} + 7t^3 \neq 0$. It is easy to see that in this case, all L_t^* are isomorphic as Lie algebras. Thus L_t^* is a trivial family.

The classification of nilpotent Lie algebras in higher dimensions (>7) remains wide open. It is known that there are one-parameter families of non-isomorphic nilpotent Lie algebras (but no two-parameter families) in seven dimension, while there are no such families in dimension less than seven. And the existence of such families is known in dimension greater than seven. However, such examples are hard to construct ([41]). As a corollary of Theorem A and Theorem B, we obtain non-trivial one-parameter families of 11-dimensional and 12-dimensional solvable (resp. 10-dimensional and 11-dimensional nilpotent) Lie algebras associated to $\tilde{E}_7$ and $\tilde{E}_8$ respectively.

Yau and Zuo ([52]) formulated a sharp upper estimate conjecture for the Yau number of weighted homogeneous isolated hypersurface singularities and validated this conjecture for binomial isolated hypersurface singularities. A natural question is: what is the numerical relation between the new analytic invariant $\lambda^*(V)$ and the Yau number $\lambda(V)$? We propose the following conjecture:

Conjecture 4.1. Let $(V,0)$ be an isolated hypersurface singularity defined by $f \in \mathcal{O}_n$, $n \geq 2$, and multiplicity greater than or equal to 3. Let $\lambda^*(V)$ be the dimension of $L^*(V) := \mathrm{Der}(A^*(V), A^*(V))$, then $\lambda^*(V) = \lambda(V)$.

The above conjecture is true when the isolated singularity $(V,0)$ is not quasi-homogeneous. Recall the beautiful result of Saito ([39], Corollary 3.8): let $f \in \mathcal{O}_n$ be a germ of a holomorphic function which defines a hypersurface with an isolated singularity at 0, then f is not quasi-homogeneous, precisely when

$$\mathrm{Det}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j=1,\cdots,n} \in \left(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right).$$

Consequently, for non-quasi-homogeneous isolated hypersurface singularities, $A(V) = A^*(V)$. It follows that $L^*(V) = L(V)$ and $\lambda^*(V) = \lambda(V)$.

In [8], we have also obtained the following results. The Conjecture 4.1 is verified when $n \leq 4$:

Theorem C ([8]). *Let f be a weighted homogeneous polynomial in $\mathbb{C}[x_1, x_2, \ldots, x_n]$ $(n \geq 2)$ with respect to weight system $(w_1, w_2, \ldots, w_n; 1)$ and with $mult(f) \geq 3$. Suppose that f defines an isolated singularity $(V,0)$, then*

$$\lambda^*(V) \leq \lambda(V).$$

Furthermore, the equality holds when $n \leq 4$.

Elashvili and Khimshiashvili [13] proved the following result: if X and Y are two simple singularities except the pair A_6 and D_5, then $L(X) \cong L(Y)$ as Lie algebras, if and only if X and Y are analytically isomorphic. Finally, we shall also show that the simple hypersurface singularities can be characterized completely by the new Lie algebra $L^*(V)$.

Theorem D ([8]). *If X and Y are two simple hypersurface singularities, then $L^*(X) \cong L^*(Y)$ as Lie algebras, if and only if X and Y are analytically isomorphic.*

The proof follows directly from the computation performed in section 6 by a straightforward analysis of the new Lie algebras.

5. Fewnomial Singularities

In this subsection we recall the definition of fewnomial isolated singularities [30].

Definition 5.1. We say that a polynomial $f \in \mathbb{C}[z_1, z_2, \cdots, z_n]$ is fewnomial if the number of monomials in f does not exceed n.

Obviously, the number of monomials in f may depend on the system of coordinates. In order to obtain a rigorous concept, we shall only allow linear transformations of coordinates. The f (or rather its germ at the origin) is called a k-nomial if k is the smallest natural number such that f consists of k nomials after (possibly) a linear transformation of coordinates. An isolated hypersurface singularity V is called k-nomial if there exists an isolated hypersurface singularity Y analytically isomorphic to V which can be defined by a k-nomial f. It was shown in [10] that an $(n-1)$-dimensional isolated hypersuface singularity defined by a fewnomial f is isolated, only if f is a n-nomial in n variables when its multiplicity is at least 3.

Definition 5.2. We say that an isolated hypersurface singularity V is fewnomial if it is defined by a fewnomial polynomial f. V is called a weighted homogenous fewnomial isolated singularity, if it is defined by a weighted homogenous fewnomial polynomial f. The 2-nomial (resp. 3-nomial) isolated hypersurface singularity is also called binomial (resp. trinomial) singularity.

The following proposition and corollary tell us that each simple singularity belongs to one of the following three types.

Proposition 5.1 ([52]). *Let f be a weighted homogeneous fewnomial isolated hypersurface singularity with multiplicity at least 3. Then f is analytically equivalent to a linear combination of the following three series:*

Type A. $x_1^{a_1}+x_2^{a_2}+\cdots+x_{n-1}^{a_{n-1}}+x_n^{a_n}$, $n\geq 1$,

Type B. $x_1^{a_1}x_2+x_2^{a_2}x_3+\cdots+x_{n-1}^{a_{n-1}}x_n+x_n^{a_n}$, $n\geq 2$,

Type C. $x_1^{a_1}x_2+x_2^{a_2}x_3+\cdots+x_{n-1}^{a_{n-1}}x_n+x_n^{a_n}x_1$, $n\geq 2$.

Corollary 5.1 ([52]). *Each binomial isolated singularity is analytically equivalent to one of the three series: A) $x_1^{a_1}+x_2^{a_2}$, B) $x_1^{a_1}x_2+x_2^{a_2}$, and C) $x_1^{a_1}x_2+x_2^{a_2}x_1$.*

6. Computing the New Lie Algebras $L^*(V)$

In [8], we computed the new Lie algebra for binomial singularities, which includes the simple singularities as a special case. As an application, we proved that the simple hypersurface singularities can be characterized completely by the new Lie algebra. The following propositions are obtained in [8].

Proposition 6.1. *Let $(V,0)$ be a weighted homogeneous fewnomial isolated singularity of type A, defined by $f=x_1^{a_1}+x_2^{a_2}$ ($a_1\geq 2$, $a_2\geq 3$) with weight type $(\frac{1}{a_1},\frac{1}{a_2};1)$. Then*

$$\lambda^*(V)=\begin{cases} 2a_1a_2-3(a_1+a_2)+4, & a_1\geq 3, a_2\geq 3\\ a_2-3, & a_1=2, a_2\geq 3.\end{cases}$$

Remark 6.1. Since our new Lie algebra is not defined for Milnor number $\mu(f)=1$, the restriction $a_1\geq 2$, $a_2\geq 3$ in Proposition 6.1 follows from $\mu(f)\geq 2$. Similar restrictions also appear in Proposition 6.2 and Proposition 6.3 below.

Proposition 6.2. *Let $(V,0)$ be a binomial isolated singularity of type B defined by $f=x_1^{a_1}x_2+x_2^{a_2}$ ($a_1\geq 2$, $a_2\geq 2$) with weight type $(\frac{a_2-1}{a_1a_2},\frac{1}{a_2};1)$. Then*

$$\lambda^*(V)=\begin{cases} 2a_1a_2-2a_1-3a_2+5, & a_1\geq 2, a_2\geq 3\\ 2a_1-3, & a_1\geq 2, a_2=2.\end{cases}$$

Proposition 6.3. *Let $(V,0)$ be a binomial isolated singularity of type C, defined by $f=x_1^{a_1}x_2+x_2^{a_2}x_1$ ($a_1\geq a_2\geq 2$) with weight type $(\frac{a_2-1}{a_1a_2-1},\frac{a_1-1}{a_1a_2-1};1)$. Then*

$$\lambda^*(V)=\begin{cases} 2a_1a_2-2a_1-2a_2+6, & a_1\geq a_2\geq 3,\\ 2a_1, & a_1\geq a_2=2\end{cases}$$

In order to prove Theorem D, we need the following proposition.

Proposition 6.4. *The following three pairs of new Lie algebras arising from simple hypersurface singularities are not isomorphic:*

$$L^*(D_7)\ncong L^*(E_6),\ L^*(A_{10})\ncong L^*(E_7),\ \text{and}\ L^*(D_{10})\ncong L^*(E_8).$$

It is easy to see that, from Propositions 6.1 and 6.2, we get $\dim L^*(A_k)=k-2$, $\dim L^*(D_k)=k$, $\dim L^*(E_6)=7, \dim L^*(E_7)=8$, and $\dim L^*(E_8)=10$. The Cartan subalgebras of $L^*(A_k)$ and $L^*(D_k)$ are generated by $<x_2\partial_2>$ and $<x_1\partial_1,x_2\partial_2>$ respectively. It is then easy to verify that $\mathrm{rk}L^*(A_k)=\mathrm{rk}L^*(E_7)=1$ while $\mathrm{rk}L^*(E_6)=\mathrm{rk}L^*(E_8)=\mathrm{rk}L^*(D_k)=2$. When the dimensions or ranks of the new Lie algebras for all simple singularities are different, then they are certainly not isomorphic, so we only need to treat the three pairs of Lie algebras $(L^*(A_{10}),L^*(E_7)),(L^*(E_6),L^*(D_7)),(L^*(E_8),L^*(D_{10}))$ which have the same dimensions and ranks. It follows from the Proposition 6.4 that these three pairs are non-isomorphic. Therefore we have the following Theorem.

Proposition 6.5 (i.e. Theorem D). *If X and Y are two simple hypersurface singularities, then $L^*(X)\cong L^*(Y)$ as Lie algebras if and only if X and Y are analytically isomorphic.*

In fact, we have obtained the following theorem which generalizes Theorem D.

Theorem 6.1 ([8]). *Conjecture 4.1 is true for binomial singularities.*

Proof. In order to prove Conjecture 4.1, i.e., $\lambda^*(V)=\lambda(V)$, we need the following propositions from [52].

Proposition 6.6 ([52]). *Let $(V,0)$ be a weighted homogeneous fewnomial isolated singularity of type A defined by $f=x_1^{a_1}+x_2^{a_2}+\cdots+x_n^{a_n}$ with weight type $(\frac{1}{a_1},\frac{1}{a_2},\cdots,\frac{1}{a_n};1)$. Then the Yau number is*

$$\lambda(V)=n\prod_{i=1}^{n}(a_i-1)-\sum_{i}^{n}(a_1-1)(a_2-1)\cdots\widehat{(a_i-1)}\cdots(a_n-1),$$

where $\widehat{(a_i-1)}$ means that a_i-1 is omitted.

Proposition 6.7 ([52]). *Let $(V,0)$ be a binomial isolated singularity of type B defined by $f=x_1^{a_1}x_2+x_2^{a_2}$ with weight type $(\frac{a_2-1}{a_1a_2},\frac{1}{a_2};1)$. Then the Yau number is*

$$\lambda(V)=2a_1a_2-2a_1-3a_2+5.$$

Proposition 6.8 ([52]). *Let $(V,0)$ be a binomial isolated singularity of type C defined by $f=x_1^{a_1}x_2+x_2^{a_2}x_1$ with weight type $(\frac{a_2-1}{a_1a_2-1},\frac{a_1-1}{a_1a_2-1};1)$. If $mult(f)\geq 4$, i.e., $a_1,a_2\geq 3$, then the Yau number is*

$$\lambda(V)=2a_1a_2-2a_1-2a_2+6.$$

If $mult(f)=3$, i.e., $f=x_1^2x_2+x_2^{a_2}x_1$, then the Yau number is $\lambda(V)=2a_2$.

Comparing the Yau number $\lambda(V)$ with the new analytic invariant $\lambda^*(V)$ in the case of binomial isolated singularities of type A, type B and type C (see Propositions 6.1–6.3), it is easy to see that the Conjecture 4.1 holds for binomial isolated singularities, i.e.

$$\lambda^*(V) = \lambda(V),$$

and hence Theorem 6.1 is proved. □

7. k-th Yau Algebra

Let V be a hypersurface with an isolated singularity at the origin defined by the holomorphic function $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$. In [24], we introduced a new series of Lie algebras, i.e., k-th Yau algebras $L^k(V)$, $k\geq 0$, which are a generalization of Yau algebra as follows.

Recall that we have the following theorem.

Theorem 7.1 ([15], Theorem 2.26). *Let $f,g\in\mathfrak{m}\subset\mathcal{O}_n$. The following are equivalent:*

1) $(V(f),0)\cong(V(g),0)$;

2) For all $k\geq 0$, $\mathcal{O}_n/(f,\mathfrak{m}^kJ(f))\cong\mathcal{O}_n/(g,\mathfrak{m}^kJ(g))$ as $\mathbb{C}$-algebra;

3) There is some $k\geq 0$ such that $\mathcal{O}_n/(f,\mathfrak{m}^kJ(f))\cong\mathcal{O}_n/(g,\mathfrak{m}^kJ(g))$ as $\mathbb{C}$-algebra,

where $J(f)=(\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n})$.

In particular, if $k=0$ and $k=1$ above, then the claim of the equivalence of 1) and 3) is exactly same as the Mather-Yau theorem.

Based on Theorem 7.1, it is natural for us to introduce the new series of k-th Yau algebras $L^k(V)$ which are defined to be the Lie algebra of derivations of the k-th moduli algebra $A^k(V)=\mathcal{O}_n/(f,\mathfrak{m}^kJ(f)),k\geq 0$, i.e., $L^k(V)=\mathrm{Der}(A^k(V),A^k(V))$. Its dimension is denoted as $\lambda^k(V)$. This number $\lambda^k(V)$ is a new numerical analytic invariant of a singularity. We call it the k-th Yau number. In particular, $L^0(V)$ is exactly the Yau algebra, thus $L^0(V)=L(V),\lambda^0(V)=\lambda(V)$. Therefore, we have reasons to believe that these new Lie algebras $L^k(V)$ and numerical invariants $\lambda^k(V)$ will also play an important role in the study of singularities.

In [24], we obtained the weak Torelli-type theorems of simple elliptic singularities using Lie algebras $L^1(V)$ and $L^2(V)$. We have also characterized the simple singularities completely using $L^1(V)$.

Theorem E ([24]). *$L^2(\tilde{E}_6)$, $L^1(\tilde{E}_7)$, $L^2(\tilde{E}_7)$, $L^1(\tilde{E}_8)$ and $L^2(\tilde{E}_8)$ are non-trivial one-parameter families. Thus the weak Torelli-type theorems hold for simple elliptic singularities $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$.*

Theorem F ([24]). *If X and Y are two simple hypersurface singularities, then $L^1(X)\cong L^1(Y)$ as Lie algebras, if and only if X and Y are analytically isomorphic.*

Remark 7.1. *Using the same method as in our proof, though the calculations are extremely complicated, we conjecture that Theorem E and Theorem F are still true for $k>2$ and $k>1$ respectively.*

8. Isolated Complete Intersection Singularities

In [25], we generalized the construction of Yau algebra $L(V)$ to the case of isolated complete intersection singularity (this will be abbreviated in the sequel to ICIS) $(V,0)$.

In this section, we consider isolated complete intersection singularities $(V^n,0)$ in $(\mathbb{C}^m,0)$ which are defined by weighted homogeneous polynomials. Recall that a polynomial $f(x_1,\dots,x_m)$ is said to be weighted homogeneous with weights $(w_1,\dots,w_m)$, $w_j\in\mathbb{Q}$ and $w_j>1$, if for every monomial $\alpha x_1^{a_1}\dots x_m^{a_m}$ one has

$$a_1w_1+\cdots+a_mw_m=1.$$

We say that f has weight type $(w_1,\cdots,w_m;1)$. Sometimes we also use integer weight type in the following manner. Write $w_j=u_j/v_j$, where u_j and v_j are positive integers without common factor. Let $d=\mathrm{lcm}(v_1,\dots,v_m)$, and define $w'_j=dw_j$. Then f has integer weight type $(w'_1,\cdots,w'_m;d)$.

In this section we consider weighted homogenous complete intersection variety with isolated singularity $V=V(f_1,\dots,f_k)=V(f_1)\cap\dots\cap V(f_k)\subset\mathbb{C}^m$, where (i) each f_i is a weighted homogeneous polynomial with weights $(w_1,\dots,w_m)$ independent of i, and (ii) for all $V(f_1,\dots,f_k)$ is a complete intersection with an isolated singularity at the origin in $\mathbb{C}^m$. The weight type of $V=V(f_1,\dots,f_k)$ is denoted as $(w_1,\cdots,w_m;d_1,\cdots,d_m)$ where d_i is weight degree of f_i with respect to $(w_1,\cdots,w_m)$.

Observe that if we let $f:\mathbb{C}^m\to\mathbb{C}^k$ be the holomorphic function with coordinates $f_1,\dots,f_k$, then (ii) implies that the $k\times m$ Jacobian matrix $(\partial f_i/\partial z_j)$ has rank k everywhere in some neighborhood of the origin in V except possibly at the origin itself.

Let V be an analytic space at the origin of $\mathbb{C}^m$ defined by an ideal $I_V=(f_1,\cdots,f_k)\subset\mathfrak{m}^2$ as the fiber of the corresponding map germ $f:(\mathbb{C}^m,0)\to(\mathbb{C}^k,0)$. It is well-known [46] that, in the case $m\geq k$, the map germ f is finitely contact determined if and only if $(V,0)$ is an ICIS. Thus the ICIS $(V,0)$ is determined by the Artinian $\mathbb{C}$-algebra $\mathcal{O}_n/(I_V+\mathfrak{m}^{d+1})$ where d is the order of contact-determinacy of the map germ f. In the remaining part of the paper, we consider a different Artinian $\mathbb{C}$-algebra. More precisely, if V is an ICIS defined by an ideal I_V as above, then one can consider the singular subspace of V, which is the analytic space germ SV defined by the ideal $SI_V\subset\mathfrak{m}$ generated by the f_i and all the $k\times k$ minors in the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$, $i=1,\cdots,k;j=1,\cdots,m$. Since SV depends only on the isomorphism class of V, it follows that $\mathcal{O}_m/SI_V$ (the coordinate ring of SV) is an invariant of

$(V,0)$. In one-dimensional ICIS case, the $\dim(\mathcal{O}_m/SI_V)$ is exactly the Tjurina number of $(V,0)$. That is to say that if $f,g \in \mathbb{C}\{x,y,z\}$ are analytic functions defining an isolated curve singularity $(V,0)$, then the Tjurina number of $(V,0)$ (i.e., the dimension of the tangent space of the base space of the semiuniversal deformation of $(V,0)$), $\tau(V,0) = \dim_{\mathbb{C}}(\mathbb{C}\{x,y,z\}/(f,g,M_1,M_2,M_3))$, where M_1,M_2,M_3 are the 2-minors of the Jacobian matrix of f,g, i.e.,

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix},$$

and

$$M_1 = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}, M_2 = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial z} \end{vmatrix}, M_3 = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix}.$$

In the theory of isolated singularities, one always wants to find invariants associated to the isolated singularities. Hopefully with enough invariants found, one can distinguish between isolated singularities. However, not many invariants are known for ICIS. In [25], we generalize the construction of Yau algebra $L(V)$ to the case of ICIS $(V,0)$. We introduce a new derivation Lie algebra as follows.

Definition 8.1. For each ICIS $(V,0)$ in $(\mathbb{C}^m,0)$, the new derivation Lie algebra $\mathcal{NL}(V)$ is defined to be the Lie algebra of derivations of the local Artinian algebra $\mathcal{O}_m/SI_V$, i.e., $\mathcal{NL}(V) = \mathrm{Der}(\mathcal{O}_m/SI_V, \mathcal{O}_m/SI_V)$. Its dimension is denoted as $\upsilon(V)$.

This number $\upsilon(V)$ is also a new numerical analytic invariant. The new Lie algebra is a generalization of the Yau algebra $L(V)$.

It is interesting to bound the Yau number with a number which depends on weight type. In [52], we firstly proposed the sharp upper estimate conjecture that bound the Yau number $\lambda(V)$. They also proved that this conjecture holds in case of binomial isolated hypersurface singularities. Furthermore, in [18], this conjecture was verified for trinomial singularities (the definitions of fewnomial, binomial, and trinomial singularities can be found in [52]).

In [24], we proposed the following conjecture:

Conjecture 8.1. For each k, let $h_k(a_1,\cdots,a_m)$ denote $\lambda_k(\{x_1^{a_1}+\cdots+x_m^{a_m}=0\})$. Let $(V,0)=\{(x_1,x_2,\cdots,x_m)\in\mathbb{C}^m : f(x_1,x_2,\cdots,x_m)=0\},(m\geq 2)$ be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1,x_2,\cdots,x_m)$ of weight type $(w_1,w_2,\cdots,w_m;1)$. Then $\lambda_k(V)\leq h_k(1/w_1,\cdots,1/w_m)$.

The Conjecture 8.1 tells us that the Brieskorn singularity has the maximal dimension of derivation Lie algebra of L_k when fixing the weight type of the singularity. This gives a sharp upper bound for $\lambda_k(V)$. The conjecture was only proven for $k=0$, and 1 for binomial and trinomial singularities in [52], [18], and [24] respectively.

It is well-known that the ICIS of Brieskorn type is a generalization of Brieskorn hypersurface singularities. It is a natural question to consider the similar properties of Brieskorn ICIS.

Recall that a germ $(W,0)\subset(\mathbb{C}^m,0)$ of an n-dimensional Brieskorn ICIS is defined by

$$W=\left\{(x_i)\in\mathbb{C}^m \mid q_{j1}x_1^{a_1}+\cdots+q_{jm}x_m^{a_m}=0, j=1,\ldots,k\right\}$$

where $a_i\geq 2$ are integers and $n=m-k$. $(W,0)$ has an isolated singularity at 0 if and only if every maximal minor of the matrix (q_{ji}) does not vanish (cf. [[28], Section 7]).

A natural interesting question is: whether one can give a sharp bound for the new introduced $\upsilon(V)$ of an ICIS $(V,0)$. We proposed the following sharp upper estimate conjecture which is a natural generalization of Conjecture 8.1 to ICIS.

Conjecture 8.2. Let

$$\begin{aligned}(V,0)=\{&(x_1,x_2,\cdots,x_m)\in\mathbb{C}^m : f_i(x_1,x_2,\cdots,x_m)\\ &=0, 1\leq i\leq k\},(m\geq 2)\end{aligned}$$

be an n-dimensional $(n=m-k)$ ICIS defined by the weighted homogeneous polynomials $f_1,\cdots,f_k$ of weight type $(w_1,\cdots,w_m;d_1,\cdots,d_k)$. Let $(W,0)$ be an n-dimensional Brieskorn ICIS defined by polynomials

$$W=\left\{(x_i)\in\mathbb{C}^m \mid q_{j1}x_1^{a_1}+\cdots+q_{jm}x_m^{a_m}=0, j=1,\ldots,k\right\},$$

where

$$d=max\{d_1,\cdots,d_k\}, a_l=\lceil d/w_l\rceil, 1\leq l\leq m.$$

Then

$$\upsilon(V)\leq\upsilon(W).$$

Recall that the classifications of contact simple and unimodal complete intersection singularities were done by Giusti [14] and Aleksandrov [1]. The classification of the contact simple complete intersection (SCI) which is not hypersurface singularities (i.e., with modality 0) is as follows [14].

(1) Zero-dimensional simple complete intersection singularities.

Type $F_{q+r-1}^{q,r}$ $(xy,\ x^q+y^r);\ q,r\geq 2,$

Types $\left\{\begin{array}{ll} G_5 & (x^2,\ y^3), \\ G_7 & (x^2,\ y^4), \end{array}\right.$

Type H_μ $(x^2+y^{\mu-3},\ xy^2),\ \mu\geq 6,$

Types $\left\{\begin{array}{ll} I_{2q-1} & (x^2+y^3,\ y^q),\ q\geq 4, \\ I_{2r+2} & (x^2+y^3,\ xy^r),\ r\geq 3, \end{array}\right.$

(2) Simple complete intersection curve singularities.

Type S_μ $(x^2+y^2+z^{\mu-3},\ yz),\ \mu \geq 5$,

$$\text{Types} \begin{cases} T_7 & (x^2+y^3+z^3,\ yz), \\ T_8 & (x^2+y^3+z^4,\ yz), \\ T_9 & (x^2+y^3+z^5,\ yz), \end{cases}$$

$$\text{Types} \begin{cases} U_7 & (x^2+yz,\ xy+z^3), \\ U_8 & (x^2+yz+z^3,\ xy), \\ U_9 & (x^2+yz,\ xy+z^4), \end{cases}$$

$$\text{Types} \begin{cases} W_8 & (x^2+z^3,\ y^2+xz), \\ W_9 & (x^2+yz^2,\ y^2+xz), \end{cases}$$

$$\text{Types} \begin{cases} Z_9 & (x^2+z^3,\ y^2+z^3), \\ Z_{10} & (x^2+yz^2,\ y^2+z^3). \end{cases}$$

Note that all SCI singularities are weighted homogeneous singularities.

Aleksandrov [[1], page 21] and Wall [45] have obtained following classification for weighted homogeneous unimodal complete intersection singularities (here we use the notations in Aleksandrov's article).

Type(V)	Equations
T_{10}	$\{(x^2+y^3+z^6,\ ax+yz),\ 27a^6+4\neq 0\}$
T_k	$\{(x^2+y^3+z^{k-4},\ yz),\ k\geq 11\}$
R_9	$\{(x^2+y^4+z^4,\ ax+yz),\ 4a^4-1\neq 0\}$
R_k	$\{(x^2+y^4+z^{k-5},\ yz),\ k\geq 10\}$
$L_{2,q,r}$	$\{(x^2+y^q+z^r,\ yz),\ q,r\geq 5\}$
U_{11}	$\{(x^2+yz+z^4+axz^2,\ xy),\ a^2-4\neq 0\}$
U_{13}	$(x^2+yz,\ xy+z^6)$
U_{14}	$(x^2+z^5+yz,\ xy)$
U_{15}	$(x^2+yz,\ xy+z^7)$
V_{10}	$\{(x^3+yz+z^3+ax^2z,\ xy),\ 4a^3+27\neq 0\}$
V_{12}	$(x^4+yz+z^3,\ xy)$
V_{13}	$(yz+z^3,\ xy+x^4)$
V_{11}	$(x^5+yz+z^3,\ xy)$
Q_{13}	$(x^3+yz,\ xy+z^4)$
Q_{11}	$(x^3+yz+z^4,\ xy)$
$L_{3,2,4}$	$(x^4+y^2+z^3,\ xy)$
$L_{3,2,5}$	$(x^5+y^2+z^3,\ xy)$
Y_{11}	$\{(x^2+y^3+z^3+ay^2z,\ xy),\ 4a^3+27\neq 0\}$
G_{14}	$(x^2+y^3z+z^3,\ xy)$
H_{13}	$(x^2+y^2z,\ xy+z^3)$
H_{14}	$(x^2+y^3,\ xy+z^3)$
M_{11}	$\{(x^2+z^4,\ y^2+z^3+axz),\ a^2+1\neq 0\}$
M_{12}	$(x^2+yz^3,\ y^2+z^3)$
M_{13}	$(x^2+z^5,\ y^2+z^3)$
M_{14}	$(x^2+yz^4,\ y^2+z^3)$
N_{13}	$(x^2+yz^3,\ y^2+xz)$
N_{14}	$(x^2+z^5,\ y^2+xz)$

In [25], we verified the Conjecture 8.2 for simple and unimodal isolated complete intersection singularities. We also construct several new one-parameter families of solvable Lie algebras from T_{10}, R_9, U_{11}, V_{10}, Y_{11}, and M_{11} singularities and show that the weak Torelli-type theorem holds. We obtain the following results.

Theorem G. *The Conjecture 8.2 is true for the following classes of singularities:*

1) Contact simple complete intersection zero-dimensional singularities,

2) Contact simple complete intersection curve singularities,

3) Weighted homogeneous contact unimodal complete intersection curve singularities.

Theorem H. *The new derivation Lie algebras $\mathcal{NL}(V)$ of T_{10}, R_9, U_{11}, V_{10}, Y_{11}, and M_{11} are non-trivial one-parameter families. Thus the weak Torelli-type theorems hold for these one-parameter families of singularities T_{10}, R_9, U_{11}, V_{10}, Y_{11}, and M_{11}.*

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