Immersions of Manifolds and Homotopy Theory

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Abstract. The interface between the study of the topology of differentiable manifolds and algebraic topology has been one of the richest areas of work in topology since the 1950's. In these notes I will focus on one aspect of that interface: the problem of studying embeddings and immersions of manifolds using homotopy theoretic techniques. I will discuss the history of this problem, going back to the pioneering work of Whitney, Thom, Pontrjagin, Wu, Smale, Hirsch, and others. I will discuss the historical applications of this homotopy theoretic perspective, going back to Smale's eversion of the 2-sphere in 3-space. I will then focus on the problems of finding the smallest dimension Euclidean space into which every n-manifold embeds or immerses. The embedding question is still very much unsolved, and the immersion question was solved in the 1980's. I will discuss the homotopy theoretic techniques involved in the solution of this problem, and contributions in the 60's, 70's and 80's of Massey, Brown, Peterson, and myself. I will also discuss questions regarding the best embedding and immersion dimensions of specific manifolds, such has projective spaces. Finally, I will end by discussing more modern approaches to studying spaces of embeddings due to Goodwillie, Weiss, and others.

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Contents

Introduction 89
1 Vector Bundle Obstructions to Embeddings and Immersions 91
2 Foundational Work of Whitney, Smale, and Hirsch 92
2.1 Smale-Hirsch Theory, and "Turning a Sphere Inside out" 92
2.2 Whitney's Embedding and Immersion Theorems, and Translating Immersion Questions into Homotopy Theory 93
3 The Immersion Conjecture I: Work of W. Massey, R. Brown, and the Program of E. H. Brown Jr. and F. P. Peterson 94
3.1 Cohomology and Cobordism 94
3.2 Cobordisms, Spectra, and the Steenrod Algebra 95
3.3 The Brown-Peterson Approach to the Immersion Conjecture 99
4 The Immersion Conjecture II: Outline of Its Solution: The Homotopy Theory 104
5 Manifolds with Structure, Projective Spaces, and the Goodwillie-Weiss Embedding Calculus 107
5.1 Manifolds with Structure 107
5.2 Immersions of Projective Spaces 107
5.3 The Goodwillie-Weiss Calculus for Studying Embeddings 108
References 110

Introduction

In these notes we will discuss topics at the interface between the study of differentiable manifolds and of algebraic topology. This interface has been one of the richest areas of study in topology since the

1950's, with the pioneering work of R. Thom, L. Pontrjagin, J. Milnor, S. Smale, S. Novikov, M. Atiyah, R. Bott, F. Hirzebruch, as well as many others.

At one time the fields of Differential Topology and Algebraic Topology were separate and somewhat disjunct. Now there is no clear boundary between these fields. The study of manifolds has progressed remarkably with the use of homotopy theoretic techniques, and conversely, the study of manifolds has inspired algebraic topologists to develop and use techniques, including recent ones, that have found applications not only to differential topology, but to differential geometry, algebraic geometry, number theory, and even statistics and data analysis.

The focus of these notes will be on the following types of questions: Given two C^∞ manifolds M and N, does M embed in N? How can one tell when two embeddings are isotopic? More generally, what can one say about the topology of the space of embeddings, $Emb(M,N)$? These questions are quite hard, and in the more than 75 years since Whitney's pioneering work on embeddings, progress has been quite limited. There are also analogous questions about immersions, and there has been much more success in their study, primarily because, due to powerful results of Smale and Hirsch [38] [39] [25], these questions can be translated into questions in homotopy theory, and doing so brings powerful tools to bear.

These notes are organized as follows. We begin in Section 1 by discussing how vector bundle theory can be used to find obstructions to the existence of embeddings and immersions. More specifically, notice that if an embedding or an immersion of one manifold into another exists, then there will be an associated normal bundle. This is a vector bundle of fiber-dimension equal to the codimension of the embedding or immersion, and it satisfies some very specific properties. Therefore if one can show that no vector bundle of the right fiber-dimension satisfying these properties exists, then one would have an obstruction to the existence of the embedding or immersion. We then describe the remarkable result of Smale and Hirsch which essentially says that for immersions, the normal bundle obstruction is a *complete* obstruction to the existence of the immersion. As an application we discuss the first, and probably still the most startling of the applications of this result, Smale's eversion of S^2 in $\mathbb{R}^3$. That is to say, Smale's theorem says that one can turn the sphere in $\mathbb{R}^3$ "inside out" through a one-parameter family of immersions.

In Section 2 we begin our focus of studying embeddings and immersions of closed manifolds into Euclidean space. We recall Whitney's famous embedding and immersion theorems, and describe how, using Smale-Hirsch theory for immersions and the study of classifying spaces, questions about immersions of manifolds can be translated into homotopy theoretic questions. In Section 3 we begin our focus on the question of finding the smallest number $\phi(n)$ with the property every closed, smooth, n-dimensional manifold immerses in $\mathbb{R}^{n+\phi(n)}$. We describe Massey's theorem, in which he uses a characteristic class argument to show that $\phi(n) \geq n-\alpha(n)$, where $\alpha(n)$ is the number of ones in the binary expansion of n. We in particular show that this result is best possible by explicitly describing n-dimensional manifolds that immerse in $\mathbb{R}^{2n-\alpha(n)}$ but do not immerse in $\mathbb{R}^{2n-\alpha(n)-1}$. This led Massey to conjecture that $\phi(n) = n-\alpha(n)$, which is to say that every closed n-manifold immerses in $\mathbb{R}^{2n-\alpha(n)}$. This became known as the "Immersion Conjecture", and we outline its solution in the remainder of Section 3 and Section 4. We begin by recalling Thom's cobordism theorem, and show how it can be used to prove a theorem of R. Brown stating that the immersion conjecture is true "up to cobordism". That is, every closed n-manifold is cobordant to one that immerses in $\mathbb{R}^{2n-\alpha(n)}$. We then describe the Brown-Peterson program for the solution of the immersion conjecture and their remarkable contributions, which ultimately reduced the conjecture to the study of the homotopy type of particular spaces called "BO/I_n" which in some sense encode all the normal bundle obstructions to the immersion conjecture being true. We then describe the proof of the immersion conjecture given by the author in the early 1980's which studies the homotopy types of the Brown-Peterson space BO/I_n in great detail.

We begin Section 5 by describing characteristic class obstructions that have been computed for manifolds with structure e.g orientation, almost complex structures, and spin structures. This includes old work of Massey-Peterson, Papastavridis, and Koonce, as well as some quite recent work of Davis and Wilson. The author is grateful to Donald Davis for bringing some of this work to his attention. We then turn to immersion questions about specific manifolds, most notably projective spaces, and describe a strong nonimmersion theorem due to Davis. We finish with a short a description of a relatively new kind of homotopy theoretical application to the study of embeddings: the "Goodwillie-Weiss embedding calculus". After giving a brief description of the theory, we discuss a variety of results obtained over the last 20 years using this theory.

The author is grateful to Professor S.T. Yau for his invitation to give a lecture on this subject in the Harvard Center of Mathematical Sciences and Applications Math-Science Literature Lecture Series, and his encouragement to write these notes.

These notes are dedicated to Professor E. H. Brown Jr. in recognition of his many contributions to both algebraic and differential topology, and for putting up with the author as his PhD advisee in the mid-1970's.

1. Vector Bundle Obstructions to Embeddings and Immersions

Let M^n and N^{n+k} be smooth (C^∞) manifolds of dimensions n and $n+k$ respectively. We will assume $k \geq 1$. Recall that a smooth embedding of M^n into N^{n+k} is a C^∞-differentiable map which we denote by

$$e : M^n \hookrightarrow N^{n+k}$$

that maps M^n diffeomorphically onto its image. Associated to such an embedding is a k-dimensional *normal bundle* $\nu_e^k \to M^n$. A conceptually easy way to think of this normal bundle is by first endowing the ambient manifold N^{n+k} with a Riemannian metric, and then defining the normal space $\nu_e(x)$ for $x \in M^n$, to be the orthogonal complement of the image of the n-dimensional tangent space, $De(T_xM^n)$ in $T_{e(x)}N^{n+k}$. We write

$$\nu_e(x) = De(T_xM^n)^\perp \subset T_{e(x)}N^{n+k}.$$

Notice that a Riemannian metric is not crucial in the definition of this bundle, since there is an isomorphism from the orthogonal complement to the quotient space,

$$De(T_xM^n)^\perp \cong T_{e(x)}N^{n+k}/De(T_xM^n).$$

Therefore the normal bundle could be defined as the quotient bundle $e^*(TN^{n+k})/TM^n$, where $e^*(TN^{n+k})$ is the pull-back of the tangent bundle of N^{n+k} to M^n via the embedding e. Notice that the definition of this quotient bundle does not require the use of a metric. This in particular implies that the more conceptual definition, using a choice of metric, has an isomorphism type that is independent of that choice.

Since the normal bundle of an embedding $e : M^n \hookrightarrow N^{n+k}$ can be viewed as the orthogonal complement bundle to the image of the tangent bundle TM^n inside TN^{n+k}, it satisfies the following equation of vector bundles over M^n:

$$TM^n \oplus \nu_e^k \cong e^*(TN^{n+k}).$$

Since the isomorphism type of a pull-back vector bundle only depends on the homotopy type of the map being pulled back, we can conclude the following:

Proposition 1. *If a map $f : M^n \to N^{n+k}$ is homotopic to an embedding, then there exists a k-dimensional vector bundle $\nu \to M^n$ satisfying the equation*

$$TM^n \oplus \nu \cong f^*(TN^{n+k}).$$

If a vector bundle ν with this property exists we call it a "virtual normal bundle".

Thus the nonexistence of a virtual normal bundle is an *"obstruction"* to the map $f : M^n \to N^{n+k}$ being homotopic to an embedding.

Recall that an *immersion* $j : M^n \looparrowright N^{n+k}$ is a differentiable map whose derivative is a bundle monomorphism. That is, at every $x \in M^n$,

$$Dj_x : T_xM^n \to T_{j(x)}N^{n+k}$$

is a linear monomorphism. Recall that as a consequence of the implicit function theorem, an immersion is a local embedding. This means that around every $x \in M^n$, there is an open neighborhood $U_x \subset M^n$ so that the restriction of j to U_x is an embedding $j : U_x \hookrightarrow M^n$. In particular this means that immersions have normal bundles as well. Again, we can define it as the quotient bundle

$$\nu_j^k = j^*(TN^{n+k})/TM^n,$$

which again is isomorphic to the orthogonal complement bundle defined just as it is for embeddings. Thus the nonexistence of a virtual normal bundle is an obstruction to the existence of an immersion, just as it is to the existence of an embedding. That is, we can strengthen Proposition 1 to include immersions as well as embeddings. But a theorem of Hirsch and Smale, which we examine more closely in the next section, says that in the case of immersions, the virtual normal bundle obstruction is a *complete* obstruction to the existence of an immersion. In particular the following theorem holds:

Theorem 2 (Hirsch and Smale, 1959 [25] [38]). *If M^n is a closed, smooth n-manifold, with $n \geq 2$, and N^{n+k} is any smooth $(n+k)$-dimensional manifold with $k \geq 1$, then a map $f : M^n \to N^{n+k}$ is homotopic to an immersion if and only if there exists a k-dimensional vector bundle $\nu \to M^n$ satisfying the equation*

$$TM^n \oplus \nu \cong f^*(TN^{n+k}).$$

In other words, for every virtual normal ν there is an immersion $\tilde{f} : M^n \looparrowright N^{n+k}$ which is homotopic to f, whose normal bundle $\nu_{\tilde{f}}$ is isomorphic to ν. Furthermore, two immersions $j_0 : M^n \looparrowright N^{n+k}$ and $j_1 : M^n \looparrowright N^{n+k}$ are isotopic (i.e there is a continuous, one parameter family of immersions $h_t : M^n \looparrowright N^{n+k}, t \in [0,1]$, so that $h_0 = j_0, h_1 = j_1$) if and only if their normal bundles are isomorphic $\nu_{j_0} \cong \nu_{j_1}$.

As we will see in the next section, Hirsch and Smale actually proved a generalization of this theorem that is extremely powerful in its applications. But an important special case of the above theorem occurs when the target (ambient) manifold is $\mathbb{R}^{n+k}$. Since all maps $f : M^n \to \mathbb{R}^{n+k}$ are homotopic to a constant map, and the tangent bundle of Euclidean space is trivial, we have the following corollary of Theorem 2.

Corollary 3. *A closed smooth n-dimensional manifold M^n admits an immersion into $\mathbb{R}^{n+k}$ if and only if there is a k-dimensional vector bundle ν^k over M^n with the property that*

$$TM^n \oplus \nu^k \cong \epsilon^{n+k}.$$

Here ϵ^{n+k} is the trivial $(n+k)$-dimensional bundle, $\epsilon^{n+k} = M^n \times \mathbb{R}^{n+k}$. We refer to such a bundle ν^k as a "k-dimensional inverse" of the tangent bundle.

Furthermore, isotopy classes of immersions $M^n \looparrowright \mathbb{R}^{n+k}$ are in bijective correspondence with isomorphism classes of k-dimensional inverse bundles of TM^n.

Comments. Using standard vector bundle theory and in particular the theory of classifying spaces that we will describe in the next section, one can show that any finite dimensional vector bundle $\zeta \to X^{(n)}$ over an n-dimensional finite CW-complex $X^{(n)}$ has a k-dimensional inverse $\nu^k \to X^{(n)}$ for $k \geq n$. Moreover any two such k-dimensional inverses are isomorphic if $k \geq n+1$. Now by Morse theory one knows that any closed n-dimensional manifold is homotopy equivalent to an n-dimensional, finite CW-complex. Thus Corollary 3 implies that every closed n-manifold immerses in $\mathbb{R}^{2n}$, and any two immersions in $\mathbb{R}^{n+k}$ for $k > n$ are isotopic.

2. Foundational Work of Whitney, Smale, and Hirsch

2.1 Smale-Hirsch Theory, and "Turning a Sphere Inside out"

As mentioned in the last section, an amazing fact due to Smale and Hirsch (late 1950's) is that the normal bundle is a *complete* invariant of an immersion.

We will now describe a more general theorem that they proved. To do so, throughout this section, M^n will denote a smooth (C^∞) closed, n-dimensional manifold with $n \geq 2$, and N^{n+k} will be any smooth manifold of dimension $n+k$ with $k \geq 1$. Consider the space of all immersions, $Imm(M^n, N^{n+k})$. This space is topologized as a subspace of the space of continuous maps from M^n to N^{n+k}, which in turn is endowed with the compact-open topology.

As we recalled earlier, the derivative of an immersion $j : M^n \looparrowright N^{n+k}$. is a bundle monomorphism between their tangent bundles

$$Dj : TM^n \to TN^{n+k}.$$

So we now consider the space of all bundle monomorphisms, $Mono(TM^n, TN^{n+k})$. Recall that a bundle monomorphism between any two bundles $\zeta \to X$ and $\xi \to Y$ is a pair of maps $f : X \to Y$ and $\phi : \zeta \to \xi$ that make the following diagram commute

$$\begin{tikzcd} \zeta \arrow[r, "\phi"] \arrow[d] & \xi \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

and where ϕ is a linear monomorphism on each fiber, $\phi_x : \zeta_x \hookrightarrow \xi_x$. We therefore topologize the space of bundle monomorphisms $Mono(\zeta, \xi)$ to be a subspace of the product of the space of continuous maps from X to Y, and the space of continuous maps from ζ to ξ, both of which are endowed with the compact-open topology.

With these topologies, one may think of the derivative as a continuous map

$$D : Imm(M^n, N^{n+k}) \to Mono(TM^n, TN^{n+k}).$$

The following amazing theorem was proved by Smale [38] in the case when M^n is a sphere, and then generalized by Hirsch [25].

Theorem 4 (Hirsch and Smale, 1959 [38] [25]). *The derivative map*

$$\begin{aligned} D : Imm(M^n, N^{n+k}) &\to Mono(TM^n, TN^{n+k}) \\ f &\to Df : TM^n \hookrightarrow TN^{n+k} \end{aligned}$$

is a (weak) homotopy equivalence.

Here are a couple simple consequences of this result:

Consequences:

1. The space $Imm(M^n, N^{n+k})$ is nonempty if and only if the space $Mono(TM^n, TN^{n+k})$ is nonempty. In particular if one can, using vector bundle theory, find a bundle monomorphism between their tangent bundles, then one knows their exists an immersion of M^n into N^{n+k}.

2. The fact that the path components of $Imm(M^n, N^{n+k})$ and of $Mono(TM^n, TN^{n+k})$ are in bijective correspondence means that two immersions $j_1 : M^n \looparrowright N^{n+k}$ and $j_2 : M^n \looparrowright N^{n+k}$ are isotopic (which is equivalent to them living in the same path component of $Imm(M^n, N^{n+k})$), if and only if their derivatives are in the same path component of $Mono(TM^n, TN^{n+k})$. Furthermore this is true if and only if the pull-back bundles over M^n, $j_1^*(TN^{n+k})$ and $j_2^*(TN^{n+k})$ are isomorphic.

Now consider the special case when the target manifold is Euclidean space, $Imm(M^n, \mathbb{R}^{n+k})$. In this case, the derivative of an immersion $j : M^n \looparrowright \mathbb{R}^{n+k}$ assigns to every point $x \in M^n$ a linear monomorphism,

$T_xM^n \hookrightarrow \mathbb{R}^{n+k}$. We can think of this construction in terms of a fiber bundle

$$V_{n,n+k} \to \mathcal{V}(TM^n) \xrightarrow{p} M^n$$

Here $\mathcal{V}(TM^n)$ is the space of pairs (x,u), where $x \in M^n$ and $u : T_xM^n \to \mathbb{R}^{n+k}$ is a linear monomorphism. The map $p : \mathcal{V}(TM^n) \to M^n$ is defined by $p(x,u) = x$. Notice that the fibers of this fiber bundle are all homeomorphic to the Stiefel manifold $V_{n,n+k}$ of all linear monomorphisms $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k}$. In this case, since $\mathbb{R}^{n+k}$ is contractible, the space of bundle monomorphisms $Mono(M^n,\mathbb{R}^{n+k})$ is homotopy equivalent to the space of sections of this fiber bundle, which we denote by $\Gamma(\mathcal{V}(TM^n))$. This allows one to compute the homotopy type of the space of bundle monomorphisms, and thus by Theorem 4 the space of immersions of M^n into $\mathbb{R}^{n+k}$ in terms of the homotopy type of the Stiefel manifold.

In particular Smale proved the important special case of Theorem 4 when $M^n = S^n$ and $N^{n+k} = \mathbb{R}^{n+k}$, and a consequence of which became one of the most celebrated works of the 20th century. He proved the following:

Theorem 5 (Smale, 1958 [38]). *The set of isotopy classes of immersions of S^n into $\mathbb{R}^{n+k}$ is in bijective correspondence with the set of path components, $\pi_0(Imm(S^n,\mathbb{R}^{n+k}))$, and for $k > 1$,*

$$\pi_0(Imm(S^n,\mathbb{R}^{n+k})) \cong \pi_n(V_{n,n+k}).$$

For $k = 1$, there is a surjection

$$\pi_n(V_{n,n+1}) \to \pi_0(Imm(S^n,\mathbb{R}^{n+1})).$$

Moreover $V_{n,n+1} \simeq SO(n+1)$, the special orthogonal group.

Smale then observed that since $SO(3)$ is homeomorphic to the projective space, $\mathbb{RP}^3$, and since its universal covering space is the sphere S^3, one knows that the second homotopy group is trivial

$$\pi_2(V_{2,3}) = \pi_2(SO(3)) \cong \pi_2(\mathbb{RP}^3) \cong \pi_2(S^3) = 0.$$

From this theorem one can conclude that the space $Imm(S^2,\mathbb{R}^3)$ is path connected. This means that any two immersions of S^2 in $\mathbb{R}^3$ are isotopic! In particular one can isotop the identity immersion of S^2 as the unit sphere to its opposite $(t_1,t_2,t_3) \to (-t_1,-t_2,-t_3)$. Such an isotopy (or regular homotopy) is called an "eversion" of S^2. So one can "turn a sphere inside out!".

Remarks:

1. The Hirsch-Smale theorem was an early example of a type of theorem that is now known as an "h-principle". Over the years these have been studied and greatly generalized by Gromov, Eliashberg, Mishachev, Vassiliev, and many others.
2. This homotopy theoretic argument for the existence of an eversion of S^2 in $\mathbb{R}^3$ is, of course, nonconstructive. In fact in Smale's paper he remarked that he did not know how such an eversion might be constructed. However explicit constructions of eversions were eventually discovered by Shapiro, Phillips, Morin, Thurston, and others.

2.2 Whitney's Embedding and Immersion Theorems, and Translating Immersion Questions into Homotopy Theory

Recall Whitney's famous embedding and immersion theorem:

Theorem 6 (Whitney, 1944 [47] [48]).

- *Let M^n be a closed n-dimensional manifold, $n \geq 2$. Then there is an embedding $e : M^n \hookrightarrow \mathbb{R}^{2n}$.*
- *Any two embeddings of M^n into $\mathbb{R}^{2n+k}$ for $k \geq 1$ are isotopic.*
- *There is an immersion $j : M^n \looparrowright \mathbb{R}^{2n-1}$.*
- *Any two immersions of M^n into $\mathbb{R}^{2n+k}$ for $k > 0$ are isotopic.*

Let $e(n)$ be the smallest integer so that every closed n-manifold embeds in $\mathbb{R}^{n+e(n)}$. By Whitney's theorem one knows that $e(n) \leq n$. Notice that for $n = 1$ or 2, $e(n) = n$ as the circle embedded in $\mathbb{R}^2$ and the Klein bottle embedded in $\mathbb{R}^4$ demonstrate. More generally, Whitney also knew that $\mathbb{RP}^{2^k}$ does not embed in $\mathbb{R}^{2^{k+1}-1}$ by a characteristic class argument. (This was, perhaps, the earliest characteristic class argument regarding embeddings.) So Whitney's theorem is the best possible in dimensions equal to a power of 2.

Whitney's result can be improved to $e(n) \leq n-1$ unless n is a power of 2. This is a result of Haefliger and Hirsch [26], [24] (for $n > 4$) and C. T. C. Wall [43] (for $n = 3$). In general, though, unless n is a power of 2, a closed formula for $e(n)$ is still not known, and it is a difficult and deep question.

Consider the corresponding question about immersions. Let $\phi(n)$ be the smallest integer so that every closed n-manifold *immerses* in $\mathbb{R}^{n+\phi(n)}$.

By the Hirsch-Smale Theorem 4, this can be translated to a question about vector bundle theory. Namely, we have the following corollary:

Corollary 7 (Hirsch-Smale). *$\phi(n)$ is equal to the smallest integer so that $Mono(TM^n, T\mathbb{R}^{n+\phi(n)})$ is nonempty for every closed n-manifold M^n.*

Given a bundle monomorphism $j : TM^n \to T\mathbb{R}^{n+k}$ and a point $x \in M^n$, consider the linear embedding $j_x : T_xM^n \hookrightarrow \mathbb{R}^{n+k}$. Let $\nu_x \subset \mathbb{R}^{n+k}$ be the orthogonal complement of the image of j_x. Then the collection $\{\nu_x :$

$x \in M^n\}$ defines a k-dimensional vector bundle $\nu \to M^n$ with the property that

$$TM^n \oplus \nu \cong \epsilon^{n+k}$$

where ϵ^{n+k} is the trivial bundle of dimension $n+k$. In other words, ν is a k-dimensional inverse of TM^n.

Hirsch-Smale theory says that this "virtual normal bundle" $\nu \to M^n$ is isomorphic to an honest normal bundle $\nu_f \to M^n$ of an immersion

$$f : M^n \looparrowright \mathbb{R}^{n+k}.$$

Corollary 8 (Hirsch-Smale)**.**

- *M^n immerses in $\mathbb{R}^{n+k}$ if and only if M^n has a "virtual normal bundle" of dimension k.*
- *$\phi(n)$ is equal to the smallest integer for which every closed n-manifold M^n admits a $\phi(n)$-dimensional virtual normal bundle (i.e a $\phi(n)$ dimensional inverse to TM^n).*

Hirsch-Smale theory (Theorem 4) thus reduces the problem of finding the best Euclidean space immersion dimension for any n-manifold ($\phi(n)$) to a question in vector bundle theory.

We now want to reduce the bundle theory question to a question of homotopy theory, via the use of classifying spaces. A quick introduction to the theory of classifying spaces can be found in [17].

A basic result in this theory states that for any topological group G, there is a "universal principal G-bundle" $G \to EG \xrightarrow{p} BG$. The term "universal" comes from the following property.

Given a map $f : X \to BG$, consider the pullback bundle $G \to f^*(EG) \to X$, where

$$f^*(EG) = \{(x,u) \in X \times EG : f(x) = p(u)\}$$

This pullback construction induces a set map

$$\rho_{EG} : [X, BG] \to Prin_G(X).$$

Here $[X, BG]$ means homotopy classes of maps from X to BG, and $Prin_G(X)$ is the set of isomorphism classes of principal G-bundles over X. The statement that $p : EG \to BG$ is universal means that ρ_{EG} is a bijection for every space X of the homotopy type of a CW-complex.

Universal bundles always exist, and are unique up to fiberwise homotopy equivalence. BG is called a "classifying space" of the group G.

For $G = O(n)$ there is a bijection between isomorphism classes of principal $O(n)$ bundles and isomorphism classes of n-dimensional vector bundles,

$$Prin_{O(n)}(X) \xrightarrow{\cong} Vect^n(X).$$
$$(E \to X) \to (E \times_{O(n)} \mathbb{R}^n \to X)$$

This implies that there is a bijection,

$$[X, BO(n)] \cong Vect^n(X).$$

So bundle theory can be studied via homotopy theory. We now apply this fact to immersion theory.

A corollary to Whitney's Immersion Theorem 6 states that any two immersions of a closed n-manifold M^n into $\mathbb{R}^L$ are isotopic ("regularly homotopic") if $L > 2n$. So by combining Hirsch-Smale theory with Whitney's theorem, we can conclude that for L large, every manifold M^n is equipped with a map, which is well defined up to homotopy,

$$\nu_M^L : M^n \to BO(L)$$

that classifies a normal bundle of an immersion into codimension L Euclidean space. By taking the limit over L, we call the resulting space BO, and we call the map,

$$\nu_M : M^n \to BO$$

the "stable normal bundle" map for M^n. Again, it is well-defined up to homotopy.

We then get the following interpretation of the Hirsch-Smale Theorem 4 when applied to this setting. Notice that it describes the immersion problem entirely in terms of homotopy theory:

Corollary 9. *A closed n-manifold M^n admits an immersion into $\mathbb{R}^{n+k}$ if an only if there is a map $\nu^k : M^n \to BO(k)$ making the following diagram homotopy commute:*

$$\begin{tikzcd} M^n \arrow[r, "\nu^k"] \arrow[d, "="'] & BO(k) \arrow[d] \\ M^n \arrow[r, "\nu_M"'] & BO \end{tikzcd}$$

Thus the Euclidean space immersion problem has been entirely translated to a question of "homotopy lifting" the stable normal bundle map.

3. The Immersion Conjecture I: Work of W. Massey, R. Brown, and the Program of E. H. Brown Jr. and F. P. Peterson

3.1 Cohomology and Cobordism

In order to get an idea for what is the smallest integer $\phi(n)$ for which the stable normal bundle map $\nu_{M^n} : M^n \to BO$ of any closed n-manifold M^n lifts to $BO(\phi(n))$, we look for cohomological obstructions.

We begin by recalling the following cohomology calculations. All coefficients will be $\mathbb{Z}/2$, and the ring structures of the cohomologies come from the standard cup products.

$$H^*(BO;\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1,\cdots,w_i,\cdots]$$
$$H^*(BO(k);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1,\cdots,w_k]$$

where $w_i \in H^i(BO;\mathbb{Z}/2)$ is known as the i^{th} *Stiefel-Whitney characteristic class*. The inclusion map $BO(k) \hookrightarrow BO$ induces a ring homomorphism in cohomology which sends w_j to 0 for $j > k$,

We write $\bar{w}_k(M) = \nu^*_{M^n}(w_k) \in H^k(M^n;\mathbb{Z}/2)$. This is known as the k^{th} *normal Stiefel-Whitney class of* M^n. Since the homotopy type of the stable normal bundle map $\nu_{M^n} : M^n \to BO$ is a well defined invariant of the manifold M^n, the normal Stiefel-Whitney classes are also cohomological invariants of M^n. We therefore have the following corollary of these cohomological calculations:

Corollary 10. *If $\bar{w}_k(M^n) \neq 0$ then M^n does not immerse in $\mathbb{R}^{n+k-1}$.*

Example. It is a well-known, standard calculation that $\bar{w}_{2^k-1}(\mathbb{RP}^{2^k}) \neq 0$ in $H^{2^k-1}(\mathbb{RP}^{2^k};\mathbb{Z}_2)$. A good reference is the text by Milnor and Stasheff [33]. This calculation was first done by Whitney in [47]. By his immersion theorem, $\mathbb{RP}^{2^k} \looparrowright \mathbb{R}^{2^{k+1}-1}$, but since $\bar{w}_{2^k-1}(\mathbb{RP}^{2^k}) \neq 0$, then $\mathbb{RP}^{2^k}$ *does not* immerse in $\mathbb{R}^{2^{k+1}-2}$.

Notice that as a consequence of Whitney's immersion theorem, which can be interpreted as saying that $\phi(n) \leq n-1$, and this example, which implies that $\phi(2^k) \geq 2^k - 1$, we may conclude that $\phi(2^k) = 2^k$. This implies that Whitney's theorem is best possible for $n = 2^k$.

Continuing to look for cohomological obstructions to immersing manifolds, we note that in 1960 W. Massey [31] made the following important calculation, which involved inputting Poincaré duality into Stiefel-Whitney class calculations. The following is his result.

Theorem 11 (Massey, 1960 [31])**.** *For M^n a closed n-manifold, $\bar{w}_i(M^n) = 0$ for $i > n - \alpha(n)$, where $\alpha(n) =$ the number of ones in the binary expansion of n.*

Furthermore, this result is the best possible as the following example demonstrates:

Write n as a sum of distinct powers of 2:

$$n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_r}.$$

So in this case $r = \alpha(n)$.

Let $M^n = \mathbb{RP}^{2^{i_1}} \times \cdots \times \mathbb{RP}^{2^{i_r}}$. We observe that there is a Stiefel-Whitney class obstruction to M^n immersing in $\mathbb{R}^{2n-\alpha(n)-1}$. To see this one uses a product formula for Stiefel-Whitney classes (the "Cartan formula"), to conclude that

$$\begin{aligned}\bar{w}_{n-\alpha(n)}(M^n) &= \bar{w}_{(2^{i_1}-1)+(2^{i_2}-1)+\cdots+(2^{i_r}-1)}(\mathbb{RP}^{2^{i_1}} \times \cdots \times \mathbb{RP}^{2^{i_r}}) \\ &= \bar{w}_{(2^{i_1}-1)}(\mathbb{RP}^{2^{i_1}}) \times \bar{w}_{(2^{i_2}-1)}(\mathbb{RP}^{2^{i_2}}) \times \cdots \\ &\quad \times \bar{w}_{(2^{i_r}-1)}(\mathbb{RP}^{2^{i_r}}) \\ &\neq 0.\end{aligned}$$

Using Corollary 10 we conclude that this n-dimensional manifold $M^n = \mathbb{RP}^{2^{i_1}} \times \cdots \times \mathbb{RP}^{2^{i_r}}$ does not immerse in $\mathbb{R}^{2n-\alpha(n)-1}$. In other words, this example, together with Whitney's Immersion Theorem 6 demonstrates hat the best immersion dimension $n + \phi(n)$ for all n-manifolds M^n, satisfies

$$n - \alpha(n) \leq \phi(n) \leq n-1. \tag{1}$$

Equivalently, using the calculations of $H^*(BO(k);\mathbb{Z}/2)$ given above, Massey's theorem can be interpreted as saying that the following cohomological theorem is true.

Theorem 12 (Massey [31])**.** *For every n-manifold there exists a homomorphism of graded rings,*

$$\theta_{M^n} : H^*(BO(n-\alpha(n));\mathbb{Z}_2) \to H^*(M^n;\mathbb{Z}_2)$$

making the following diagram commute:

$$\begin{tikzcd} H^*(BO;\mathbb{Z}_2). \arrow[r, "\nu^*_{M^n}"] \arrow[d] & H^*(M^n;\mathbb{Z}_2) \arrow[d, "="] \\ H^*(BO(n-\alpha(n));\mathbb{Z}_2). \arrow[r, "\theta_{M^n}"] & H^*(M^n;\mathbb{Z}_2) \end{tikzcd}$$

Notice that if the homomorphism θ_{M^n} can be realized by a map $\tilde{\nu}_{M^n} : M^n \to BO(n-\alpha(n))$ that lifts the stable normal bundle map $\nu_{M^n} : M^n \to BO$, then by Hirsch-Smale theory, $\tilde{\nu}_{M^n}$ would classify the normal bundle of an immersion $j : M^n \looparrowright \mathbb{R}^{2n-\alpha(n)}$. This leads to the following conjecture, originally due to Massey.

Immersion Conjecture (Massey) $\phi(n) = n - \alpha(n)$. That is, every closed n-manifold $M^n \looparrowright \mathbb{R}^{2n-\alpha(n)}$.

By the above example, ($M^n = \mathbb{RP}^{2^{i_1}} \times \cdots \times \mathbb{RP}^{2^{i_r}}$), this conjecture is as strong as possible.

3.2 Cobordisms, Spectra, and the Steenrod Algebra

We now begin the description of a program that eventually led to a solution of this conjecture. As one does with many questions in differential topology, we will start with R. Thom's work on cobordism theory. Thom's work was one of the real breakthroughs in manifold theory, and in particular showed how the disciplines of differential topology and algebraic topology are inseparable. In particular Thom's results spurred on the development of stable homotopy theory, an area that is still extremely active, and an area that is under constant development.

We begin with the notion of the *Thom space* of a vector bundle $\zeta \to X$, which we will denote by X^ζ. We will assume the bundle has been given a Euclidean metric, and one defines X^ζ by

$$X^\zeta = D(\zeta)/S(\zeta)$$

where $D(\zeta)$ is the unit disk bundle, $D(\zeta) = \{v \in \zeta : |v| \leq 1\}$, and $S(\zeta)$ is the unit sphere bundle $S(\zeta) = \{v \in \zeta : |v| = 1\}$. Notice that if the base space X is compact, the Thom space X^ζ is homeomorphic to the one-point compactification, $\zeta \cup \infty$.

The following is the classical Thom isomorphism theorem (with $\mathbb{Z}/2$-coefficients).

Theorem 13. *Let ζ^k be a k-dimensional vector bundle over a connected space X. The Thom space X^{ζ^k} satisfies the following properties.*

1. $$H^k(X^{\zeta^k};\mathbb{Z}/2) \cong \mathbb{Z}/2$$

   generated by a class $u_k \in H^k(X^{\zeta^k};\mathbb{Z}/2)$ called the Thom class.
2. *For every $n \geq 0$, there is an isomorphism of $H^n(X;\mathbb{Z}/2)$ with $H^{n+k}(X^{\zeta^k};\mathbb{Z}/2)$ given by the cup product with u_k:*

$$\begin{aligned}\cup u_k : H^n(X;\mathbb{Z}/2) &\cong H^n(D(\zeta^k);\mathbb{Z}/2)\\ &\xrightarrow{\cong} H^{n+k}(D(\zeta^k), S(\zeta^k);\mathbb{Z}/2)\\ &\cong \tilde{H}^{n+k}(X^{\zeta^k};\mathbb{Z}/2).\end{aligned}$$

Let $\gamma_n \to BO(n)$ be the universal vector bundle

$$\gamma_n = EO(n) \times_{O(n)} \mathbb{R}^n \to BO(n).$$

This has a concrete description as follows. A good model for the universal principal bundle

$$EO(n) \to BO(n)$$

is to let $EO(n)$ be the infinite dimensional Stiefel manifold of linear monomorphisms $\mathbb{R}^n \hookrightarrow \mathbb{R}^\infty$, and $BO(n)$ can be taken to be the resulting infinite dimensional Grassmannian manifold of n-dimensional linear subspaces of $\mathbb{R}^\infty$. The map $EO(n) \to BO(n)$ is defined by taking the image subspace of a linear monomorphism. With these models, the universal vector bundle $EO(n) \times_{O(n)} \mathbb{R}^n$ is the space of pairs (V, v) where $V \subset \mathbb{R}^\infty$ is an n-dimensional subspace, and $v \in V$ is a vector. Then, of course the map $EO(n) \times_{O(n)} \mathbb{R}^n \to BO(n)$ simply maps (V, v) to V, viewed as an element of the Grassmannian.

Using Thom's original notation, we let $MO(n)$ be the Thom space $MO(n) = BO(n)^{\gamma_n}$.

Consider the inclusion map

$$\iota : BO(k) \to BO(k+1).$$

Observe that the pull-back of γ_{k+1} over $BO(k)$ via the map ι, $\iota^*(\gamma_{k+1})$ is simply the $(k+1)$-dimensional bundle $\gamma_k \oplus \epsilon^1$. The Thom space of this bundle is the suspension $\Sigma MO(k)$. Therefore on the Thom space level the inclusion map ι induces a map

$$\iota_k : \Sigma MO(k) \to MO(k+1). \tag{2}$$

These maps give the collection of space $\{MO(k); k \geq 0\}$ the structure of a *spectrum.* For our purposes we use the following definition of a spectrum.

Definition 1. *A spectrum $\mathbb{E}$ is a sequence of spaces $\{E_n\}$ together with maps $e_n : \Sigma E_n \to E_{n+1}$. These maps are called the structure maps of the spectrum $\mathbb{E}$.*

The above is the classical definition of spectrum, going back to Lima [29] and Whitehead [46]. In the current literature the above structure is often referred to as a "prespectrum". Studying categories of spectra satisfying appropriate properties is of great importance in modern homotopy theory, but we will not be concerned with the strict definitions of these categories in this expository paper.

Spectra have homotopy and homology groups. They are defined by

$$\begin{aligned}\pi_q\mathbb{E} &= \varinjlim_{k\to\infty} \pi_{q+k}E_k\\ H_q\mathbb{E} &= \varinjlim_{k\to\infty} \tilde{H}_{q+k}E_k\end{aligned} \tag{3}$$

where the limits are defined using the structure maps e_n and the suspension homomorphisms.

The following are perhaps the most important examples of spectra:

Examples.

1. For a space X with a basepoint $x_0 \in X$, we define its suspension spectrum by

$$\Sigma^\infty X = \{\Sigma^n X, id\}.$$

Notice that by the suspension isomorphism, $H_*(\Sigma^\infty X) = \tilde{H}_*(X)$ and $\pi_*(\Sigma^\infty X)$ are the stable homotopy groups of X. When $X = S^0$, the zero dimensional sphere (i.e the two-point space), then $\Sigma^\infty(S^0)$ is called the *sphere spectrum,* which we denote by $\mathbb{S}$. The n^{th}-space of the sphere spectrum is the n-dimensional sphere, S^n.

2. Let G be an abelian group, and let $K(G,n)$ be an Eilenberg-MacLane space of type (G,n) for $n > 0$. This means that $K(G,n)$ is a space with

$$\pi_q(K(G,n)) = \begin{cases} G, & \text{if} \quad q = n \\ 0 & \text{otherwise.}\end{cases}$$

It is a well known property of Eilenberg-MacLane spaces that if X is any space of the homotopy type of a CW-complex with basepoint, then the set of homotopy classes of basepoint preserving maps $[X, K(G,n)]$

is isomorphic to the cohomology group, $H^n(X;G)$. This leads to the fact that

$$H^n(K(G,n);G) \cong Hom(G,G)$$

and there is a fundamental class $\iota_n \in H^n(K(G,n);G)$ corresponding to the identity homomorphism. Since, by the suspension isomorphism,

$$H^{n+1}(\Sigma K(G,n);G) \cong H^n(K(G,n);G) \cong Hom(G,G)$$

there is a map, well defined up to homotopy

$$\iota_n : \Sigma K(G,n) \to K(G,n+1)$$

corresponding to the fundamental class. The collection $\{K(G,n),\iota_n\}$ defines a spectrum called the "Eilenberg-MacLane spectrum", which we denote by $\mathbb{H}G$.

3. Let $\mathbb{MO} = \{MO(n),\iota_n\}$ be the Thom spectrum defined by (2).

As described in [1], given a spectrum $\mathbb{E}$, one can suspend or desuspend $\mathbb{E}$, and study homotopy classes of maps of any degree from spaces to $\mathbb{E}$. (A map of degree one from X to $\mathbb{E}$ is a map from X to $\Sigma\mathbb{E}$.) In particular $\mathbb{E}$ defines a generalized homology and cohomology theories a follows. Given a space X of the homotopy type of a CW-complex define

$$\mathbb{E}^q(X) = [X_+,\mathbb{E}]^q = \varinjlim_k [\Sigma^k(X_+),E_{k+q}],$$
$$\mathbb{E}_q(X) = \pi_q(\mathbb{E}\wedge X_+) = \varinjlim_k \pi_{q+k}(E_k \wedge X_+),$$

where X_+ denotes X with a disjoint basepoint. (**Note.** Given a based space Y and a spectrum $\mathbb{E}$ one can define the smash product spectrum $\mathbb{E}\wedge Y$ to be the sequence of spaces $\{E_n \wedge Y\}$ and structure maps $e_n \wedge 1 : \Sigma(E_n \wedge Y) = \Sigma E_n \wedge Y \to E_{n+1}\wedge Y$.)

When $\mathbb{E} = \mathbb{H}G$, a classical result of Whitehead [46] states that the generalized (co)homology this spectrum represents is simply ordinary (co)homology with coefficients in G. When $\mathbb{E} = \mathbb{MO}$, the associated (co)homology theory is called the (co)bordism groups of a space X.

The following theorem, and its proof, have had a huge impact on algebraic and differential topology.

Theorem 14 (Thom, 1954 [41]). *There is an isomorphism between the homotopy groups of the Thom spectrum,*

$$\pi_n(\mathbb{MO}) = \lim_{k\to\infty} \pi_{n+k}(MO(k))$$

and the set of cobordism classes of closed n-manifolds, η_n. This is defined to be the set of equivalence classes of n-dimensional closed manifolds, defined by saying M_1^n is cobordant to M_2^n if there is an $(n+1)$ dimensional manifold with boundary, W^{n+1}, with

$$\partial W^{n+1} = M_1^n \sqcup M_2^n.$$

The abelian group structure on η_n corresponding to the group structure on stable homotopy groups is simply induced by disjoint union. The identity element in this group is the empty set $\emptyset$ (by convention $\emptyset$ can be viewed as a manifold of any dimension). Notice that this group consists entirely of elements of order 2, which one sees because for any closed n-manifold M^n, the disjoint union $M^n \sqcup M^n$ is cobordant to the empty set $\emptyset$ because it is the boundary of $W^{n+1} = M^n \times [0,1]$. Furthermore, the graded abelian groups $\eta_* \cong \pi_*^s(\mathbb{MO})$ actually form a graded ring, with the product given by cartesian product of manifolds.

Thom also did a complete calculation of these graded rings.

Theorem 15 ([41]).

$$\eta_* \cong \mathbb{Z}_2[b_2,b_4,b_5,\cdots,b_r,\cdots : r \neq 2^k-1].$$

In other words, η_* is a polynomial algebra over the field $\mathbb{Z}/2$ with one generator b_r of dimension $r>0$ so long as r is not of the form 2^k-1 for any integer $k>0$.

In fact Thom gave a complete description of the homotopy type of the spectrum $\mathbb{MO}$.

Theorem 16 ([41]). *The spectrum $\mathbb{MO}$ has the homotopy type of a wedge of Eilenberg-MacLane spectra,*

$$\mathbb{MO} \simeq \bigvee_{\omega\in I} \Sigma^{|\omega|}\mathbb{H}\mathbb{Z}/2$$

where the indexing set I consists of all monomials in $\mathbb{Z}/2[b_2,b_4,\cdots,b_r\cdots,: r \neq 2^k-1]$. The notation $|\omega|$ refers to the dimension of the monomial $b_\omega \in \mathbb{Z}/2[b_2,b_4,\cdots,b_r\cdots,: r\neq 2^k-1]$.

In order to understand more about the immersion conjecture and how it was proved, it is important to recall a bit about how Thom proved this theorem. His main tool was the Steenrod algebra, which we now discuss.

Recall that the Steenrod squaring operations, Sq^i, $i \geq 0$, satisfy the following axioms:

Axioms.

1. Sq^i is a natural transformation of abelian group valued functors

   $$Sq^i : H^n(-;\mathbb{Z}/2) \to H^{n+i}(-;\mathbb{Z}/2)$$

   for every n,
2. $Sq^0 = 1$ the identity transformation
3. $Sq^i(x) = 0$ if the dimension of x is less than i
4. $Sq^i(x) = x^2$ if the dimension of x equals i
5. The Steenrod satisfy the product formula known as the "Cartan formula":

   $$Sq^i(xy) = \sum_j (Sq^j x)(Sq^{i-j} y).$$

6. Sq^1 is the Bockstein homomorphism of the coefficient sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0.$$

7. The Steenrod squares satisfy the "Adem relations":
For $a < 2b$,

$$Sq^a Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j$$

where the binomial coefficients are taken mod 2.

Axioms (6) and (7) can be shown to be consequences of axioms (1)–(5). The Steenrod operations act on the cohomology of spectra as well as spaces. One of their important features is how they are related to the Stiefel-Whitney characteristic classes. Recall from Theorem 13 that if one is given a k-dimensional vector bundle $\zeta^k \to X$ then the Thom class $u_k \in H^k(X^{\zeta^k};\mathbb{Z}/2)$ defines the Thom isomorphism,

$$\cup u_k : H^q(X;\mathbb{Z}/2) \xrightarrow{\cong} \tilde{H}^{q+k}(X^{\zeta^k};\mathbb{Z}/2)$$

Then the Steenrod squaring operations, when applied to the Thom class are related to the Stiefel-Whitney classes of the bundle ζ^k by the formula:

$$w_i(\zeta^k) \cup u_k = Sq^i(u_k) \in H^{k+i}(X^{\zeta^k};\mathbb{Z}/2). \tag{4}$$

The mod 2 Steenrod algebra $\mathcal{A}$ is the algebra generated operations Sq^i subject to the Adem relations. From the axioms it is not difficult to construct an additive basis for $\mathcal{A}$. Namely, if $I = (i_1, \cdots, i_q)$ is a finite sequence of positive integers, let Sq^I be the product

$$Sq^I = Sq^{i_1} Sq^{i_2} \cdots Sq^{i_q}.$$

We say that the sequence I is *admissible* if $i_s \geq 2i_{s+1}$ for every $s = 1, \ldots, q-1$.

For any space (or spectrum) X, $H^*(X;\mathbb{Z}/2)$ has the structure of an $\mathcal{A}$-module by axiom (1) above. By using the axioms to study this module structure on the cohomology of products of infinite dimensional projective spaces, $\mathbb{RP}^\infty \times \cdots \times \mathbb{RP}^\infty$, one can prove the following without much difficulty.

Proposition 17.

1. $\{Sq^I : I\,admissible\}$ *is a basis for* $\mathcal{A}$ *as a graded vector space over* $\mathbb{Z}/2$.
2. $\{Sq^{2^r}\, r \geq 0\}$ *generates* $\mathcal{A}$ *as a graded algebra over* $\mathbb{Z}/2$.

$\mathcal{A}$ has more structure as well. It is a "Hopf algebra", meaning that it is both an algebra and a coalgebra, and the coproduct is a map of algebras. The coproduct map

$$\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$$

is defined to be the map of algebras induced by the Cartan formula,

$$Sq^i \to \sum_j Sq^j \otimes Sq^{i-j}.$$

One can check directly that this does give a well-defined map of algebras by seeing that it respects the Adem relations.

A calculation of the cohomology of Eilenberg-MacLane spaces by H. Cartan and J.P. Serre showed that the Steenrod algebra $\mathcal{A}$ is indeed the algebra of *all* cohomology operations, which is to say all natural transformations from cohomology with $\mathbb{Z}/2$-coefficients to itself, viewed as a functor from the category of spaces of the homotopy type of CW-complexes to the category of graded abelian groups. The representing spectrum of cohomology with $\mathbb{Z}/2$-coefficients is the Eilenberg-MacLane spectrum $\mathbb{HZ}/2$, so Cartan's calculation can be interpreted as saying that the Steenrod algebra $\mathcal{A}$ is the cohomology of $\mathbb{HZ}/2$,

$$\mathcal{A} \cong H^*(\mathbb{HZ}/2;\mathbb{Z}/2).$$

Thom proved Theorem 14 by a general construction, now known as the "Pontrjagin-Thom construction". This establishes that any cobordism theory, i.e where one might insist that the manifolds have certain structures such as an orientation, or an almost complex structure, can be described in terms of the homotopy groups of a certain Thom spectrum. The primary work in Thom's proof of Theorems 15 and 16, was to show that the mod 2 cohomology of the Thom spectrum $\mathbb{MO}$ is a free module over the Steenrod algebra, with one generator corresponding to every monomial basis element in the polynomial algebra $\mathbb{Z}/2[b_2, b_4, \cdots, b_r \cdots, : r \neq 2^k - 1]$.

$$H^*(\mathbb{MO};\mathbb{Z}/2) \cong \bigoplus_{\omega \in I} \Sigma^{|\omega|}\mathcal{A} \tag{5}$$

where, as above, the indexing set I consists of all monomials in $\mathbb{Z}/2[b_2, b_4, \cdots, b_r \cdots, : r \neq 2^k - 1]$. The notation $\Sigma^{|\omega|}\mathcal{A}$ means that the grading of the Steenrod algebra $\mathcal{A}$ is shifted by the dimension of the monomial $b_I \in \mathbb{Z}/2[b_2, b_4, \cdots, b_r \cdots, : r \neq 2^k - 1]$.

From this it is a rather formal argument to show that the spectrum $\mathbb{MO}$ is homotopy equivalent to a wedge of Eilenberg-MacLane spectra (proving Theorem 16), thus determining its homotopy groups (and proving Theorem 15).

Furthermore, from Thom's calculations one can describe examples of manifolds B_r representing generators b_r of the cobordism ring.

We define B_{2^i} to be $\mathbb{RP}^{2^i}$. For general n, suppose recursively that B_k has been defined for $k < n$. Write n as a sum of distinct powers of 2,

$$n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_r} \quad \text{with} \quad i_1 < i_2 < \cdots < i_r.$$

Notice again that $r = \alpha(n)$.

We can then write $n = 2^{i_1} + 2m$, where $m = 2^{i_2-1} + \cdots + 2^{i_r-1}$. We then define

$$B_n = S^{2^{i_1}} \times_{\mathbb{Z}_2} B_m \times B_m.$$

Using the fact that these manifolds are iterated $\mathbb{Z}/2$-equivariant products of spheres and projective spaces, it is possible to directly show inductively, using Whitney's immersion $\mathbb{RP}^{2^j} \looparrowright \mathbb{R}^{2^{j+1}-1}$, that these generators admit immersions

$$B_m \looparrowright \mathbb{R}^{2m-\alpha(m)}.$$

Furthermore, by taking disjoint unions and products of these manifolds and immersions, one can prove the following theorem due to R, Brown [10]. This argument was carried out in [16].

Theorem 18 (Brown [10]). *Every closed n-manifold is cobordant to one that immerses in $\mathbb{R}^{2n-\alpha(n)}$.*

Notice that this gives more evidence for the truth of the immersion conjecture.

3.3 The Brown-Peterson Approach to the Immersion Conjecture

This brings us to the program of E.H. Brown Jr. and F.P. Peterson that eventually led to a proof of the immersion conjecture.

Consider the stable normal bundle map

$$\nu_M : M^n \to BO.$$

Consider the exact sequence in cohomology:

$$0 \to I_{M^n} \to H^*(BO;\mathbb{Z}/2) \xrightarrow{\nu_M^*} H^*(M^n;\mathbb{Z}/2)$$

Here I_{M^n} is the kernel of ν_M^*, and is an ideal in $H^*(BO;\mathbb{Z}/2) \cong \mathbb{Z}_2[w_1, \cdots w_i, \cdots]$.

Define

$$I_n = \bigcap_{M^n} I_{M^n}.$$

One may view this as the ideal of all relations among the normal Stiefel-Whitney classes of all n-manifolds. Massey's Theorem 11 above can be interpreted to say that $w_i \in I_n$ for $i > n - \alpha(n)$.

In 1963 Brown and Peterson calculated the ideal $I_n \subset H^*(BO;\mathbb{Z}/2)$ explicitly. It is easier to state their result in terms of the Thom isomorphic image of the ideal,

$$\phi(I_n) \subset H^*(\mathbb{MO};\mathbb{Z}/2)$$

where $\phi = \cup u_n : H^*(BO;\mathbb{Z}/2) \xrightarrow{\cong} H^*(\mathbb{MO};\mathbb{Z}/2)$ is the Thom isomorphism.

In fact we will describe the quotient module, $H^*(\mathbb{MO};\mathbb{Z}/2)/\phi(I_n)$. Now recall Thom's cohomology calculation (5) of $H^*(\mathbb{MO};\mathbb{Z}/2)$. In particular it is a free module over the Steenrod algebra $\mathcal{A}$, with a very explicit basis. The quotient module $H^*(\mathbb{MO};\mathbb{Z}/2)/\phi(I_n)$ was shown by Brown and Peterson to split as a sum of cyclic modules over the Steenrod algebra, indexed by the same basis. In order to describe these cyclic modules over $\mathcal{A}$, we begin by recalling that being a connective Hopf algebra, $\mathcal{A}$ admits a canonical antiautomorphism. More explicitly, given the connectivity of $\mathcal{A}$, we can write the coproduct of an element $a \in \mathcal{A}$ in the form

$$\Delta(a) = a \otimes 1 + 1 \otimes a + \sum_i a_i \otimes b_i \in \mathcal{A} \otimes \mathcal{A}$$

where the gradings of a_i and b_i are both positive for all i in this sum. Then the canonical antiautomorphism $\chi : \mathcal{A} \to \mathcal{A}$ is defined recursively (using the grading) by the rules $\chi(1) = 1$ and

$$\chi(a) + a + \sum_i a_i \chi(b_i) = 0. \tag{6}$$

It is easy to see from this definition that

$$\begin{aligned} \chi^2 &= 1 \quad \text{and} \\ \chi(ab) &= \chi(b)\chi(a) \quad \text{forall } a, b \in \mathcal{A}. \end{aligned} \tag{7}$$

The canonical antiautomorphism $\chi : A \to A$ plays an important role in understanding how the action of the Steenrod algebra behaves with respect to Poincaré duality. More specifically, let M^n be a closed n-manifold with stable normal bundle ν_{M^n} having Thom spectrum $\mathbb{T}\nu_{M^n}$. This spectrum is defined as follows. Let $e : M^n \hookrightarrow \mathbb{R}^L$ be an embedding of M^n into some large dimension Euclidean space. Let ν_e be its normal bundle and M^{ν_e} the corresponding Thom space. Consider its suspension spectrum $\Sigma^\infty(M^{\nu_e})$. Then the Thom spectrum is defined to be the desuspension

$$\mathbb{T}\nu_{M^n} = \Sigma^{-L}\Sigma^\infty(M^{\nu_e}).$$

Notice that in the cohomology of the Thom spectrum, the Thom class has dimension zero:

$$u_{M^n} \in H^0(\mathbb{T}\nu_{M^n};\mathbb{Z}/2).$$

Define the left ideal $\tilde{J}(M^n) \subset \mathcal{A}$ by

$$\tilde{J}(M^n) = \{a \in \mathcal{A} : au_{M^n} = 0\}.$$

We can then take the intersection of these ideals to define the left ideal

$$\tilde{J}_n = \bigcap_{M^n} \tilde{J}(M^n)$$

where the intersection is taken over all closed n-manifolds M^n. The following is the main calculational result that Brown and Peterson needed to compute all relations among the normal Stiefel- Whitney classes of n-manifolds.

Theorem 19 (Brown and Peterson [5] [6]).

$$\tilde{J}_n = J_{[\frac{n}{2}]} = A\{\chi(Sq^i) : 2i > n\}.$$

Outline of proof.

This theorem was proved by considering the following composite isomorphism

$$D : H_q(M^n;\mathbb{Z}/2) \xrightarrow{P.D} H^{n-q}(M^n;\mathbb{Z}/2) \xrightarrow{\Phi} H^{n-q}(\mathbb{T}\nu_{M^n})$$

where $P.D$ is the Poincaré duality isomorphism and Φ is the Thom isomorphism. They heavily use the following identity originally due to Wu.

Lemma 20. *For $a \in \mathcal{A}$ having degree i,*

$$D(\chi(a)_*([M^n])) = a(u_{M^n}) \in H^i(\mathbb{T}(\nu_{M^n};\mathbb{Z}/2),$$

where $[M^n] \in H_n(M^n;\mathbb{Z}/2)$ is the fundamental class, and if $b \in \mathcal{A}$ is an operation in mod 2 cohomology of degree i, $b : H^q(-;\mathbb{Z}/2) \to H^{q+i}(-;\mathbb{Z}/2)$, then b_* denotes the dual operation in mod 2 homology, $b_* : H_r(-;\mathbb{Z}/2) \to H_{r-i}(-;\mathbb{Z}/2)$.

From this Brown and Peterson were able to show the following:

Lemma 21. *Let $a \in \mathcal{A}$ have degree i. Then $a \in \tilde{J}_n$ if and only if*

$$\chi(a) : H^{n-i}(X;\mathbb{Z}/2) \to H^n(X;\mathbb{Z}/2)$$

is zero for every space X.

From this Theorem 19 followed from rather standard calculations.

This theorem allowed Brown and Peterson to describe the ideal $I_n \subset H^*(BO;\mathbb{Z}/2)$ indirectly by explicitly describing the quotient space after applying the Thom isomorphism. Namely, they proved the following (compare with the cohomology calculation of $H^*(\mathbb{MO};\mathbb{Z}/2)$ given in (5)).

Theorem 22 (Brown and Peterson [5] [6]). *Let I be the indexing set of monomials in the cobordism ring $\eta_* \cong \pi_*(\mathbb{MO}) = \mathbb{Z}/2[b_2, b_4, \cdots, b_r \cdots, : r \neq 2^k - 1]$.*

$$H^*(\mathbb{MO};\mathbb{Z}/2)/\phi(I_n) = \bigoplus_{\omega \in I, |\omega| \leq n} \Sigma^{|\omega|}\mathcal{A}/J_{[\frac{n-|\omega|}{2}]},$$

where, for $\omega \in I$, $|\omega|$ is the grading of the monomial $b_\omega \in \mathbb{Z}/2[b_2, b_4, \cdots, b_r \cdots, : r \neq 2^k - 1]$.

The next major step toward the proof of the immersion conjecture was accomplished by Brown and Gitler [4] in 1973. In that paper, Brown and Gitler proved the following:

Theorem 23. *There exist spectra B_m, $m \geq 0$, satisfying the following properties:*

1. $H^*(B_m;\mathbb{Z}/2) \cong \mathcal{A}/J_m = \mathcal{A}/\mathcal{A}\{\chi(Sq^i) : i > m\}$ *as modules over the Steenrod algebra*
2. *Let $u_m : B_m \to \mathbb{HZ}/2$ represent the generator of $H^*(B_m;\mathbb{Z}/2)$ as a module over the Steenrod algebra. Then if X is any space of the homotopy type of a CW-complex,*

$$u_{m_*} : \pi_q(B_m \wedge X_+) \to \pi_q(\mathbb{HZ}/2 \wedge X_+) \cong H_q(X;\mathbb{Z}/2)$$

is surjective for $q \leq 2m+1$.

Furthermore, these properties characterize the homotopy type of the spectra B_m.

For any connected, closed n-manifold M^n, there is a well-known "Spanier-Whitehead duality" between M^n and the Thom spectrum of its stable normal bundle, $\mathbb{T}\nu_{M^n}$. A consequence of this duality implies that

$$H^q(\mathbb{T}\nu_{M^n};\mathbb{Z}/2) \cong H_{n-q}(M^n;\mathbb{Z}/2), \quad \text{forall } q \geq 0.$$

Using this duality one can conclude the following:

Corollary 24. *Let M^n be a connected, closed n-dimensional manifold. Let $\alpha_r \in H^r(\mathbb{T}\nu_{M^n};\mathbb{Z}/2)$, $0 \leq r \leq n$, be any cohomology class, represented by a map of spectra which by abuse of notation we also call*

$$\alpha_r : \mathbb{T}\nu_{M^n} \to \Sigma^r\mathbb{HZ}/2.$$

Then there is a map of spectra

$$\tilde{\alpha}_r : \mathbb{T}\nu_{M^n} \to \Sigma^r B_{[\frac{n-r}{2}]},$$

such that the composition

$$\mathbb{T}\nu_{M^n} \xrightarrow{\tilde{\alpha}_r} \Sigma^r B_{[\frac{n-r}{2}]} \xrightarrow{u_{[\frac{n-r}{2}]}} \Sigma^r\mathbb{HZ}/2$$

is homotopic to α_r. Here $[\frac{n-r}{2}]$ denotes the integral part of the real number $\frac{n-r}{2}$. In particular the Thom class

$$u_{M^n} : \mathbb{T}\nu_{M^n} \to \mathbb{HZ}/2$$

lifts to a map

$$\tilde{u}_{M^n} : \mathbb{T}\nu_{M^n} \to B_{[\frac{n}{2}]}.$$

This theorem was proved by Brown and Gitler by a rather complicated obstruction theory argument. They basically proved that there are no obstructions to the existence of spectra B_n satisfying these properties. The relevance of these properties to the immersion conjecture, and specifically the homotopy lifting properties needed to prove the immersion conjecture is the following corollary of this theorem.

Let $\nu_{M^n} : M^n \to BO$ be the stable normal bundle map of a connected, closed n-dimensional manifold, and let

$$\nu^t_{M^n} : T\nu_{M^n} \to \mathbb{MO}$$

is the induced map of Thom spectra. Define the spectrum $\mathbb{MO}/I_n$ to be the wedge of Brown-Gitler spectra indexed by the monomial basis of the cobordism ring:

Definition 2.

$$\mathbb{MO}/I_n = \bigvee_{\omega\in I, |\omega|\le n} \Sigma^{|\omega|} B_{[\frac{n-|\omega|}{2}]}$$

Notice that by Theorem 22 we have that

$$(8) \qquad \begin{aligned} H^*(\mathbb{MO}/I_n;\mathbb{Z}/2) &\cong H^*(\mathbb{MO};\mathbb{Z}/2)/\phi(I_n) \\ &\cong \bigoplus_{\omega\in I,|\omega|\le n} \Sigma^{|\omega|}\mathcal{A}/J_{[\frac{n-|\omega|}{2}]} \end{aligned}$$

as modules over the Steenrod algebra. This property is what motivated the notation of "$\mathbb{MO}/I_n$" for this spectrum.

As above, let $u_m : B_m \to \mathbb{HZ}/2$ be a map that represents the generator of $H^*(B_m;\mathbb{Z}/2) \cong \mathcal{A}/J_m$ as a (cyclic) module over the Steenrod algebra. Taking a wedge of these maps produces a map

(9)

$$v_n : \mathbb{MO}/I_n = \bigvee_{\omega\in I,|\omega|\le n} \Sigma^{|\omega|} B_{[\frac{n-|\omega|}{2}]} \xrightarrow{\vee u_{[\frac{n-|\omega|}{2}]}} \bigvee_{\omega\in I_n} \Sigma^{|\omega|}\mathbb{HZ}/2 = \mathbb{MO}.$$

As a consequence of Corollary 24 one immediate has the following.

Corollary 25. *Let M^n be a closed n-manifold with stable normal bundle $\nu_{M^n} : M^n \to BO$ and induced Thom spectrum map $\nu^t_{M^n} : \mathbb{T}\nu_{M^n} \to \mathbb{MO}$. Then there is a map of spectra*

$$\tilde{\nu}^t_{M^n} : \mathbb{T}\nu_{M^n} \to \mathbb{MO}/I_n$$

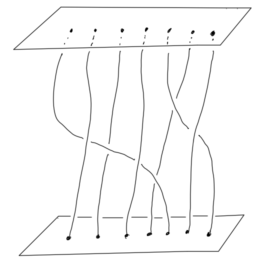
that lifts $\nu^t_{M^n} : \mathbb{T}\nu_{M^n} \to \mathbb{MO}$ in the sense that the composition

$$\mathbb{T}\nu_{M^n} \xrightarrow{\tilde{\nu}^t_{M^n}} \mathbb{MO}/I_n \xrightarrow{v_n} \mathbb{MO}$$

is homotopic to $\nu^t_{M^n}$.

Because of this corollary, the spectrum $\mathbb{MO}/I_n$ can be viewed as a "universal spectrum for the Thom spectra of stable normal bundles of n-manifolds". In order to pursue these ideas one needed a way of going from this kind of structure on the level of Thom spectra, to structure on the level of the stable normal bundles themselves. Brown and Peterson eventually accomplished this as well [9]. But before we describe how this was done, we go back to Brown-Gitler spectra and describe an explicit construction of them that is related to Artin's braid groups, which play an important role in knot theory and geometric group theory. This description was also useful in the proof of the immersion conjecture.

Let β_k be Artin's braid group on k strings. An element $b \in \beta_k$ can be thought of as a configuration of k strings, connecting two sets of k fixed points, each set lying in parallel planes in $\mathbb{R}^3$. Thus one can picture $b \in \beta_k$ as follows:



More precisely, an element $b \in \beta_k$ is an isotopy class of such configurations. The group multiplication in β_k is given by juxtaposition of braids. The clearest way of making this definition precise is by defining β_k to be the fundamental group of the configuration space of k unordered points in $\mathbb{R}^2$. That is, if we let

$$F_k = \{(t_1, \cdots, t_k) \in (\mathbb{R}^2)^k : t_i \ne t_j \quad \text{if } i \ne j\},$$

and we let C_k be the orbit space of the natural, free action of the symmetric group Σ_k on F_k given by permutation of coordinates,

$$C_k = F_k/\Sigma_k.$$

Then β_k is defined to be the fundamental of C_k:

$$(10) \qquad \beta_k = \pi_1(C_k).$$

It is also not difficult to see that the configuration spaces C_k are Eilenberg-MacLane spaces:

$$C_k = K(\beta_k, 1)$$

See [34] for example.

The configuration space C_k also comes equipped with a natural k-dimensional vector bundle which we call γ_k. It is defined by

$$\gamma_k = F_k \times_{\Sigma_k} \mathbb{R}^k \to F_k/\Sigma_k = C_k.$$

Alternatively, γ_k is defined by the k-dimensional representation of the braid group β_k defined by associating to a braid the permutation matrix given by the permutation of the endpoints of the strings.

Notice that the Thom space of γ_k is given by

$$C_k^{\gamma_k} = F_{k_+} \wedge_{\Sigma_k} S^k$$

where we are thinking of the sphere S^k as $S^k = \mathbb{R}^k \cup \infty$, with the action of Σ_k given by permuting coordinates (and ∞ is a fixed point).

The relevance of the braid groups and the spaces C_k and $C_k^{\gamma_k}$ to Brown-Gitler spectra and the Immersion Conjecture are the following two results:

Theorem 26 (Mahowald [30]). *Let $\mathbb{T}\gamma_k$ be the Thom spectrum*

$$\mathbb{T}\gamma_k = \Sigma^{-k}\Sigma^\infty(C_k^{\gamma_k}) = \Sigma^{-k}\Sigma^\infty\left(F_{k_+} \wedge_{\Sigma_k} S^k\right).$$

That is, $\mathbb{T}\gamma_k$ is the spectrum whose k-fold suspension is the suspension spectrum of the Thom space,

$$\Sigma^k \mathbb{T}\gamma_k = \Sigma^\infty \left(F_{k_+} \wedge_{\Sigma_k} S^k \right).$$

Then as modules over the Steenrod algebra,

$$H^*(\mathbb{T}\gamma_k; \mathbb{Z}/2) \cong A/J_{[\frac{k}{2}]}.$$

Notice that this is the same cohomology as the Brown-Gitler spectrum. Mahowald also conjectured that $\mathbb{T}\gamma_k$ is indeed homotopy equivalent to the corresponding Brown-Gitler spectrum. This conjecture was proved by Brown and Peterson.

Theorem 27 (Brown and Peterson [8])**.** *There is a map of spectra*

$$g_k : \mathbb{T}\gamma_k \to B_{[\frac{k}{2}]}$$

that induces an isomorphism in cohomology with coefficients in $\mathbb{Z}/2$. (Such a map is called a "2-primary weak homotopy equivalence".)

Putting these results together we have the following:

Corollary 28. *There is a 2-primary weak homotopy equivalence*

$$\begin{aligned} \mathbb{MO}/I_n &\simeq \bigvee_{\omega \in I, |\omega| \le n} \Sigma^{|\omega|} \mathbb{T}\gamma_{n-|\omega|} \\ &= \bigvee_{\omega \in I, |\omega| \le n} \Sigma^{2|\omega|-n} \left(F_{n-|\omega|_+} \wedge_{\Sigma_{n-|\omega|}} S^{n-|\omega|} \right). \end{aligned}$$

This in turn, using Corollary 25, implies the lifting, on the level of Thom spectra, of stable normal bundle maps to these Thom spectra of the braid group representations. This is related to the notion of "braid orientations" of manifolds which was studied by F. Cohen [11] and the author [14].

We now go back to the Brown-Peterson program. Notice that by the definition of the ideal $I_n \subset H^*(BO; \mathbb{Z}/2)$, together with Massey's calculation, one can conclude that in cohomology, for any n-manifold M^n, there is a commutative diagram

$$\begin{array}{ccc} H^*(BO) & \xrightarrow{\nu_M^*} & H^*M^n \\ \downarrow & & \uparrow \\ H^*(BO(n-\alpha(n))) & \xrightarrow[\text{Massey}]{} & H^*(BO)/I_n. \end{array} \tag{11}$$

Basically, the Brown-Peterson program was to show that one could realize this diagram by a diagram of maps between spaces. This is broken down into the following steps:

1. Show that there exist spaces "BO/I_n" together with maps $\rho_n : BO/I_n \to BO$ satisfying the following properties:

   (a). In cohomology the map $\rho_n^* : H^*(BO; \mathbb{Z}/2) \to H^*(BO/I_n; \mathbb{Z}/2)$ is surjective with kernel $I_n \subset H^*(BO; \mathbb{Z}/2)$, and

   (b) For every n manifold M^n, there is a map $\tilde{\nu}_{M^n} : M^n \to BO/I_n$ making the following diagram homotopy commute:

$$\begin{array}{ccc} M^n & \xrightarrow{\tilde{\nu}_{M^n}} & BO/I_n \\ = \downarrow & & \downarrow \rho_n \\ M^n & \xrightarrow[\nu_M]{} & BO. \end{array}$$

2. There is a map $\tilde{\rho}_n : BO/I_n \to BO(n-\alpha(n))$ that lifts (up to homotopy) the map $\rho_n : BO/I_n \to BO$.

Notice that if these steps could be completed, then for every closed n-manifold, one would have the following homotopy commutative diagram, realizing the cohomology diagram (11) above:

$$\begin{array}{ccc} M^n & \xrightarrow{\tilde{\nu}_{M^n}} & BO/I_n \\ \nu_{M^n} \downarrow & & \downarrow \tilde{\rho}_n \\ BO & \longleftarrow & BO(n-\alpha(n)). \end{array} \tag{12}$$

We now have the information necessary to do this program on the Thom spectrum level. Namely we will prove the following. This was originally proved by Brown and Peterson in [7].

Theorem 29 (Brown and Peterson, 1977 [7])**.** *Let M^n be a closed n-dimensional manifold. Let $\nu_{M^n} : M^n \to BO$ be the stable normal bundle map, and let $T\nu_{M^n} : \mathbb{T}\nu_{M^n} \to \mathbb{MO}$ be the corresponding map of Thom Spectra. Let $\mathbb{MO}(k) = \Sigma^{-k}\Sigma^\infty MO(k)$ be the Thom spectrum of the universal bundle over $BO(k)$. Then there is a map of spectra*

$$\tilde{T}\nu_{M^n} : \mathbb{T}\nu_{M^n} \to \mathbb{MO}(n-\alpha(n))$$

so that the composition

$$\mathbb{T}\nu_{M^n} \xrightarrow{\tilde{T}\nu_{M^n}} \mathbb{MO}(n-\alpha(n)) \to \mathbb{MO}$$

is homotopic to $T\nu_{M^n} : \mathbb{T}\nu_{M^n} \to \mathbb{MO}$.

Notice that by Corollary 25, in order to prove this theorem it suffices to prove the following "universal" result:

Theorem 30. *There is a map*

$$\tilde{\nu}_n : \mathbb{MO}/I_n \to \mathbb{MO}(n-\alpha(n))$$

so that the composition

$$\mathbb{MO}/I_n \xrightarrow{\tilde{\nu}_n} \mathbb{MO}(n-\alpha(n)) \to \mathbb{MO}$$

is homotopic to the map $\nu_n : \mathbb{MO}/I_n \to \mathbb{MO}$ described in equation (9).

Note. The reason that Theorem 30 implies Theorem 29 is that by Corollary 25, the stable normal bundle map $\nu_{M^n} : M^n \to BO$ has induced map of Thom spectra $\nu^t_{M^n} : \mathbb{T}\nu_{M^n} \to \mathbb{MO}$ that factors through a map $\tilde{\nu}^t_{M^n} : \mathbb{T}\nu_{M^n} \to \mathbb{MO}/I_n$. Furthermore, Theorem 30 supplies us with a map $\tilde{\nu}_n : \mathbb{MO}/I_n \to \mathbb{MO}(n-\alpha(n))$, so we have the resulting composition

$$\tilde{T}\nu_{M^n} : \mathbb{T}\nu_{M^n} \xrightarrow{\tilde{\nu}^t_{M^n}} \mathbb{MO}/I_n \xrightarrow{\tilde{\nu}_n} \mathbb{MO}(n-\alpha(n))$$

that satisfies Theorem 29.

Proof. (Sketch). Recall that from Definition 2 and Theorem (16), we have that

$$\mathbb{MO}/I_n = \bigvee_{\omega\in I, |\omega|\le n} \Sigma^{|\omega|} B_{\frac{n-|\omega|}{2}} \quad \text{and} \quad \mathbb{MO} \simeq \bigvee_{\omega \in I} \Sigma^{|\omega|}\mathbb{HZ}/2.$$

Furthermore the map $\nu_n : \mathbb{MO}/I_n \to \mathbb{MO}$ is given by a wedge of maps of the form

$$\begin{aligned}(13)\quad \nu_{n,\omega} : \Sigma^{|\omega|} B_{\frac{n-|\omega|}{2}} = S^{|\omega|} \wedge B_{\frac{n-|\omega|}{2}} &\xrightarrow{1\wedge j_{\frac{n-|\omega|}{2}}} S^{|\omega|} \wedge \mathbb{HZ}/2 \\ &= \Sigma^{|\omega|}\mathbb{HZ}/2 \hookrightarrow \mathbb{MO},\end{aligned}$$

where $j_k : B_k \to \mathbb{HZ}/2$ represents the generator of $H^*(B_k;\mathbb{Z}/2)$ as a module over the Steenrod algebra, $\mathcal{A}$.

In order to understand this map better, we recall some multiplicative structure possessed by the Thom spectrum $\mathbb{MO}$. Consider the Whitney sum map on the level of classifying spaces,

$$\mu : BO(k) \times BO(r) \to BO(k+r)$$

On the vector bundle level, this is the map that classifies the Whitney sum of vector bundles. On the group level this map is induced by the pairing

$$O(k) \times O(r) \to O(k+r)$$

given by "block sum". That is, it takes a $k \times k$-matrix and an $r \times r$- matrix and puts them in the upper left hand $k \times k$ – block and the lower right hand $r \times r$ – block, respectively, of a $(k+r)\times(k+r)$-dimensional matrix, with all other entries being zero. On the Thom space level, this defines a map

$$\mu^t : MO(k) \wedge MO(r) \to MO(k+r)$$

and on the Thom spectrum level this induces a product, which by abuse of notation we also call μ^t,

$$\mu^t : \mathbb{MO} \wedge \mathbb{MO} \to \mathbb{MO}.$$

This gives the spectrum $\mathbb{MO}$ the structure of a "ring spectrum".

Note. The fact that one can take smash products of spectra in an appropriately associative and functorial way is, perhaps surprisingly, technically quite difficult. But the technology necessary to do this is now part of every homotopy theorist's "tool kit".

This structure allows us to understand the splitting of $\mathbb{MO}$ as a wedge of Eilenberg-MacLane spectra a bit better.

Let $\omega \in I$ be a monomial basis element of the cobordism ring η_*. By Thom's Theorem 14, $\eta_* \cong \pi_*(\mathbb{MO})$, so we may let

$$b_\omega : S^{|\omega|} \to \mathbb{MO}$$

represent the homotopy class defined by $\omega \in \eta_*$. Now let

$$\iota_\omega : \Sigma^{|\omega|}\mathbb{HZ}/2 \to \mathbb{MO}$$

be the inclusion given by the Thom splitting of $\mathbb{MO}$ (Theorem 16). This splitting map is given by the composition

$$\iota_\omega : S^{|\omega|} \wedge \mathbb{HZ}/2 \xrightarrow{b_\omega \wedge \iota_1} \mathbb{MO} \wedge \mathbb{MO} \xrightarrow{\mu^t} \mathbb{MO}.$$

To prove Theorem 29 we need to show that the map $\nu_n : \mathbb{MO}/I_n \to \mathbb{MO}$, factors through $\mathbb{MO}(n-\alpha(n))$. By the above, it therefore suffices to show that the maps

$$\begin{aligned}\nu_{n,\omega} : S^{|\omega|} \wedge B_{\frac{n-|\omega|}{2}} &\xrightarrow{1\wedge j_{\frac{n-|\omega|}{2}}} S^{|\omega|} \wedge \mathbb{HZ}/2 \\ &\xrightarrow{b_\omega \wedge \iota_1} \mathbb{MO} \wedge \mathbb{MO} \xrightarrow{\mu^t} \mathbb{MO}\end{aligned}$$

factors through $\mathbb{MO}(n-\alpha(n))$.

To do this we first observe that the homotopy group interpretation of R. Brown's Theorem 18 about every n-manifold being cobordant to one that immerses in $\mathbb{R}^{2n-\alpha(n)}$, is that

$$(14)\qquad \pi_k(\mathbb{MO}(n-\alpha(n))) \to \pi_k(\mathbb{MO})$$

is surjective. Therefore the homotopy class b_ω lifts to a class

$$\tilde{b}_\omega : S^{|\omega|} \to \mathbb{MO}(|\omega| - \alpha(|\omega|)).$$

Also, the calculations done of the mod 2 cohomology of the Brown Gitler spectra B_k, as well as odd primary calculations about Brown-Gitler spectra and braid groups done in [13] and [12], say that B_k has the weak homotopy type of a finite CW-spectrum (i.e a spectrum made up of CW-complexes and cellular structure maps), of dimension $2k-\alpha(k)$. Obstruction theory then tells us that the generating map

$$B_k \xrightarrow{j_k} \mathbb{HZ}/2 \xrightarrow{\iota_1} \mathbb{MO}$$

factors through $\mathbb{MO}(2k-\alpha(k))$:

$$\tilde{j}_k : B_k \to \mathbb{MO}(2k-\alpha(k)).$$

Therefore the map

$$\nu_{n,\omega}: S^{|\omega|} \wedge B_{\frac{n-|\omega|}{2}} \xrightarrow{1\wedge j_{\frac{n-|\omega|}{2}}} S^{|\omega|} \wedge \mathbb{HZ}/2$$
$$\xrightarrow{b_\omega \wedge \iota_1} \mathbb{MO} \wedge \mathbb{MO} \xrightarrow{\mu^t} \mathbb{MO}$$

lifts to the composition

(15)

$$\tilde{\nu}_{n,\omega}: S^{|\omega|} \wedge B_{\frac{n-|\omega|}{2}}$$
$$\xrightarrow{\tilde{b}_\omega \wedge \tilde{j}_{\frac{n-|\omega|}{2}}} \mathbb{MO}(|\omega| - \alpha(|\omega|) \wedge \mathbb{MO}(n - |\omega| - \alpha(n-|\omega|)$$
$$\xrightarrow{\mu^t} \mathbb{MO}(n - \alpha(|\omega|) - \alpha(n-|\omega|)) \to \mathbb{MO}(n-\alpha(n)),$$

where the last map is the inclusion that exists because

$$\alpha(k) + \alpha(r) \geq \alpha(k+r).$$

As argued above, this is what is needed to complete the proof of this theorem. □

As mentioned above, Theorem 29 gives us the Thom spectrum analogue of the immersion conjecture. More precisely, this theorem tells us that the stable normal bundle map of a closed n-manifold, $\nu_{M^n}: M^n \to BO$ has induced map of Thom spectra, $\nu^t_{M^n}: \mathbb{T}\nu_{M^n} \to \mathbb{MO}$ that factors through $\mathbb{MO}(n-\alpha(n))$. The immersion conjecture would be proved once one shows that the actual stable normal bundle map factors through $BO(n-\alpha(n))$.

To do this, Brown and Peterson's program is to essentially "de-Thom-ify" the above constructions and arguments. The first major step in this was completed by Brown and Peterson [9] in 1979. For each n they constructed what they called a "universal space for normal bundles of n-manifolds", "BO/I_n", together with maps

$$\rho_n: BO/I_n \to BO$$

that satisfies the following properties:

1. $H^*(BO/I_n;\mathbb{Z}/2) \cong H^*(BO;\mathbb{Z}/2)/I_n$ and $\rho_n^*: H^*(BO;\mathbb{Z}/2) \to H^*(BO/I_n;\mathbb{Z}/2)$ is the projection map.
2. The Thom spectrum of $\rho_n: BO/I_n \to BO$ is $\mathbb{MO}/I_n$, as defined above.
3. For every n manifold M^n, there is a map $\tilde{\nu}_M: M^n \to BO/I_n$ making the following diagram homotopy commute:

$$\begin{tikzcd} M^n \arrow[r, "\tilde{\nu}_M"] \arrow[d, "="'] & BO/I_n \arrow[d, "\rho_n"] \\ M^n \arrow[r, "\nu_M"'] & BO. \end{tikzcd}$$

This Brown-Peterson construction of BO/I_n was obstruction theoretic. They used a kind of Moore-Postnikov tower to show that no obstructions to the existence to these spaces with these properties exist. They did not construct explicit models for these spaces. But the existence of these spaces allows one to reduce the study of the Immersion Conjecture about the best immersion dimensions of all n-manifolds, to a homotopy theoretic question about these spaces, and the maps $\rho_n: BO/I_n \to BO$. Namely their program proceeds with the following question:

Question. Is there a map $\tilde{\rho}_n: BO/I_n \to BO(n-\alpha(n))$ lifting $\rho_n: BO/I_n \to BO$?

Notice that if the answer is yes, then by Brown and Peterson's theorem, for any n-manifold, the composition

$$M^n \xrightarrow{\tilde{\nu}_M} BO/I_n \xrightarrow{\tilde{\rho}_n} BO(n-\alpha(n))$$

would be a lifting of the stable normal bundle map $\nu_M: M^n \to BO$, and by Hirsch-Smale, this would classify the normal bundle of an immersion

$$M^n \looparrowright \mathbb{R}^{2n-\alpha(n)}.$$

Theorem 31 (C. 1985). *Such a lifting $\tilde{\rho}_n: BO/I_n \to BO(n-\alpha(n))$ exists, and therefore the immersion conjecture is true.*

4. The Immersion Conjecture II: Outline of Its Solution: The Homotopy Theory

The proof of Theorem 31 was homotopy theoretic. Since the spaces BO/I_n were only understood in terms of their homotopy theoretic and manifold theoretic properties, the existence of the required liftings $\tilde{\rho}_n: BO/I_n \to BO(n-\alpha(n))$ was proved using a homotopy theoretic, and indeed an obstruction theoretic, argument. It was quite technical. In this section we review the ingredients of that proof given in [15], and mention a couple of places where the proof might be simplified, given a more modern understanding of the relevant homotopy theory.

The basic object of study in the proof of the immersion conjecture was the *Moore- Postnikov tower* for the inclusion map $\iota_{n-\alpha(n)}: BO(n-\alpha(n)) \to BO$. This is a tower of fibrations of the form

(16)
$$\begin{tikzcd} & & K_j \arrow[d] & & K_1 \arrow[d] & \\ BO(n-\alpha(n)) \arrow[r] & \cdots & X_j \arrow[r] & X_{j-1} \cdots \arrow[r] & X_1 \arrow[r] & BO \end{tikzcd}$$

where each $K_j \to X_j \to X_{j-1}$ is a fibration with fiber K_j being an Eilenberg-MacLane space, where $X_0 = BO$. The tower converges to $BO(n-\alpha(n))$.

The idea of the proof is to use an induction argument that assumes that the map $\rho_n : BO/I_n \to BO$ lifts to a map $\rho_{n,j-1} : BO/I_n \to X_{j-1}$ satisfying certain properties, and then show that the inductive step of finding an appropriate map $\rho_{n,j} : BO/I_n \to X_j$ could be completed.

$$\begin{array}{ccccc} & K_j & & K_1 & \\ & \downarrow & & \downarrow & \\ BO(n-\alpha(n)) \to \cdots & X_j \to X_{j-1} & \cdots \to & X_1 & \to BO \\ & & \uparrow \rho_{n,j-1} & & \uparrow \rho_n \\ & & BO/I_n & \xrightarrow{=} & BO/I_n \end{array}$$

The first step in completing the inductive argument was to study the corresponding diagram on the level of Thom spectra:

$$\begin{array}{ccccc} & L_j & & L_1 & \\ & \downarrow & & \downarrow & \\ \mathbb{MO}(n-\alpha(n)) \to \cdots & (X_j)^\gamma \to (X_{j-1})^\gamma & \cdots \to & (X_1)^\gamma & \to \mathbb{MO} \\ & & \uparrow T\rho_{n,j-1} & & \uparrow T\rho_n \\ & & \mathbb{MO}/I_n & \xrightarrow{=} & \mathbb{MO}/I_n \end{array} \tag{17}$$

Here L_j is the (homotopy) fiber of the induced map of Thom spectra, $(X_j)^\gamma \to (X_{j-1})^\gamma$.

Now from Theorem 30 we know that the map of Thom spectra $T\rho_n : \mathbb{MO}/I_n \to \mathbb{MO}$ has a lifting all the way up the tower to $\mathbb{MO}(n-\alpha(n))$. The idea is to then use this lifting to show that an appropriate lifting on the level of base spaces exists. In order to understand the relationship between the obstructions to obtaining liftings on the Thom spectrum level and liftings on the base space level, one needs to understand how the successive homotopy fibers L_j of the map of Thom spectra $(X_j)^\gamma \to (X_{j-1})^\gamma$ compare to the homotopy fibers K_j of the map of base spaces $X_j \to X_{j-1}$. This was studied by Brown and Peterson in their paper constructing the BO/I_n spaces [9]. Central in their study was understanding how Steenrod algebra interacts with the Thom isomorphism. We recall their result now.

Let $f : B \to BO$ be a map that induces an isomorphism in homotopy groups through dimension k. Let V be a graded $\mathbb{Z}/2$-vector space with $V_q = 0$ for $q \le k$. Let $K(V)$ be the corresponding Eilenberg-MacLane space and suppose $\gamma : B \to K(V)$ is a map with homotopy fiber B_1. So we have a "two stage system over BO":

$$\begin{array}{ccccc} B_1 & \xrightarrow{\iota} & B & \xrightarrow{f} & BO \\ & & \gamma\big\downarrow & & \\ & & K(V) & & \end{array}$$

Let $\mathbb{T}$ and $\mathbb{T}_1$ be the associated Thom spectra of the maps $f : B \to BO$ and $f \circ \iota : B_1 \to BO$.

Consider the induced (co)fibration sequence of spectra:

$$\mathbb{T}_1 \to \mathbb{T} \to \mathbb{T}/\mathbb{T}_1.$$

The goal is to understand, at least through a range of dimensions, the homotopy type of the cofiber $\mathbb{T}/\mathbb{T}_1$ in terms of the Eilenberg-MacLane space $K(V)$. Brown and Peterson showed that, through a range of dimensions, the cohomology $H^*(\mathbb{T}/\mathbb{T}_1;\mathbb{Z}/2)$ can be described in terms of the vector space V, the Steenrod algebra $\mathcal{A}$, and the cohomology of BO, $H^*(BO;\mathbb{Z}/2)$. We now describe their result more carefully.

Let $\mathcal{A}(BO)$ be the semi-tensor product of the Steenrod algebra $\mathcal{A}$ with $H^*(BO;\mathbb{Z}/2)$. That is,

$$\mathcal{A}(BO) = \mathcal{A} \otimes H^*(BO;\mathbb{Z}/2)$$

with the algebra structure defined by

$$(a \otimes u)(b \otimes v) = \sum_i ab'_i \otimes (\chi(b''_i)u)v$$

where if $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is the Cartan diagonal map, then $\Delta(b) = \sum_i b'_i \otimes b''_i$. To remember this twisted multiplication we denote $a \otimes u$ by $a \circ u$.

Given any map $g : X \to BO$, then of course $H^*(X;\mathbb{Z}/2)$ has the structure of a graded module over the graded algebra $H^*(BO;\mathbb{Z}/2)$. This induces a $\mathcal{A}(BO)$ module structure on the cohomology of the Thom spectrum $\mathbb{T}_g$:

$$\begin{gathered} \mathcal{A}(BO) \otimes H^*(\mathbb{T}_g;\mathbb{Z}/2) \to H^*(\mathbb{T}_g;\mathbb{Z}/2) \\ (a \circ u)(\phi(x)) = a(\phi(u \cup x)) \end{gathered}$$

where $x \in H^*(X;\mathbb{Z}/2)$, $\phi : H^*(X;\mathbb{Z}/2) \xrightarrow{\cong} H^*(\mathbb{T}_g;\mathbb{Z}/2)$ is the Thom isomorphism.

Consider the map

$$\psi : (\mathcal{A}(BO) \otimes V)^* \to H^{*+1}(\mathbb{T}_g/\mathbb{T}_{g_1}).$$

defined by

$$\psi(a \circ u \otimes v) = a(u \cup \phi(\tilde{\gamma}^*(v)))$$

where we are identifying $v \in V$ with the corresponding cohomology class $v \in H^*(K(V);\mathbb{Z}/2)$, and here ϕ denotes the relative Thom isomorphism. In [5] Brown and Peterson proved the following:

Theorem 32 ([5]). *The map*

$$\psi : (\mathcal{A}(BO) \otimes V)^q \to H^{q+1}(\mathbb{T}_g/\mathbb{T}_{g_1})$$

is an isomorphism for $q \le 2k$.

Moreover, since $\mathcal{A}(BO) \otimes V$ is a free module over the Steenrod algebra $\mathcal{A}$, one can conclude that the cofiber $\mathbb{T}_g/\mathbb{T}_{g_1}$ has the homotopy type of a wedge of Eilenberg-MacLane spectra through dimension $2k$.

Applying this to the Postnikov tower 16, one can conclude that the homotopy (co)fibers L_j of the induced tower of Thom spectra 17 have the homotopy type of Eilenberg-MacLane spectra through dimension $2(n-\alpha(n)) \geq n$ (assuming $n > 3$). In particular the homotopy type of these spectra are determined, through this range, as free $\mathcal{A}(BO)$-modules on the homotopy type of the fibers K_j of Postnikov system (16). This was a crucial fact in the obstruction theory arguments of [15] in knowing when lifts on the level of Thom spectra “de-Thom-ify” to give lifts on the level of base spaces.

There were two other ingredients in the obstruction theory arguments (i.e lifting arguments) of [15].

1. Stable homotopy properties of the spaces BO/I_n. This produced a “stable lifting” of the map $\rho_n : BO/I_n \to BO(n-\alpha(n))$.
2. The existence of certain multiplicative properties of the disjoint union of the space $\amalg_n BO/I_n$. These showed how the liftings of the BO/I_k’s for $k < n$ force liftings on a large skeleton of BO/I_n.

The first of these ingredients can be stated in the following theorem.

Theorem 33 ([15]). *For sufficiently large $N \geq 0$, the N-fold suspension of the map $\rho_n : BO/I_n \to BO$ lifts to $\Sigma^N BO(n-\alpha(n))$. That is, there is a map*

$$\rho_n^N : \Sigma^N BO/I_n \to \Sigma^N BO(n-\alpha(n))$$

making the following diagram homotopy commute:

$$\begin{tikzcd}
\Sigma^N BO/I_n \arrow[r, "\rho_n^N"] \arrow[d, "="'] & \Sigma^N BO(n-\alpha(n)) \arrow[d] \\
\Sigma^N BO/I_n \arrow[r, "\Sigma^N \rho_n"'] & \Sigma^N BO.
\end{tikzcd}$$

Proof. (Sketch). The obstruction to the existence of a stable lifting map $\rho_n^N : \Sigma^N BO/I_n \to \Sigma^N BO(n-\alpha(n))$ is the composition

$$\Sigma^N BO/I_n \xrightarrow{\Sigma^N \rho_n} \Sigma^N BO \to \Sigma^N (BO/BO(n-\alpha(n))).$$

That is, one can find such a lifting if and only if this composition is null-homotopic. Now one can show in a rather direct way, using a stable splitting theorem of Snaith [40] stated below, that the quotient space $BO/BO(n-\alpha(n))$ has the same homotopy type, through dimension n, as a product of mod-2 Eilenberg-MacLane spaces. Since BO/I_n has the same homotopy type as an n-dimensional CW-complex, this obstruction is entirely cohomological. But the fact that all cohomological obstructions vanish follows from Massey’s result (Theorem 11 above) and the definition of the ideal I_n. □

Here is Snaith’s splitting result referenced above:

Theorem 34 (Snaith [40]). *There is a weak homotopy equivalence of suspension spectra,*

$$\begin{aligned}\Sigma^\infty BO \simeq \Sigma^\infty (&BO(1) \vee BO(2)/BO(1) \vee \cdots \\ &\vee BO(m)/BO(m-1) \vee \cdots)\end{aligned}$$

At this point have the required lifting on the Thom spectrum level, and stably (i.e after taking suspension spectra). And we also know the result at the level of cobordism theory (i.e that every n-manifold is cobordant to one that immerses in $\mathbb{R}^{2n-\alpha(n)}$).

The final ingredient we need is that there are “multiplicative structures”

$$\begin{aligned}(18) \qquad BO/I_k \times BO/I_m &\to BO/I_{k+m} \\ S^1 \times_{\mathbb{Z}_2} (BO/I_k)^2 &\to BO/I_{2k}.\end{aligned}$$

These multiplicative structures were produced using the universal normal space properties of the Brown-Peterson spaces BO/I_n as well as cobordism theory arguments. It also used the description of the Brown-Gitler spectra as Thom spectra of bundles over the classifying spaces of braid groups (Theorem 27) and the well-known multiplicative structure that these classifying spaces possess.

These multiplicative structures, including the induced structures on homology, were used in the following way. If one strengthened the inductive assumptions to assume that there exist liftings of the spaces BO/I_k to $BO(k-\alpha(k))$ for $k < n$ that respect, in an appropriate sense, these multiplicative structures, then this forces the obstructions to the lifting of BO/I_n to $BO(n-\alpha(n))$ to be zero on the $(n-1)$-dimensional skeleton of any CW-complex of the homotopy type of BO/I_n. The n-dimensional cells were analyzed and shown to not contribute an obstruction by using the existence of the stable lifting. This argument then produced, via a complicated inductive argument, a lifting of the map $\rho_n : BO/I_n \to BO$ up the Moore-Postnikov tower and thereby the required lifting

$$\tilde{\rho}_n : BO/I_n \to BO(n-\alpha(n)).$$

Comments. The primary reason for this complicated inductive obstruction theoretic argument was the fact that the Brown-Peterson spaces BO/I_n were not constructed explicitly. That is to say there are no explicit models known for them. They were shown to exist with the appropriate properties by an obstruction theoretic argument. In particular one does not have a clear cell decomposition of the spaces BO/I_n that would allow for a more concrete obstruction theory argument for the required lifting to $BO(n-\alpha(n))$. However in the 35–40 years since the writing of [15], much has been learned by the algebraic topology community. For example the multiplicative structures described above (18) suggest that the disjoint union $\amalg_n BO/I_n$ has the structure of an algebra over an E_2 operad. The conjecture of this structure was made by Mike Hopkins. Indeed if one takes the operad of little 2-dimensional disks, which are models of the classifying spaces of braid groups, one might be able to find explicit models of the BO/I_n's that come equipped with cell decompositions that respect this E_2-structure. The recent work of Galatius, Kupers, and Randal-Williams on E_2-cell decompositions [20] might be relevant. If such explicit models can be found, surely the obstruction theory argument needed to prove the immersion conjecture could be simplified greatly.

5. Manifolds with Structure, Projective Spaces, and the Goodwillie-Weiss Embedding Calculus

Up until now, the bulk of this paper has been a discussion of the techniques used in the proof of the immersion conjecture. In this section we comment on three topics. The first has to do with manifolds with structures such as orientations, spin structures, or stably almost complex structures. The second is about the question of finding the best immersion dimensions for specific manifolds, and in particular real projective spaces. Finally we give a brief description of another more modern approach to studying the question of embeddings of manifolds originally introduced by Goodwillie and Weiss, which relies heavily on homotopy theoretic techniques.

5.1 Manifolds with Structure

At the end of the author's lecture on "Immersions of manifolds and homotopy theory" at the Mathematics Science Literature Lecture Series of Harvard University in 2020 (upon which these notes are based), the moderator, M.J. Hopkins asked if there were results similar to the immersion conjecture, or perhaps any of its motivating preliminary results, known for other classes of manifolds, such as orientable, Spin, or stably almost complex manifolds. Of course, as pointed out above, one of the most important motivating factors in the original immersion conjecture were the calculations done by Massey and Brown-Peterson of the relations among the normal Stiefel-Whitney classes of n-manifolds. In response to the Hopkins's question, I mentioned that many years ago, a PhD student of mine, A. Koonce did some calculations analogous to the Brown-Peterson Stiefel-Whitney class calculations [5], that computed relations among the $K(n)$-characteristic classes of almost complex manifolds. Here the $K(n)$ are "Morava K-theory" spectra that have proven to be extremely important in homotopy theory over the last 45 years. Koonce's results can be found in [27].

At the time the question was asked, I did not remember that Massey and Peterson [32] and Papastavridis [36] did some calculations of Stiefel-Whitney classes of orientable manifolds and of manifolds with spin structures. I was reminded of this shortly after the lecture by Donald Davis. Davis and Wilson [19] then wrote a paper giving a much cleaner exposition of these old results, clarified their implications, and extended them in significant ways. Among their results is the following:

Theorem 35 (Davis and Wilson [19]). *Let $\epsilon_n = 0$ if n is congruent to 1 mod 4, otherwise let $\epsilon_n = 1$. As above, let $\bar{w}_j(M^n)$ denote the j^{th} Stiefel-Whitney class of the stable normal bundle of a closed manifold M^n. Then there exists a closed orientable n-manifold M^n with $\bar{w}_{n-k}(M^n) \neq 0$ if and only if $k \geq \alpha(n) + \epsilon_n$.*

The following is an immediate corollary.

Corollary 36. *There exists a closed orientable n-manifold which cannot be immersed in $\mathbb{R}^{2n-\alpha(n)-\epsilon_n-1}$.*

Notice that says that for n congruent to 1 mod 4, one cannot find a better general immersion theorem for orientable n-manifolds than what the immersion conjecture guarantees for all n-manifolds. For n not congruent to 1 mod 4, the best result one might conjecture is that it might be possible to immerse orientable n-manifolds in $\mathbb{R}^{2n-\alpha(n)-1}$. This is a fascinating open problem.

5.2 Immersions of Projective Spaces

The immersion conjecture is a statement about all closed n-dimensional manifolds. But particular n-manifolds may immerse in a much lower dimension than is guaranteed by the immersion conjecture. To state an obvious example, the n-dimensional sphere S^n has a standard immersion, indeed embedding, into $\mathbb{R}^{n+1}$. So given a particular n-manifold M^n, finding it's best immersion dimension using Smale-Hirsch theory as well as the homotopy theory of classifying spaces is an important and often difficult problem. More specifically one would like to answer the following question:

Question: Given a fixed n-dimensional closed manifold M^n, what is the smallest k such that $M^n \looparrowright \mathbb{R}^{n+k}$.

By Smale-Hirsch theory and the theory of classifying spaces discussed above, this is equivalent to the following homotopy theoretic question.

Question: Give a fixed n-dimensional closed manifold M^n find the smallest k such that there exists a map

$$\nu^k_{M^n} : M^n \to BO(k)$$

that lifts the stable normal bundle map $\nu_{M^n} : M^n \to BO$. That is, the composition

$$M^n \xrightarrow{\nu^k_{M^n}} BO(k) \to BO$$

is homotopic to the stable normal bundle map ν_{M^n}.

Probably the most studied of such specific manifolds are projective spaces, $M^n = \mathbb{RP}^n$. In these cases there has been much work over many years. But still the final general answer is not known. Prominent among the contributers to our knowledge about this problem include, J. Adem, L. Astey, A. Berrick, D. Davis, S. Gitler, I. James, M. Mahowald, R.J. Milgram, and others. All of their work uses homotopy theoretic obstruction theory of different types, including K-theory, other generalized cohomology theories, stable and unstable homotopy theory, etc. Indeed, so many different types of obstruction theory have been used to study this problem that the projective space immersion problem became known not only as an important example of how homotopy theory can be used to study a basic question about manifolds, but conversely it became a testing ground for new homotopy theoretic technology. Indeed, around 1980, Mark Mahowald, one of the leading homotopy theorists of the second half of the twentieth century and beyond, told the author that to him, the main value of the projective space immersion problem is that it is a good test for the efficacy of an obstruction theory.

One of the strongest results along these lines is a "nonimmersion" result proved by D.M. Davis in 1984. It uses an obstruction theory based on a spectrum usually denoted by "$BP\langle 2\rangle$" which is an offshoot of the Brown-Peterson spectrum "BP". BP, in turn represents a cohomology theory that is a summand of (almost) complex cobordism theory, localized at a prime.

Davis's theorem states the following.

Theorem 37 (Davis [18]). *For all m, $\mathbb{RP}^{2m}$ does not immerse in $\mathbb{R}^{4m-4d-2\alpha(m-d)}$ where d is the smallest nonnegative integer such that $\alpha(m-d) \leq d+1$.*

Examples: (1) When $m = 2^k + 1$, one concludes that $\mathbb{RP}^{2m}$ does not immerse in $\mathbb{R}^{4m-6}$. The immersion conjecture implies that it does immerse in $\mathbb{R}^{2m-2}$, so the best possible immersion dimension for such a manifold lies in dimensions between $2m-5$ and $2m-2$.

(2) A more general, interesting collection of examples occurs when n is of the form $n = 2^{2^k+k+2} - 3\cdot 2^k$. Then this theorem implies $\mathbb{RP}^n$ does not immerse in $\mathbb{R}^{2^{2^k+k+3}-6\cdot 2^{k+1}-2}$. So for example, when $k=1$ this says that $\mathbb{RP}^{26}$ does not immerse in $\mathbb{R}^{38}$, yet the immersion conjecture says it does immerse in $\mathbb{R}^{49}$. When $k=2$ this says that $\mathbb{RP}^{244}$ does not immerse in $\mathbb{R}^{462}$, whereas the immersion conjecture says it does immerse in $\mathbb{R}^{483}$.

5.3 The Goodwillie-Weiss Calculus for Studying Embeddings

As mentioned in the first section, the study of immersions of manifolds has traditionally been much more tractable using the techniques of homotopy theory, than the study of embeddings of manifolds. This is primarily due to two facts. The first is the theory of Smale and Hirsch which reduces the study of immersions to the study of vector bundles. The second is the old result of algebraic topology that says that vector bundles can be understood in terms of the homotopy type of mapping spaces where the targets are classifying spaces of the form BG, where G is typically one of the groups $O(n)$, $SO(n)$, or $U(n)$. Since Smale-Hirsch theory does not apply to embeddings of manifolds, for many years homotopy theoretic methods were of limited use in studying spaces of embeddings.

A new homotopy theoretic approach was discovered in the 1990's by Goodwillie and Weiss [44], [45], [22] with subsequent extensions and generalizations by many others, including Klein, Sinha, Arone, Lambrechts, Turchin, Volic, and others (see for example [23], [37], [3]).

This theory has become known as the "Goodwillie-Weiss Embedding Calculus".

The basic viewpoint in this theory is the following. Given an n-manifold M^n, let $\mathcal{O}_M$ be the poset of open subsets of M^n, partially ordered by inclusion. Given an L-manifold N^L, one considers the contravariant functor ("cofunctor")

$$\begin{aligned} Emb(-,N^L) : \mathcal{O}_M &\to Spaces \\ V &\to Emb(V,N^L) \end{aligned}$$

Of course one can also view the space of immersions as a cofunctor,

$$\begin{aligned} Imm(-,N^L) : \mathcal{O}_M &\to Spaces \\ V &\to Imm(V,N^L) \end{aligned}$$

The immersion cofunctor is a sheaf, in that

$$\begin{array}{ccc} Imm(V_1 \cup V_2, N^L) & \longrightarrow & Imm(V_1,N^L) \\ \downarrow & & \downarrow \\ Imm(V_2,N^L) & \longrightarrow & Imm(V_1 \cap V_2, N^L) \end{array} \qquad (19)$$

is a pullback square for any open subspaces V_1 and V_2 of M^n.

What is less obvious, is that this square is a *homotopy pullback* square as well. This means that for example, if one takes homotopy groups of the four spaces in this square, one gets a Mayer-Vietoris long exact sequence. This was proved by Weiss [44] [45] and is, in a sense that can be made precise, a reconstituted form of Smale-Hirsch theory. In homotopy theory a cofunctor with this Mayer-Vietoris property is called *"excisive"*, and by borrowing terminology from Goodwillie's calculus of homotopy functors (developed a bit prior to Weiss's work) one can summarize this excisive property by saying that the immersion cofunctor is a "polynomial cofunctor of degree ≤ 1".

From this viewpoint, what makes the cofunctor $Imm(-;N^L)$ more calculable than $Emb(-;N^L)$ is that, being of degree ≤ 1, the homotopy type of the immersion cofunctor is determined by this Mayer-Vietoris property as well as how it behaves on open sets diffeomorphic to a disk D^n. Furthermore, in a sense that Goodwillie and Weiss make precise, the natural transformation

$$Emb(-;N^L) \to Imm(-;N^L)$$

is the "best approximation" to the embedding cofunctor by a cofunctor of degree ≤ 1. The immersion cofunctor is therefore denoted in this theory by

$$Imm(-;N^L) = T_1(Emb(-;N^L))$$

where the notation is meant to conjure up the notion of being the "degree 1 Taylor polynomial" approximation to the embedding cofunctor.

The basic idea in the Goodwillie-Weiss embedding calculus is to construct a tower of cofunctors

(20)
$$\begin{aligned} Emb(-;N^L) &\to \cdots \to T_k Emb(-;N^L) \to T_{k-1}Emb(-;N^L) \\ &\to \cdots T_1 Emb(-;N^L) = Imm(-;N^L) \end{aligned}$$

that, in an appropriate sense converges to the embedding cofunctor, and so that the natural transformation

$$\eta_k : Emb(-;N^L) \to T_k Emb(-;N^L)$$

is the "best" approximation to the embedding cofunctor by a "polynomial cofunctor of degree $\leq k$". That is, $T_k Emb(-;N^L)$ is the "degree k-Taylor polynomial" cofunctor approximation to the embedding cofunctor. We will not give the precise definition of a polynomial cofunctor of degree $\leq k$, but such cofunctors are distinguished by the property that their homotopy types are determined by its values on tubular neighborhoods of subsets S of M^n of cardinality $\leq k$ (i.e open subsets diffeomorphic to a disjoint union of $\leq k$ open disks, D^n).

In order to understand the notion of "the best" approximation by a polynomial functor of degree $\leq k$ we introduce a bit more terminology and recall a theorem of Weiss [45].

Definition 3. *A cofunctor $\mathcal{F} : \mathcal{O}_M \to Spaces$ is said to be good if*

1. *it takes isotopy equivalences to homotopy equivalences, and*
2. *for any sequence $\{V_i : i \geq 0\}$ of objects in $\mathcal{O}_M$ (i.e open subsets of M^n), the canonical map*

$$F(\cup_i V_i) \to holim_i F(V_i)$$

is a homotopy equivalence. (Here "$holim$ refers to the homotopy inverse limit.)

Now let $\mathcal{F}$ be the category of good cofunctors. Weiss showed that the construction of T_k extends to all of $\mathcal{F}$ and proved the following.

Theorem 38 ([45]). *The functor $T_k : \mathcal{F} \to \mathcal{F}$ and the natural transformations $\eta_k : id_{\mathcal{F}} \to T_k$ satisfy the following properties:*

- *T_k takes equivalences to equivalences*
- *$T_k F$ is polynomial of degree $\leq k$, for all $F \in \mathcal{F}$,*
- *If F is polynomial of degree $\leq k$, the $\eta_F : F \to T_k F$ is an equivalence, and*
- *For every $F \in \mathcal{F}$, the map $T_k(\eta_k) : T_k F \to T_k T_k F$ is an equivalence.*

These properties of the construction of the "k^{th}" Taylor approximation $T_k F$ are what is meant by the statement that it gives the "best" approximation to the functor F by a polynomial functor of degree $\leq k$.

The notion of convergence in this theory is sometimes known as "analyticity", and is proved under the appropriate hypotheses using the notion of "multiple disjunction lemmas" for concordance embeddings and diffeomorphisms by Goodwillie [21] and Goodwillie-Klein-Weiss [23].

Now in order for the tower (20) to be useful for calculations, one needs to be able to compute the homotopy types of the homotopy fibers of the successive terms:

$$L_k(F(V)) \to T_k F(V) \to T_{k-1}F(V).$$

The functor $L_k : \mathcal{O}_{M^n} \to Spaces$ is known as a *homogeneous polynomial functor of degree* k, in that its lower degree Taylor approximations, $T_j L(V)$ are contractible for all $j < k$.

An impressive part of the Goodwillie-Weiss theory is that they can completely classify the homogeneous polynomial cofunctors $\mathcal{O}_{M^n} \to Spaces$. For example, let $C(M^n,k)$ be the configuration space of k-distinct, unordered points in M^n. That is, an element of $C(M^n,k)$ is a subset of M^n of cardinality k. Notice that this space can be viewed as the complement of the fat diagonal in the k-fold symmetric product $M^n \times \cdots \times M^n / \Sigma_k$. Let

$$p : Z \to C(M^n,k)$$

be a fibration with partial section $s : C(M^n,k) \cap Q \to Z$ where Q is a neighborhood of the fat diagonal in the symmetric product. For $V \subset \mathcal{O}_{M^n}$ one can define $F(V)$ to be the space of sections of p which are defined on $C(V,k)$ and agree with s on $C(V,k) \cap Q'$ for some neighborhood of the fat diagonal $Q' \subset Q$. The cofunctor F defined this way is a homogeneous polynomial cofunctor of degree k and the remarkable theorem of Goodwillie and Weiss is that *all* homogeneous polynomial cofunctor of degree k come about this way, for some fibration p and section s.

Now the topology of the configuration spaces $C(M^n,k)$ have been well-studied over the last fifty years, so the homotopy type of the homogeneous degree k polynomial cofunctors L_kF lends itself to calculation.

So in the case of the embedding cofunctor, one has a tower of fibrations (20) whose base is the space of immersions, that "converges" to the space of embeddings (under appropriate hypotheses), and the homotopy fibers $L_k(Emb(-;N^L))$ have tractable homotopy types. One can view this as a "resolution" of the embedding cofunctor $Emb(-;N^L)$ in terms of the immersion cofunctor $Imm(-;N^L)$ where the "layers" of the resolution lend themselves to calculation. Of course how these layers fit together is another, often very difficult problem. Nonetheless over the past 25 years the Goodwillie-Weiss machinery and subsequent extensions and generalizations have lead to a good bit of progress on understanding the homotopy type of the embedding spaces $Emb(M^n,N^L)$. We end with a quick description of some results that have been obtained using this theory.

Examples of results using the Goodwillie-Weiss embedding calculus

1. B. Munson [35] gave a complete obstruction for an immersion $M^n \looparrowright \mathbb{R}^L$ being isotopic to an embedding for $3L > 4n+4$, extending classical work of Haefliger.
2. I. Volic [42] studied knots, $Emb(S^1,\mathbb{R}^3)$, and he related the Goodwillie-Weiss obstruction theory to the finite-type knot invariants of Vassiliev.
3. Arone, Lambrechts, and Volic [2] used this theory to study the rational homotopy type of the "difference" (homotopy fiber) between embedding and immersion spaces,
$$\tilde{Emb}(M^n;\mathbb{R}^L) \to Emb(M^n;\mathbb{R}^L) \to Imm(M^n;\mathbb{R}^L)$$
when $L > 2\cdot$ embedding dimension of M^n.
4. G. Arone, P. Lambrechts, V. Turchin, and I. Volic [3] combined this theory with work of Kontsevich to calculate the rational homotopy type of the space of "long knots" in $\mathbb{R}^L$, $L \geq 4$.
5. R. Koytcheff [28] related the Goodwillie-Weiss obstruction theory to the "Bott-Taubes integrals", giving invariants of knots.
6. D. Sinha [37] adapted the Goodwillie-Weiss theory to construct a cosimplicial model for $Emb([0,1],N)$ where N is simply connected and has dimension ≥ 4.

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