On Kiyoshi Oka's Unpublished Papers in 1943

For the 120th Anniversary of Kiyoshi Oka's Birth

by Junjiro Noguchi[*][†]

Abstract. In 1943 from September to December Kiyoshi Oka wrote a series of papers numbered from VII to XI, as the research reports to Teiji Takagi (then, Professor of Tokyo Imperial University), in which he solved affirmatively the so-called Levi Problem (Hartogs' Inverse Problem termed by Oka) for unramified Riemann domains over $\mathbf{C}^n$. This problem which had been left open for more than thirty years then, was the last one of the Three Big Problems summarized by Behnke–Thullen 1934. The papers were hand-written in Japanese, consist of pp. 108 in total, and have not been published by themselves. The aim of the present article is to provide an English translation of the most important, last paper (Part II) with preparation (Part I). At the end of Part I we will discuss a problem which K. Oka left and is still open.

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Part I

In this Part I we discuss Kiyoshi Oka's unpublished series of five papers, VII–XI in 1943 ([26]), which were hand-written in Japanese and consist of pp. 108 in total. In Part II we present the English translation of the last one XI of [26] that contains the most important main results. Part II is the main part of the present article. In Part I it is not intended to survey the developments of the subject since the time of Oka and thereafter, but rather is aimed to serve for the preparations or a sort of appendices, so that Part II is readable for general readers without specific knowledge of the subject at the time. Moreover, if one gets into the proofs described in XI, he will still find methods that have not been presented in published references, so far by the author's knowledge, and are original and have interests even from the present viewpoint. For general references about the developments of the present subject, cf., e.g., Hitotsumatsu [10], Gunning–Rossi [9], Hörmander [11], Nishino [14], Lieb [12], Noguchi [15], [20].

The method of the proof of the Pseudoconvexity Problem (i.e., Hartogs' Inverse Problem, Levis's Problem) given in this series of papers 1943 is quite similar to that of Oka IX published in 1953 except for the use of Coherence Theorems: There, in the unpublished papers 1943, he proved some ideal theoretic properties of holomorphic functions, which was sufficient to prove the Jôku-Ikô (lifting principle) with estimates; then it led to the solution of the Pseudoconvexity Problem. In this series of papers, he already had in mind a project not only to settle the Pseudoconvexity Problem of general dimension, but also to deal with the problem for ramified Riemann domains; it would actually lead to the notion of "Coherence".

Reading the series of unpublished papers 1943 we see the dawn of the then unknown notion of "Coherence" or *"Idéaux de domaines indéterminés"* in

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Oka's terms, and may observe that the turn of years "*1943/1944*" was indeed the watershed in the study of analytic function theory of several variables.

1. Three Big Problems

a) K. Oka's research [22], I–IX (published) was motivated by the monograph of Behnke–Thullen [2] 1934: They summarized the main problems then in the theory of complex analytic functions of several variables, listing the following *Three Big Problems*.

(i) The Levi (Hartogs' Inverse) Problem.
(ii) Cousin (I/II) Problem.
(iii) Problem of expansions of functions (Approximation Problem).

These problems are well-known among complex analysists, but we will recall for convenience the above problems in the next subsection b), following Behnke–Thullen [2] (cf. Lieb [12]).

The difficulty of the problems was referred by H. Cartan [25] as "*quasi-surhumaine (quasi-superhuman)*" and by R. Remmert [25] as "*Er löste Probleme, die als unangreitbar galten (He solved problems which were believed to be unsolvable)*".

K. Oka solved all these problems in the opposite order. By establishing "Jôku-Ikô"[1] in [22] I–II, he proved Problem (iii) above and (ii) the Cousin I Problem, and then in [22] III, he obtained the Oka Principle, settling (ii) the Cousin II Problem. The most difficult problem (i) was first proved for univalent domains (subdomains) of $\mathbf{C}^2$ in [22] VI 1942, leaving for the general dimensional case the last paragraph of the paper:

> "*L'auteur pense que cette conclusion sera aussi indépendante des nombres de variables complexes. (The author thinks that this conclusion will be also independent of the number of complex variables.)*"

But, it was a general cognition that the higher dimensional case was still open (in Japan there seems to have been a sentiment that the higher dimensional case of univalent domains was already settled), and it was proved as follows:

(1) S. Hitotsumatsu [10] (a short note in Japanese was published), 1949 for univalent domains of $\mathbf{C}^n$ ($n \geq 2$, same as in (3) below by Weil's integral).
(2) K. Oka [22] IX, 1953 for unramified Riemann domains over $\mathbf{C}^n$ (by Coherence, Jôku-Ikô and Cauchy integral).
(3) H.J. Bremermann [5] and F. Norguet [21] 1954, independently for univalent domains of $\mathbf{C}^n$ (by Weil's integral).

[1] This consists of two (Japanese) words, and means that "*one transfers himself from the original space of the given dimension to a space of even higher dimension*". Cf. §4.1

b) (i) To get the idea of the problems we consider a univalent domain (i.e., a subdomain) Ω of $\mathbf{C}^n$. Let $\Omega' \supset \Omega$ be a domain of $\mathbf{C}^n$. If every holomorphic function in Ω is extendable to a holomorphic function in Ω', Ω' is called an *extension of holomorphy* of Ω. In the case of $n = 1$, there is no extension of holomorphy other than $\Omega' = \Omega$, but in the case case of $n \geq 2$, $\Omega' \supsetneq \Omega$ can happen (Hartogs' phenomenon, 1906–). For example, let $n \geq 2$, let $a = (a_1, \ldots, a_n) \in \mathbf{C}^n$ and define $\Omega_{\mathrm{H}}(a;\delta,\gamma) \subset \mathbf{C}^n$, so-called a Hartogs domain, as follows: With a pair of n-tuples of positive numbers, $\gamma = (\gamma_j)_{1 \leq j \leq n}$ and $\delta = (\delta_j)_{1 \leq j \leq n}$ satisfying $0 < \delta_j < \gamma_j$ $(1 \leq j \leq n)$, we set

(1.1)
$$\begin{aligned}
\mathrm{P}\Delta(a;\gamma) &= \{z = (z_1, \ldots, z_n) \in \mathbf{C}^n : |z_j - a_j| < \gamma_j, 1 \leq j \leq n\},\\
\Omega_1 &= \{z = (z_1, \ldots, z_n) \in \mathrm{P}\Delta(a;\gamma) :\\
&\qquad\qquad |z_j - a_j| < \delta_j, 2 \leq j \leq n\},\\
\Omega_2 &= \{z = (z_1, \ldots, z_n) \in \mathrm{P}\Delta(a;\gamma) : \delta_1 < |z_1 - a_1| < \gamma_1\},\\
\Omega_{\mathrm{H}}(a;\delta,\gamma) &= \Omega_1 \cup \Omega_2 \subsetneq \mathrm{P}\Delta(a;\gamma).
\end{aligned}$$

It is immediate to see that the polydisk $\mathrm{P}\Delta(a;\gamma)$ is an extension of holomorphy of $\Omega_{\mathrm{H}}(a;\delta,\gamma)$ (cf., e.g., [15] §1.2.4).

The notion of the "extension of holomorphy" is naturally generalized to the case of multi-sheeted (ramified or unramified) domains over $\mathbf{C}^n$ and this is definitely necessary in the case of $n \geq 2$; in fact, it is known that there is a subdomain of $\mathbf{C}^2$ which has an infinitely-sheeted unramified domain over $\mathbf{C}^2$ as an extension of holomorphy (cf., e.g., [15] §5.1). In this paper, domains over $\mathbf{C}^n$ are *unramified*, as far as it is not mentioned to be ramified.

Now, let Ω be a domain over $\mathbf{C}^n$. The maximal domain among the extensions of holomorphy of Ω is called the *envelope of holomorphy* of Ω, denoted by $\hat{\Omega}$. It exists, but is not necessarily univalent even if Ω is univalent as mentioned above.

If $\Omega = \hat{\Omega}$, Ω is called a *domain of holomorphy*. In the above example, $\mathrm{P}\Delta(a;\gamma)$ is the envelope of holomorphy of $\Omega_{\mathrm{H}}(a;\delta,\gamma)$ and a domain of holomorphy. Hartogs' phenomenon implies that the shape of singularities of holomorphic functions is not arbitrary; contrarily, before Hartogs it had been thought arbitrary. In the study of the shape of singularities of holomorphic functions, in other words, the shape of the boundary of a domain of holomorphy Ω, E.E. Levi found around 1910 in the case of $n = 2$ that with the assumption of the C^2-regularity of the boundary $\partial\Omega$ defined by φ so that $\Omega = \{\varphi < 0\}$, $d\varphi \neq 0$ on $\partial\Omega$, one has

$$(1.2) \qquad L(\varphi)(a) = \begin{vmatrix} 0 & \varphi_z & \varphi_w \\ \varphi_{\bar z} & \varphi_{z\bar z} & \varphi_{w\bar z} \\ \varphi_{\bar w} & \varphi_{z\bar w} & \varphi_{w\bar w} \end{vmatrix} \geq 0, \quad a \in \partial\Omega,$$

where (z,w) are the variables of $\mathbf{C}^2$. For general $n \geq 2$, J. Krzoska (1933) formulated it as with the same boundary regularity, the hermitian matrix

$$\left(\frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k}(a)\right)_{1\le j,k\le n} \qquad (a\in\partial\Omega) \tag{1.3}$$

is *positive semi-definite* on the homomorphic tangent vector space

$$\left\{(v_1,\dots,v_n)\in \mathbf{C}^n : \sum_{j=1}^{n} v_j \frac{\partial\varphi}{\partial z_j}(a)=0\right\}.$$

If $n=2$, this is reduced to (1.2). Then it is natural to ask the converse.

Levi Problem: *If $\partial\Omega$ satisfies (1.3), is Ω a domain of holomorphy?*

The property characterized by (1.2) or (1.3) is called a *pseudoconvexity* of Ω or $\partial\Omega$, which is a biholomorphically invariant property in a neighborhood of any point $a\in\partial\Omega$.

There is an inconvenience in the above characterization by φ; that is, even if φ_1,φ_2 satisfies (1.2) or (1.3), $c_1\varphi_1+c_2\varphi_2$ with positive constants c_1,c_2, does not satisfy the similar condition. This was the reason why K. Oka introduced a *pseudoconvex* function ψ in Ω such that ψ is upper semi-continuous and the restriction of ψ to the intersection of any complex affine line and Ω is subharmonic (Oka VI, 1942).[2] Pseudoconvex functions play the similar role to that of φ in (1.2) or (1.3) and still satisfies that $c_1\psi_1+c_2\psi_2$ is pseudoconvex for pseudoconvex functions ψ_j and $c_j>0$ $(j=1,2)$. If $\psi:\Omega\to\mathbf{R}$ is of C^2-class, ψ is pseudoconvex if and only if the hermitian matrix $\left(\frac{\partial^2\psi}{\partial z_j\partial\bar{z}_k}(a)\right)_{1\le j,k\le n}$ $(a\in\partial\Omega)$ is positive semi-definite.

In the unpublished papers 1943, K. Oka did not assume the boundary regularity of Ω, but defined the *pseudoconvexity* of Ω (or $\partial\Omega$) as follows: For every point $a\in\partial\Omega$ there is a neighborhood U of a in $\mathbf{C}^n$ such that if $\phi:\Omega_{\mathrm{H}}(a;\delta,\gamma)\to U\cap\Omega$ is a biholomorphic map from a Hartogs domain $\Omega_{\mathrm{H}}(a;\delta,\gamma)$ into $U\cap\Omega$, then ϕ is analytically continued to $\tilde{\phi}:\mathrm{P}\Delta(a;\gamma)\to U\cap\Omega$. It is trivial that a domain of holomorphy satisfies this pseudoconvexity, and K. Oka proved the converse: This is why he called the problem **Hartogs' Inverse Problem.** The solution naturally implies that of the Levi Problem.

(ii) Let $\Omega=\bigcup_{\alpha\in\Gamma}U_\alpha$ be an open covering. Let f_α $(\alpha\in\Gamma)$ be a meromorphic function in U_α such that $f_\alpha-f_\beta$ is holomorphic in $U_\alpha\cap U_\beta$ as far as $U_\alpha\cap U_\beta\neq\emptyset$. The pair $(\{U_\alpha\},\{f_\alpha\})$ is called a Cousin-I data on Ω.

[2] In similar time, P. Lelong defined the same notion as *plurisubharmonic* functions from potential theoretic viewpoint.

Cousin I Problem:[3] *If Ω is a domain of holomorphy, then for a Cousin-I data $(\{(U_\alpha\},\{f_\alpha\})$ on Ω, is there a meromorphic function F in Ω, called a solution of the Cousin-I data, such that $F-f_\alpha$ is holomorphic in every U_α?*

In the case of $n=1$, Mittag-Leffler's Theorem gives an affirmative answer to the problem.

Similarly, we assume that f_α are meromorphic functions, not identically zero, and that f_α/f_β is a nowhere vanishing holomorphic function in every $U_\alpha\cap U_\beta(\neq\emptyset)$. Then $(\{U_\alpha\},\{f_\alpha\})$ is called a Cousin-II data on Ω.

Cousin II Problem: *If Ω is a domain of holomorphy, then for a Cousin-II data $(\{U_\alpha\},\{f_\alpha\})$ on Ω, is there a meromorphic function F in Ω, called a solution of the Cousin-II data, such that F/f_α is nowhere zero holomorphic in every U_α?*

In the case of $n=1$, this is answered affirmatively by Weierstrass' Theorem.

(iii) Let $K\Subset\Omega$ be a compact subset and let f be a holomorphic function in a neighborhood of K.

Problem of expansion (Approximation Problem): *Assume that Ω is a domain of holomorphy. Find a condition for K such that for every such f there is a series $\sum_{\nu=1}^{\infty}f_\nu$ with holomorphic functions f_ν in Ω such that restricted on K,*

$$f=\sum_{\nu=1}^{\infty}f_\nu,$$

where the convergence is uniform on K.

In the case of $n=1$ we have Runge's Theorem. In the problems of (ii) and (iii) above, the assumption for Ω being a domain of holomorphy is necessary by examples (cf., e.g., [16] §1.2.4, §3.7).

2. Unpublished Papers VII–XI 1943

We first list the titles translated from Japanese and the numbers of pages of the papers with dates.

(i) On Analytic Functions of Several Variables VII – Two auxiliary problems on the congruence of holomorphic functions, pp. 28 (4 Sep. 1943).
(ii) On Analytic Functions of Several Variables VIII – The First Fundamental Lemma on finite domains without ramification points, pp. 11 (5 Sep. 1943).

[3] This problem was dealt with by P. Cousin [4] and affirmatively solved when the domain is a *cylinder (domain)*, which is by definition an n-product of the coordinate plane domains of $\mathbf{C}^n$: Cousin II Problem below was also solved affirmatively there when the domain is a cylinder $\prod_j D_j$ with simply connected plane domains D_j $(\subset\mathbf{C})$ except for one D_j. He used the so-called *Cousin integral* (see p. 60). They were solved affirmatively in general by K. Oka [22], I–III for univalent domains, which in Cousin II Problem yielded the *Oka Principle.*

(iii) On Analytic Functions of Several Variables IX – Pseudoconvex functions, pp. 29 (24 Oct. 1943).
(iv) On Analytic Functions of Several Variables X – The Second Fundamental Lemma, pp. 11 (12 Nov. 1943).
(v) On Analytic Functions of Several Variables XI – Pseudoconvex domains and finite domains of holomorphy: Some theorems on finite domains of holomorphy, pp. 29 (12 Dec. 1943).

K. Oka cited these papers in two places of the published papers with mentioning a further problem of ramified Riemann domains, which we quote.

(1) Introduction of [22] Oka VIII (1951, p. 204) begins with:

> Les problèmes principaux depuis le Mémoire I sont: problèmes de Cousin, problème de développement et problème des convexités.[4] Dans les Mémoires I–VI,[5] nous avons vu, disant un mot, que ces problèmes sont résolubles affirmativement pour les domaines univalents finis.[6] Et l'auteur a encore constaté quoique sans l'exposer, que ces résultats restent subsister au moins jusqu'aux domaines finis sans point critiques.[7]
> Il s'agit donc: ou bien d'introduire l'infini convenable, ou bien de permettre des points critiques; or, on retrouvera que l'on ne sais presque rien sur les domaines intérieurement ramifiés; .....

(2) Introduction 2 of [22] Oka IX (1953, p. 98) begins with:

> Dans le présent Mémoire, nous traiterons les problèmes indiqués plus haut, ainsi que les problèmes arithmétiques introduits au Mémoire VII, pour les domaines pseudoconvexes finis sans point critique intérieur; dont la partie essentielle n'est pas différente de ce que nous avons exposé en japonais en 1943.[8]
> On verra dans le Mémoire suivant que quand on admet les points critiques intérieurs, on rencontre à un problème qui m'apparaît extrêmement difficile (voir No. 23). C'est pour préparer des méthodes et pour éclaircir la figure de la difficulté, que nous avons décidé à publier le présent Mémoire, séparément.[9]

For convenience we recall their English translations by R. Narasimhan from [25]:

(1) The principal problems we have dealt with since Memoir I are the following: Cousin problems, the problem of expansions and the problem of (different types of) convexity[10] In Memoirs I–VI[11] we

[4] Ces problèmes sont fondés sur H. Behnke et P. Thullen, Theorie der Funktionen mehrerer Komplexer Veränderlichen, 1934. Nous allons les expliquer en formes précises. Soient $\mathfrak{D}, \mathfrak{D}_0$ deux domaines connexes ou non sur l'espace de n variables complexes tels que $\mathfrak{D}_0 \subseteq \mathfrak{D}$ (c'est-à-dire que $\mathfrak{D}_0$ soit un ≪Teilbereich≫ de $\mathfrak{D}$); nous appellerons que $\mathfrak{D}_0$ est holomorphe-convexe par rapport à $\mathfrak{D}$, s'il existe une fonction holomorphe dans $\mathfrak{D}$ ayant des éléments de Taylor différents aux points différents de $\mathfrak{D}_0$ et encore si, pour tout domaine connexe ou non Δ_0 tel que $\Delta_0 \Subset \mathfrak{D}_0$ (c'est-à-dire que $\Delta_0 \subset \mathfrak{D}_0$ et $\Delta_0 \ll \mathfrak{D}_0$), on peut trouver un domaine connexe ou non Δ tel que $\Delta_0 \subseteq \Delta \Subset \mathfrak{D}_0$ de façon qu'à tout point P de $\mathfrak{D}_0 - \Delta$, il corresponde une fonction f holomorphe dans $\mathfrak{D}$ telle que $|f(P_0)| > \max|f(\Delta_0)|$. Spécialement, si $\mathfrak{D}_0$ est ainsi par rapport à lui-même, nous l'appelons avec H. Behnke d'être holomorphe-convexe (regulär-konvex). Les problèmes sont alors: Problèmes de Cousin. Trouver une fonction méromorphe (ou holomorphe) admettant les pôles (ou les zéros satisfaisant à une certaine condition) donnés dans un domaine holomorphe-convexe. Problème de développement. Soit $\mathfrak{D}_0$ un domaine (connexe ou non) holomorphe-convexe par rapport à $\mathfrak{D}$; trouver, pour toute fonction holomorphe f une série de fonctions holomorphes dans $\mathfrak{D}$, convergente uniformément vers f dans tout domaine connexe ou non Δ_0 tel que $\Delta_0 \Subset \mathfrak{D}_0$. Problème des convexités. Tout domaine pseudoconvexe est-il holomorphe-convexe? Pour les domaines univalents, on peut remplacer ≪ holomorphe-convexe≫ par ≪ domaine d'holomorphie≫, grâce au théorème de H. Cartan et P. Thullen.

[5] Les Mémoires précédents sont: I–Domaines convexes par rapport aux fonctions rationnelles, 1936; II–Domaines d'holomorphie, 1937; III–Deuxième problème de Cousin, 1939 (Journal of Science of the Hiroshima University); IV–Domaines d'holomorphie et domaines rationnellement convexes, 1941; V–L'intégrale de Cauchy, 1941 (Japanese Journal of Mathematics); VI–Domaines pseudoconvexes, 1942 (Tohoku Mathematical Journal); VII–Sur quelques notions arithmétiques, 1950 (Bulletin de la Société Mathématique de France)

[6] Précisément dit, pour le deuxième problème de Cousin, nous avons montrer une condition nécessaire et suffisante pour les zéros; et pour le problème des convexités, nous l'avons expliqué pour les deux variables complexes, pour diminuer la répétition ultérieure inévitable.

[7] L'auteur l'a écrit aux détails en japonais à Prof. T. Takagi en 1943.

[8] Voir la Note à l'Introduction de Mémoire VIII. Dans ce manuscrit-ci on trouve déjà les problèmes (C_1) (C_2) (expricitement) et (E) (implicitement).

[9] cité plus haut.

[10] These problems are based on H. Behnke and P. Thullen, Theorie der Funktionen mehrerer komplexer Veränderlichen, 1934. Let us explain them in precise form. Let $\mathfrak{D}, \mathfrak{D}_0$ be two domains over the space of n complex variables connected or not such that $\mathfrak{D}_0 \subsetneqq \mathfrak{D}$ (i.e. such that $\mathfrak{D}_0$ is a "Teilbereich" of $\mathfrak{D}$). We shall say that $\mathfrak{D}_0$ is holomorph-convex with respect to $\mathfrak{D}$ if $\mathfrak{D}_0 \subsetneqq H$, H being the "Regularitätshülle" of $\mathfrak{D}_0$, and if, in addition, for every domain Δ_0, connected or not, such that $\Delta_0 \Subset \mathfrak{D}_0$ (that is, $\Delta_0 \subset \mathfrak{D}_0$ and $\Delta_0 \ll \mathfrak{D}_0$), we can find a domain Δ, connected or not such that $\Delta_0 \subset \Delta \Subset \mathfrak{D}_0$ and such that, to every point P_0 of $\mathfrak{D}_0 - \Delta$, there corresponds a function f holomorphic on $\mathfrak{D}$ with $f(P_0) > \max|f(\Delta_0)|$. In particular, if $\mathfrak{D}_\alpha$ has this property with respect to itself, we call it, with H. Behnke, holomorph-convex (regulärkonvex). The problems are then the following: Cousin problems. Find a meromorphic (or holomorphic) function having given poles (or given zeros satisfying a certain additional condition). Problem of expansions. Let $\mathfrak{D}_0$ be a domain (connected or not) holomorph-convex with respect to $\mathfrak{D}$; for any function f holomorphic on $\mathfrak{D}_0$, find a series of holomorphic functions on $\mathfrak{D}$ which converges uniformly to f on any domain Δ_0, connected or not, such that $\Delta_0 \Subset \mathfrak{D}_o$. Problem of convexity. Is every pseudoconvex domain holomorph-convex? For univalent domains, one can replace "holomorph-convex" by "domain of holomorphy" because of the theorem of H. Cartan and P. Thullen.

[11] The preceding Memoirs are: I. Rationally convex domains, 1936; II. Domains of holomorphy, 1937; III. The second Cousin problem, 1939 (Journal of Science of Hiroshima University); lV. Domains of holomorphy and rationally convex

have seen, to put it in one word, that these problems can be solved affirmatively for univalent domains without points at infinity.[12] Furthermore, the author has verified, albeit without publishing this, that these results remain valid at least as far as domains without points at infinity and without interior ramification points.[13]
We must therefore either introduce suitable points at infinity or allow points of ramification. Now, one will find that almost nothing is known about domains with interior ramification. .....

(2) In the present memoir, we shall deal with the problems indicated above, as well as the arithmetical problems introduced in Memoir VII, for pseudoconvex domains without interior ramification and without points at infinity; the essential part of this memoir is not very different from what we have expounded in Japanese in 1943.[14]
We shall see in the memoir following this one that when one permits interior points of ramification, one meets a problem which seems to me to be extremely difficult (see also No. 23 below). It is to prepare the methods and to illuminate the nature of this difficulty that we have decided to publish the present memoir separately.[15]

According to T. Nishino ([26] Vol. 1, Afterword), the original manuscripts of this series sent to T. Takagi in 1943 were lost, but fortunately, the complete set of their draft-manuscripts had been kept in Oka's home library and was found posthumously.

It is really surprising for me to learn that the way of arguments in Oka IX (published, 1953) is very similar to the one in the series of papers 1943, ten years prior, and that the part of the arguments to prove so-called *Oka's Heftungslemma*,[16] an essential step in the proof of the Levi (Hartogs' Inverse) Problem, is almost a copy of the corresponding part in unpublished Paper XI 1943.

For the English translation of Paper XI, I describe in below some supplements and recall briefly the main results that had been obtained in VII–X and used in XI.

H. Cartan once has written ([25], p. XII):

> ....
> Mais il faut avouer que les aspects techniques de ses démonstrations et le mode de présentation de ses résultats rendent difficile la tâche du lecteur, et que ce n'est qu'au prix d'un réel effort que l'on parvient à saisir la portée de ses résultats, qui est considérable. C'est pourquoi il est peut-être encore utile aujourd'hui, en hommage au grand créateur que fut Kiyoshi Oka, de présenter l'ensemble de son œuvre.
> ....

In English (by Noguchi),

> ....
> But we must admit that the technical aspects of his proofs and the mode of presentation of his results make it difficult to read, and that it is possible only at the cost of a real effort to grasp the scope of its results, which is considerable. This is why it is perhaps still useful today, for the homage of the great creator that was Kiyoshi Oka, to present the collection of his work.
> ....

The present series is no exception. The aim of the series is two folded:

(i) With an intention to deal with the problem for *ramified* Riemann domains, the conditions and the statements of lemmata, propositions etc. are made as general as possible.
(ii) In the same time, they must be satisfied and proved completely for *unramified* Riemann domains as a special case.

This approach which contains in a sense a self-confliction between "general" versus "special" seems to increase an involvedness of the presentations of the papers, but forms a motivation to invent "Coherence" or *"Idéaux de domaines indéterminés"* in terms of Oka (see §5), which is referred, e.g., as:

> Of greatest importance in Complex Analysis is the concept of a coherent analytic sheaf (Grauert–Remmert [8]).

The last Paper XI contains the most important final conclusion proving that *every pseudoconvex unramified Riemann domain over* $\mathbf{C}^n$ *is Stein (in terms of the present days).* In a year before, 1942, Oka published Oka VI ([22]), proving the result in the case of univalent domains of $\mathbf{C}^2$. In Oka VI ([22]), he used Weil's integral formula, which in n-dimensional case takes a rather involved form already in univalent domains. To deal with possibly infinitely sheeted unramified Riemann domains with his intension even to deal with ramified case, he wanted to avoid the use of Weil's integral formula, but to use simpler Cauchy's integral formula combined with "*Jôku-Ikô*" (lifting principle) which was prepared as *The First Fundamental Lemma* at the end of Paper VIII of the

domains, 1941; V. The Cauchy integral, 1941 (Japanese Journal of Mathematics); VI. Pseudoconvex domains, 1942 (Tohôku Mathematical Journal); VII. On some arithmetical concepts, 1950 (Bulletin de la Société Mathematique de France)
[12] More precisely, we obtained a necessary and sufficient condition for the second Cousin problem; and the problem of convexity was only explained for two complex variables in order to reduce the ultimate repetition which is inevitable.
[13] The author has written this out in detail in Japanese and sent it to Prof. T. Takagi in 1943.
[14] See the note in the introduction to Memoir VIII. In that manuscript, one finds already problems (C_1), (C_2) (explicitly), and problem (E) (implicitly).
[15] Cite the above.
[16] Roughly speaking, the union of two adjacent holomorphically convex domains with pseudoconvex boundary is holomorphically convex (cf., e.g., [14], Chap. 9, [1])

present series. The method of Jôku-Ikô was invented in his first two papers Oka [22] I and II.

For the proof of "Heftungslemma", he uses an integral equation of the Fredholm type similarly to Oka VI (published, [22]); in Oka IX (published, [22]) the integral equation is implicit.

Reading the series of unpublished papers VII–XI 1943, we observe not only the solution of the Levi (Hartogs' Inverse) Problem for unramified Riemann domains over $\mathbf{C}^n$ $(n \geq 2)$, but also the dawn of the then unknown notion of *"Idéaux de domaines indéterminés"* or "Coherence".

Remark 2.1. It is a nature of Oka's wording such as *Idéaux de domaines indéterminés* to represent "*a way of thinking*" rather than the formed object, similarly to the case of "*Jôku-Ikô*" (see Footnote 1 at p. 45).

3. The XI-th Paper

3.1 Some Practical Notes

This series of the present Papers VII–XI in 1943 were written as a continuation of the published papers Oka I–VI ([22]). In Part II we shall present a Japanese translation of the last Paper XI, in which at some important places, footnotes are put to remind the numbering as "Note by the translator". As a consequence, the numbering of the footnotes are different to the original.

As Oka writes "Report VI", then it means the published paper with the same number in [22]. On the other hand, Report VII to X (e.g., Report IX) is the article of the present series (not the published Oka IX in [22]).

As Oka writes "a *finite* domain", it means a multisheeted domain spread over $\mathbf{C}^n$, not containing an infinite point, say, in a compactification such as complex projective n-space.

3.2 The XI-th Paper

This is the last one of the series from VII-th, in which Oka settled affirmatively the Levi (Hartogs' Inverse) Problem for general dimensional unramified Riemann domains over $\mathbf{C}^n$, ten years before Oka [22] IX was published in 1953: There was then no notion of "*Coherence*" or "*Idéaux de domaines indéterminés*" termed by Oka. It is rather surprising to know that the Problem had been solved just after Oka VI 1942 (in the case of 2-dimensional univalent domains) by a different method, if one observes the state of advances at that time as discussed in §1.

Because of the importance, I chose the last one for the translation into English.

In this paper K. Oka begins with proving the *Cousin I/II Problems* as well as the *Problem of expansions (Approximation Problem) for unramified Riemann domains over* $\mathbf{C}^n$ $(n \geq 2)$ by a different method than those in Oka [22] I–III, using a new Jôku-Ikô prepared in Papers VII–VIII of the present series.

Let us quote the most important main result from Paper XI §10 (Part II):

> **Theorem I.** *A finite pseudoconvex domain with no interior ramification point is a domain of holomorphy.*

Remark 3.1. (i) In the published Oka I–VI the domains are assumed to be univalent. Oka first dealt with unramified multivalent domains over $\mathbf{C}^n$ systematically in the present series of VII–XI.

(ii) In the proof of Oka's Theorem I above he actually proves that such a pseudoconvex domain is holomorphically convex and satisfies the separation property by holomorphic functions (see the footnote of Theorem I, XI §10). It is noted that unramified holomorphically convex domains (multivalent in general) are domains of holomorphy; the converse holds, provided that the domains are finitely sheeted (due to Cartan–Thullen [3]). Cartan–Thullen [3] claimed the converse in general, but there was an oversight in the case of infinitely many sheeted domains. The oversight was fulfilled by the proof of Oka's Theorem I above (cf. XI §11) as a series of implications: "domain of holomorphy" $\Rightarrow$ "pseudoconvex domain" $\Rightarrow$ "holomorphically convex domain". Thus the three classes of unramified domains over $\mathbf{C}^n$ are equivalent.

4. The VII–X-th Papers

To begin with, it will be interesting and worthy to recognize Oka's own observation of the state of researches at the time to start writing the present series of papers 1943 by recalling the first paragraph of the VII-th:

> The problems discussed at the beginning of the first report[17] were solved more or less generally in the series of reports up to VI.[18] But, since these were a sort of depth sounding in a sense, we avoided domains such as not finite or non univalent, and considered some of them only in the case

[17] (Note by the present author) This is Oka [22] I 1936; the same in the sequel.

[18] (The original footnote) I began the present research with the back ground of the following monograph.
H. Behnke–P. Thullen: Theorie der Funktionen mehrerer komplexer Veränderlichen, 1934 (Egebnisse der Mathematik und ihrer Grenzgebiete).
Reports prior to this report: I, 1936; II, 1937; III, 1939, (Journal of Science of the Hiroshima University). IV, 1940; V, 1940, (Japanese Journal of Mathematics). VI, 1942, (The Tôhoku Mathematical Journal).

of two variables. While we may think of really various kinds of problems on analytic functions of several variables, it is, for a moment, our main aim of the research to get rid of these restrictions one by one. The present paper is devoted to the preparation for it.

In fact, Oka begun in the present series of papers to deal with general multi-valent domains over $\mathbf{C}^n$, systematically. Here we would like to summerize briefly what were proved in the VII–X-th papers before the XI-th paper.

The four papers were roughly classified into two groups, VII+VIII and IX+X.

4.1 VII+VIII

These two papers were devoted to the study of ideal theoretic properties of holomorphic functions. The study of this part led to the works of "*Idéaux de domaines indéterminés*" or "*Coherence*" (Oka VII, VIII, published [22]). Therefore, in Oka IX (published, [22]) the contents of this part were replaced by the more general results of Oka VII, VIII (published, [22]).

In VII he considered a domain $\mathfrak{D}$ in the space of n complex variables $x_1,\dots,x_n$. Let $\mathcal{O}(\mathfrak{D})$ denote the ring of all holomorphic functions in $\mathfrak{D}$. Let $(F) = (F_1,F_2,\dots,F_p)$ be a system of holomorphic functions in $\mathfrak{D}$. For $f(x),\varphi(x) \in \mathcal{O}(\mathfrak{D})$ we write

$$f \equiv \varphi \pmod{F_1,F_2,\dots,F_p},$$

and say that f and φ are *congruent* with respect to the function system (F) in $\mathfrak{D}$, if there are functions $\alpha_j \in \mathcal{O}(\mathfrak{D})$ $(1 \le j \le p)$ satisfying

$$f-\varphi = \alpha_1 F_1 + \alpha_2 F_2 + \cdots + \alpha_p F_p.$$

Let P be a point of $\mathfrak{D}$. We define the notion of *being congruent at* P if the above property hold in a neighborhood of P. Then it is different to say that they are congruent in $\mathfrak{D}$ and they are congruent at each point of $\mathfrak{D}$. To emphasize this difference we also say the former case to be *congruent globally in* $\mathfrak{D}$.

If $\bar{\mathfrak{D}}$ is a closed domain, we denote by $\mathcal{O}(\bar{\mathfrak{D}})$ the set of all of holomorphic functions in neighborhoods of $\mathcal{O}(\bar{\mathfrak{D}})$.

Then he formulate two problems:

Problem I. *Let $\bar{\mathfrak{D}}$ be a bounded closed domain in (x) space. For a given holomorphic function system $(F) = (F_1,F_2,\dots,F_p)$ with $F_j \in \mathcal{O}(\bar{\mathfrak{D}})$ and a given holomorphic function $\Phi(x) \in \mathcal{O}(\bar{\mathfrak{D}})$ such that $\Phi(x) \equiv 0 \pmod{F}$ at every point $P \in \bar{\mathfrak{D}}$, choose $A_j \in \mathcal{O}(\bar{\mathfrak{D}})$ so that*

$$\Phi(x) = A_1(x)F_1(x) + A_2(x)F_2(x) + \cdots + A_p(x)F_p(x), \quad x \in \bar{\mathfrak{D}}.$$

Problem II. *Let $(F) = (F_1,F_2,\dots,F_p)$ be a system of holomorphic functions defined in a neighborhood of $\bar{\mathfrak{D}}$. Suppose that for each point $P \in \bar{\mathfrak{D}}$ there are associated a polydisk (γ) with center P and a holomorphic function $\varphi(x)$ in (γ) satisfying that for two such pairs $((\gamma_j),\varphi_j)$, $j=1,2$, with $(\delta) = (\gamma_1) \cap (\gamma_2) \neq \emptyset$,*

$$\varphi_1(x) \equiv \varphi_2(x) \pmod{F_1,F_2,\dots,F_p}$$

at every point of (δ) (congruent condition). Then, find a $\Phi(x) \in \mathcal{O}(\bar{\mathfrak{D}})$ such that

$$\Phi(x) \equiv \varphi(x) \pmod{F}$$

at every point $P \in \bar{\mathfrak{D}}$.

Remark 4.1. Problem I is a sort of Syzygy type problem, and Problem II is a Cousin-I Problem for the ideal generated by $(F) = (F_1,F_2,\dots,F_p)$.

In §2 of Paper VII he defines the following property named

(A): *Let $(F_1,F_2,\dots,F_p)$ be a system of holomorphic functions in a domain $\mathfrak{D}$ of (x)-space such that $F_1 \not\equiv 0$. Let $q \in \{2,3,\dots,p\}$ and let $P \in \mathfrak{D}$ be an arbitrary point. If holomorphic functions $\alpha_j(x)$ $(j=1,2,\dots,q)$ in a neighborhood $U(\subset \mathfrak{D})$ of P satisfy*

$$\alpha_1(x)F_1(x) + \alpha_2(x)F_2(x) + \cdots + \alpha_q(x)F_q(x) = 0, \quad x \in U,$$

then

$$\alpha_q(x) \equiv 0 \pmod{F_1,F_2,\dots,F_{q-1}} \quad \text{at } P.$$

Most importantly, he shows the following for property (A):

Lemma 1. *Let X be a domain in (x)-space, and let $f_j(x)$ $(j=1,2,\dots,\nu)$ be holomorphic functions in X. Then the system of holomorphic functions $F_j(x,y) = y_j - f_j(x)$ $(j=1,2,\dots,\nu)$ satisfies property (A).*

This is intended to apply for an Oka map

$$\psi(x) = (x, f_1(x), f_2(x), \dots, f_\nu(x)) \in \Omega \times \Delta(1)^\nu \subset \Delta(R)^n \times \Delta(1)^\nu,$$

where $f_j(x) \in \mathcal{O}(X)$, Ω $(\Subset X)$ is an analytic polyhedron defined by

$$x \in X, \quad |f_j(x)| < 1, \quad j=1,2,\dots,\nu,$$

$\Delta(R)$ is the disk of radius R (>0) with center at the origin in $\mathbf{C}$ and R is chosen so that $\Omega \subset \Delta(R)^n$. This is the essential part of Oka's *Jôku-Ikô*:

Remark (Jôku-Ikô). T. Nishino [14] uses "lifting principle" for "Jôku-Ikô". It is a methodological principle termed by Oka such that

(i) one embedds a domain into a higher dimensional domain of simple shape (i.e., a polydisk) through the Oka map above;

(ii) one extends a difficult problem on the original domain to the one on the higher dimensional domain of simple shape;
(iii) by making use of the simpleness of the higher dimensional domain, one obtains a solution of the problem;
(iv) then, one restricts the solution on the embedded original domain to get a solution of the original problem.

Things do not go so simply, but this is the principal method of K. Oka all through his works.

Oka then affirmatively solves Problems I and II under this property (A) for (F).

Theorem 1. *Let $\bar{\mathfrak{D}}$ be a bounded closed cylinder domain and let $(F) = (F_1, F_2, \ldots, F_p)$ be a system of holomorphic functions in a neighborhood of $\bar{\mathfrak{D}}$ which satisfies property (A). Then, Problem I for (F) is solvable.* For a cylinder domain, see Footnote 3, p. 46.

Theorem 2. *Let $\bar{\mathfrak{D}}$ and (F) be the same as in Theorem 1 above. Then, Problem II for (F) is solvable.*

In §§8–10 of Paper VII Oka deals with Problems I and II with estimates.

Finally, at the end of Paper VIII Oka obtained

Fundamental Lemma I. *Let X be a univalent cylinder domain in (x)-space and $\Sigma \subset X$ be an analytic subset. Let V be a univalent open subset of X, containing Σ. Suppose that there are holomorphic functions $f_1(x), f_2(x), \ldots, f_p(x) \in \mathcal{O}(V)$ such that $\Sigma = \{x \in V : f_j(x) = 0, 1 \le j \le p\}$. Let $X^0 \Subset X$ be a univalent bounded cylinder domain, and set $\Sigma_0 = \Sigma \cap X^0$.*

Then, for every $\varphi(x) \in \mathcal{O}(V)$ with $|\varphi(x)| < M$ in V, there is a holomorphic function $\Phi(x) \in \mathcal{O}(X^0)$ such that at every point of Σ_0

$$\Phi(x) \equiv \varphi(x) \quad (\mathrm{mod.}\, f_1, f_2, \ldots, f_p),$$

and

$$|\Phi(x)| < KM, \quad x \in X^0,$$

where K is a positive constant independent from $\varphi(x)$.

He finishes Paper VIII with writing

> This theorem should be generalized soon later, but so far as we are concerned with finite domains without ramification points, this is sufficient for our study.

Remark 4.2. By this comment we see that he had in mind a project to deal with Levi (Hartogs' Inverse) Problem generalized to domains with ramifications.

4.2 IX+X

In these two papers Oka defines and studies *pseudoconvex* functions, equivalently plurisubharmonic functions as well strongly pseudoconvex (plurisubharmonic) functions, and investigates the boundary problem of pseudoconvex domains. The contents of these IX and X correspond to and appear in Oka IX (published, [22]), Chap. 2, §§B and C.

In these papers he deals with domains, finite and unramified over (x)-space of n complex variables $x_1, x_2, \ldots, x_n$. He begins with the notion of unramified domains over (x)-space.

Let $\mathfrak{D}$ be a domain over (x)-space and let $E \subset \mathfrak{D}$ be a subset. If the infimum of the Euclidean distances from $P \in E$ to the (ideal) boundary of $\mathfrak{D}$ is not 0, one says that E *is bounded with respect to* $\mathfrak{D}$.

He defines a *pseudoconvex domain* modeled after F. Hartogs as follows:

Definition. A domain $\mathfrak{D}$ over (x)-space is said to satisfy *Continuity Theorem* if the following condition is satisfied: Let $r = (r_j), \rho = (\rho_j)$ be n-tuples of positive numbers with $\rho_j < r_j$, and consider a polydisk $\mathrm{P}\Delta(a; r)$, $|x_j - a_j| < r_j$ with center $a = (a_j)$ and a Hartogs domain:

$$\begin{array}{lll} \Omega_{\mathrm{H}}(a; r, \rho): & & \\ & |x_j - a_j| < \rho_j, \qquad |x_n - a_n| < r_n & (j = 1, 2, \ldots, n-1), \\ \text{or} & |x_j - a_j| < r_j, \quad \rho_n < |x_n - a_n| < r_n & (j = 1, 2, \ldots, n-1). \end{array}$$

If $\phi : \Omega_{\mathrm{H}}(a; r, \rho) \to \mathfrak{D}$ is a biholomorphic map, then ϕ necessarily extends biholomorphically to $\tilde{\phi} : \mathrm{P}\Delta(a; r) \to \mathfrak{D}$.

Definition. A domain $\mathfrak{D}$ over (x)-space is said to be *pseudoconvex* if the following two conditions are satisfied:

(i) For each boundary point M of $\mathfrak{D}$ there is a positive number ρ_0 with polydisk $\mathrm{P}\Delta$ of radius ρ_0 and center $\underline{M}$ of the underlying point of M such that the maximal subdomain $\mathfrak{D}_0$ of $\mathfrak{D}$ with the boundary point M whose underlying points are contained in $\mathrm{P}\Delta$ satisfies Continuity Theorem. ($\mathfrak{D}$ satisfies locally Continuity Theorem.)
(ii) Let $\mathrm{P}\Delta_1 \subset \mathrm{P}\Delta$ be a polydisk with the same center,[19] and let $\mathfrak{D}_1$ be the maximal subdomain with the boundary point M whose underlying points are contained in $\mathrm{P}\Delta_1$. Let (T) be a one-to-one quasi-conformal[20] transform from $\mathrm{P}\Delta_1$ into (x')-space with the image denoted by Δ'_1, and $\mathfrak{D}'_1 = T(\mathfrak{D}_1)$. Then, $\mathfrak{D}'_1$ satisfies always Continuity Theorem. (The property (i) is not lost by quasi-conformal transforms.)

Remark 4.3. From the definition above one sees why he called the problem as *Hartogs' Inverse Problem*.

Then he defines a *pseudoconvex function* or a *plurisubharmonic function* valued in $[-\infty, \infty)$ so that it

[19] The radius of each variable may different.
[20] It is unclear very much what "quasi-conformal" amounts to, but it is *holomorphic*.

is upper-semicontinuous and its restriction to every complex line segment is subharmonic.

After Hartogs' holomorphic radius he defines Hartogs' radii $R_j(P)$ $(j = 1,2,\dots,n)$ at $P \in \mathfrak{D}$ by

$$R_j(P) = \sup\{r_j : \mathrm{P}\Delta(P;(r_1,\dots,r_n)) \subset \mathfrak{D},\ r_j > 0,\ 1 \le j \le n\},$$

where $\mathrm{P}\Delta(P;(r_1,\dots,r_n)) := \{(z_j) \in \mathbf{C}^n : |z_j - p_j| < r_j, 1 \le j \le n\}$ with $P = (p_j)$. He proves:

Theorem 1. *If $\mathfrak{D}$ is pseudoconvex, then $-\log R_j(P)$ is pseudoconvex in $\mathfrak{D}$. (Here the logarithm stands for the real branch.)*

Similarly, let $d(P)$ $(P \in \mathfrak{D})$ denote the supremum of radii $r > 0$ such that a ball with center P and radius r is contained in $\mathfrak{D}$, and $d(P)$ is called the *Euclidean boundary distance*. He then proves:

Theorem 3. *If $\mathfrak{D}$ is pseudoconvex, then $-\log d(P)$ is a pseudoconvex function in $\mathfrak{D}$.*

Then he consider a pseudoconvex function $\varphi(x)$ of C^2-class in general, confirming the semi-positivity of the Hermitian form

$$W(\varphi;(v_j),(w_k))(P) = \sum_{j,k} \frac{\partial^2 \varphi}{\partial x_j \partial \bar{x}_k}(P) v_j \bar{w}_k, \quad (v_j),(w_j) \in \mathbf{C}^n.$$

This form $W(\phi;\cdot,\cdot)$, which was written so in the paper and is nowadays called the *Levi form*, is due to Oka [22] VI. Then he proves in IX:

Theorem 5. *If $W(\varphi;(v_j),(w_k))(P)$ is strictly positive definite at $P = P_0$, then one can find a holomorphic polynomial function $f(x_1,x_2,\dots,x_n)$ of degree 2 such that $f(P_0) = 0$ and in a neighborhood of P_0, the analytic hypersurface $\{f = 0\}$ lies in the part $\{\varphi > 0\}$ except for P_0.*

Remark 4.4. In one variable, the situation is much simpler: If $\mathfrak{D}$ is a domain in $\mathbf{C}$ and $P_0 \in \partial\mathfrak{D}$, then $f(z) = z - P_0$. It is the purpose to construct a meromorphic function on $\mathfrak{D}$ such that its poles are only $\frac{1}{f(z)}$ near P_0. When $n \ge 2$, Oka formulated the positivity of $W(\varphi;\cdot,\cdot)$ to have $f(z)$. Later, he solves the Cousin I Problem on $\bar{\mathfrak{D}}$ with poles only $\frac{1}{f(z)}$ near P_0, and then concludes that $\mathfrak{D}$ is holomorphically convex.

Oka took a smoothing of a pseudoconvex function $\varphi(x)$ by the volume integration average, and repeat it to have a C^2-differentiable pseudoconvex function; nowadays it is more common to take a convolution integration, but the role is the same.

Finally at the end of Paper X, he obtained

Fundamental Lemma II. *Let $\mathfrak{D}$ be a pseudoconvex domain over (x)-space without ramification point. Then there is a continuous pseudoconvex function $\varphi_0(P)$ in $\mathfrak{D}$ satisfying the following two conditions:*

(i) *If $\mathfrak{D}_c := \{P \in \mathfrak{D} : \varphi_0(P) < c\}$ for every real number c, then $\mathfrak{D}_c \Subset \mathfrak{D}$.*

(ii) *There are exceptional points of $\mathfrak{D}$ with no accumulation point inside $\mathfrak{D}$ and for any other point $P_0 \in \mathfrak{D}$ than them, one can find an analytic hypersurface Σ passing P_0 in a neighborhood of P_0 such that $\varphi_0(P) > \varphi_0(P_0)$ for $P \in \Sigma \setminus \{P_0\}$.*

5. After Paper XI, and Problem Left

The series of Papers VII–XI in 1943 was not translated into French for publication, but continued to Report XII dated 26 May 1944 ([26]), titled

- On Analytic Functions of Several Variables XII – Representation of analytic sets, pp. 22.

In this manuscript, he first used *Weierstrass' Preparation Theorem* to study local properties of analytic sets. As known well, Weierstrass' Preparation Theorem plays a crucial role in the proofs of Oka's Coherence Theorems. In this sense, the turn of years *1943/1944* was indeed the *"watershed"* in the study of analytic function theory of several variables.

The research was continued more to the following and further (cf. [26]):

- XIII On the condition of Weierstrass' Preparation Theorem ([26]).

The precise date of this manuscript is unclear, but probably around 16 November 1945 due to T. Nishino's comment ([26] Vol. 2, Afterword).

Remark 5.1. These manuscripts are not completed ones, while the formers (VII–XI) are. But it is still interesting to read the introductions of the above two manuscripts.

(i) In XII he begins with "We wish to extend more the results obtained in the former reports".

(ii) In XIII, he writes at the beginning: "To consider the series of problems mentioned at the beginning of this research, we have imposed some conditions to the domains. From Reports I to VI[21] the domains were assumed to be finite and univalent, and from VII to XI[22] we excluded the infinity points and interior ramification points from the domains. As we are going to take out these conditions, it would be better to advance step by step. Putting the problem of the infinity points aside for a moment, we firstly would like to investigate what will happen if the *ramification points* are allowed to the interior of the domains."
It should be noticed that this paragraph was rephrased in the published [22] VIII, 1951.

[21] (Note by the present author.) These numbers correspond to the published papers [22].
[22] (Note by the present author.) These are the unpublished papers 1943.

Remark 5.2. The problem of the infinity points was affirmatively solved by R. Fujita [7] and A. Takeuchi [27] for unramified domains over complex projective spaces with at least one ideal boundary point; there are more extensions in the unramified case, but we stop to go further in this direction, which is away from the problem of the ramified case. A counter-example of a ramified domain over complex projective space was given by H. Grauert [13]. Therefore the problem of ramification points remained open then for domains over $\mathbf{C}^n$: Later, in 1978 J.E. Fornæss [6] gave a counter-example of a two-sheeted ramified domain over $\mathbf{C}^2$.

It is unusual not to publish such an important result obtained in the series of Papers VII–XI in 1943, which were hand-written but rather complete, ready for publication. Oka probably then noticed a shadow of an unknown concept, *"Idéaux de domaines indéterminés"* or *"Coherence"*. With a project in mind to settle the Levi (Hartogs' Inverse) Problem for domains allowing singularities and ramifications, he would have been interested more in inventing the new necessary notion for his project than the publication of the important result which was enough marvelous by itself (cf. §2).

As briefly mentioned at the end of §2, reading the series of unpublished Papers VII–XI 1943 and above XII 1944, we can see how and why Oka continued the study of the shadow of a new notion,"*Coherence*" or "*Idéaux de domaines indéterminés*" with leaving the papers unpublished, and what he really wanted to do; the problem of ramified Riemann domains left by Oka has not been settled, although the ramification case was countered by example (cf. Remark 5.2). In this sense, I think, the value of the series of the unpublished papers in 1943 has not changed. (Here, we may recall H. Cartan's words quoted in p. 48.)

K. Oka wrote his intension implicitly in a paragraph of Oka [24] ([25]) VII, Introduction, which was written and published in an interval of six or eight years after Oka [22] VI 1942, and explicitly in Oka [22] VIII, Introduction, and IX, Introduction 2 and §23 (cf. also Remark 5.1). We recall the first:

> Or, nous, devant le beau système de problèmes à F. Hartogs et aux successeurs, voulons léguer des nouveaux problèmes à ceux qui nous suivront; or, comme le champ de fonctions analytiques de plusieurs variables s'étend heureusement aux divers branches de mathématiques, nous serons permis de rêver divers types de nouveaux problèmes y préparant.

In English (from [25] VII, translation by R. Narasimhan):

> Having found ourselves face to face with the beautiful problems introduced by F. Hartogs and his successors, we should like, in turn, to bequeath new problems to those who will follow us. The field of analytic functions of several variables happily extends into divers branches of mathematics, and we might be permitted to dream of the many types of new problems in store for us.

Remark 5.3. The above paragraph was deleted in the published Oka [22] VII without notification to K. Oka in the editorial process. K. Oka was very unsatisfied with this change of the original text, so that he wrote [26] (cf. [15] Chap. 9 "On Coherence").

The series of published papers Oka [22], I–IX will be classified into two groups:

(A) I–VI+IX,
(B) VII–VIII.

In the first group he solved the Three Big Problems of Behnke–Thullen (§1). It is now known that for the solutions of those problems (even for unramified Riemann domains) one needs only a rather simple *Weak Coherence* ([18], [19]), not such general Coherence Theorems proved by Oka.

The second group (B) of VII–VIII was written beyond the Three Big Problems and was explored to a foundational theory of modern Mathematics, not only of complex analysis by H. Cartan, J.-P. Serre, H. Grauert, ...

As mentioned above, the Levi (Hartogs' Inverse) Problem for ramified domains over $\mathbf{C}^n$ was countered by example due to Fornæss [6] in 1978; in the same year K. Oka passed away. But it is unknown the cause of the failure or what is the sufficient condition for the validity of the problem in ramified case; a certain sufficient condition was lately obtained by [17].

Therefore there still remains the following interesting problem:

Oka's Problem (Dream). *What are the sufficient and/or necessary conditions with which a ramified pseudoconvex domain over* $\mathbf{C}^n$ *is Stein?*

Remark 5.4. The English word "Dream" is taken from the above Narasimhan's translation of the original 'rêver'. The problem of the pseudoconvexity was mentioned as the main problem of his research from the beginning, Oka I ([22], published, 1936). After settling the problem of pseudoconvexity for unramified domains in VII–XI (unpublished, 1943) as described above, the problem with interior ramification points had been mentioned as the principal motivation in a number of places such as, e.g., in XIII (cited above, unpublished, around 1945, cf. Remark 5.1 (ii)); in VII ([22], published, 1948[23]), the 3rd paragraph of the Introduction; in VIII ([22], published, 1951), Introduc-

[23] It is common to use the year 1948 to refer Oka VII; the actual publication year is 1950 (cf. [15] Chap. 9 "On Coherence").

tion[24] (cf. §2 too); in IX ([22], published, 1953), Introduction 2 and §23 (as already mentioned in §2 above); and later in his lecture (§5) at Yukawa Institute for Theoretical Physics, Kyoto University in March 1964 ([26]). In particular, it is noted that the above two (published) papers VII and VIII are devoted to establish his *Three Coherence Theorems* ($\mathcal{O}_{\mathbf{C}^n}$, ideal sheaves of analytic subsets,[25] and the normalizations of the structure sheaves of complex spaces); cf. [15] Chap. 9 "On Coherence" and [20] for more details.

Throughout his Mathematical research life, K. Oka seemed to intend to prove Hartogs's Inverse Problem (Levi's Problem) unconditionally for ramified domains over $\mathbf{C}^n$. H. Grauert mentioned at his talk at Conference OKA 100, Kyoto/Nara 2001[26] that the Levi problem is still open for domains ramified over $\mathbf{C}^n$.

Part II

The English translation of K. Oka's unpublished Paper XI 1943 translated by J. Noguchi.

On Analytic Functions of Several Variables

XI – Pseudoconvex Domains and Finite Domains of Holomorphy, Some Theorems on Finite Domains of Holomorphy[27]

Kiyoshi Oka

We extend the results of the first research project to unramified finite domains[28] by making use of the First and the Second Fundamental Lemmata. Here we restrict ourselves to deal with the following problems: the Problem of pseudoconvex domains being domains of holomorphy, *Cousin* I Problem, and Expansions of functions.[29]

As for *Cousin* II Problem and the integral representation, we think that they will be similarly dealt with.[30]

In the present paper, "*domains*" are assumed to be finite and to carry no ramification point in its interior: This assumption will be kept all through the paper, and will not be mentioned henceforth in general.

I – Theorems in Finitely Sheeted Domains of Holomorphy

§1

The present chapter describes *Cousin* I Problem and Expansions of functions on finitely sheeted domains of holomorphy for the preparation of what will follow in Chapter II and henceforth. The methods are due to the First Fundamental Lemma and the *H. Cartan–P. Thullen* Theorem, and so they are essentially the same as those in Report I.[31]

We first modify (the fundamental) Lemma I to a form suitable for our purpose. We recall it (Report VIII):

Lemma I. *Let (X) be a univalent cylinder domain in (x)-space, and let Σ be an analytic subset of (X). Let V be a univalent open subset of (X) with $V \supset \Sigma$. Assume that there are holomorphic functions $f_1(x), f_2(x), \ldots, f_p(x)$ in V with*

$$\Sigma = \{f_1 = \cdots = f_p = 0\}.$$

[24] The first two paragraphs of the Introduction are the rephrasing of those in XIII (cited above, unpublished) quoted in Remark 5.1 (ii).

[25] H. Cartan gave another proof for the second (cf. [15] Chap. 9 'On Coherence').

[26] Complex Analysis in Several Variables—Memorial Conference of Kiyoshi Oka's Centennial Birthday, Kyoto/Nara 2001, Ed. K. Miyajima et al., Advanced Studies in Pure Math. Vol. 42, Math. Soc. Jpn., Tokyo 2004. H. Grauert gave a talk titled 'A simple way to perform the Levi-Oka-Theory', but unfortunately, his manuscript is not included in the proceedings.

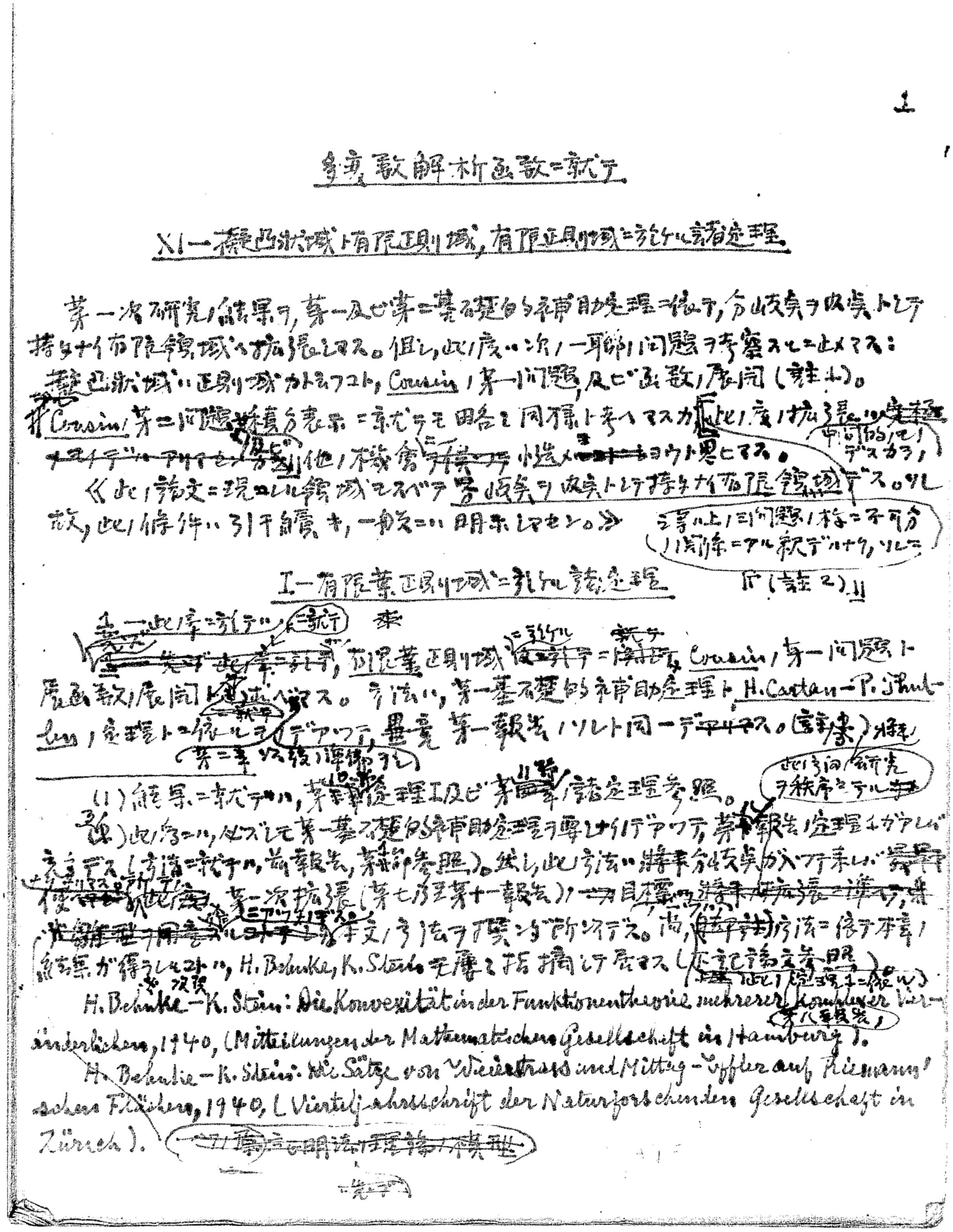
[27] This is an English translation of the original Japanese text in OKA Kiyoshi Collection, Nara Women's University Library, Unpublished manuscript, http://www.lib.nara-wu.ac.jp/oka/fram/mi.html. The handwritten original text is found in the same Collection, http://www.lib.nara-wu.ac.jp/oka/moku/html/174/001.html. © by courtesy of Mr. Hiroya Oka and OKA Kiyoshi Collection at Nara Women's University Academic Information Center.

[28] Added in translation: Oka used the term "finite domain" in the sense that it spreads over $\mathbf{C}^n$, not over a space with infinity such as $\mathbf{P}^n(\mathbf{C})$ or $(\mathbf{P}^1(\mathbf{C}))^n$.

[29] Cf. Theorem I in §10 and Theorems in §11 for the results.

[30] Since these are not in an inseparable relation as in the above three theorems, and the present extension is at an intermediate stage, we will confirm them in the next occasion.

[31] For this aim the First Fundamental Lemma is not necessarily needed, and Theorem 1 in Report VIII suffices (as for the methods, see §1 of the previous Report). This method, however, will not be effective if once a ramification point is allowed. Here it is noticed that one of the purposes of this first extension (from Reports VII–XI) is to organize the studies of this direction in future. Because of this reason we here choose the method of the present paper. And, it is was often mentioned also by H. Behnke and K. Stein that the results of the present chapter can be obtained by the method of Theorem 1 of Report VIII (cf. the papers below).
H. Behnke–K. Stein: Approximation analytischer Funktionen in vorgegebenen Bereichen des Raumes von n komplexen Veränderlichen, 1939 (Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen).
H. Behnke–K. Stein: Die Konvexität in der Funktionentheorie mehrerer komplexer Veränderlichen, 1940 (Mitteilungen der Mathematischen Gesellschaft in Hamburg).
H. Behnke–K. Stein: Die Sätze von Weierstrass und Mittag-Leffler auf Riemannschen Flächen, 1940 (Vierteljahrsschaft der Naturforschenden Gesellschaft in Zürich).



1

多変数解析函数ニ就テ

XI — 擬凸状域ト有限正則域, 有限正則域ニ於ケル諸定理

第一次研究ノ結果ヲ, 第一及ビ第二基礎的補助定理ニ依テ, 分岐点ヲ内点トシテ持タナイ有限領域ヘ拡張シマス。但シ, 此ノ度ハ次ノ一聯ノ問題ヲ考察スルニ止メマス:
擬凸状域ハ正則域カトイフコト, Cousin ノ第一問題, 及ビ函数ノ展開 (註1)。

I — 有限葉正則域ニ於ケル諸定理

H. Behnke – K. Stein: Die Konvexität in der Funktionentheorie mehrerer komplexer Veränderlichen, 1940, (Mitteilungen der Mathematischen Gesellschaft in Hamburg).

H. Behnke – K. Stein: Die Sätze von Weierstrass und Mittag-Leffler auf Riemannschen Flächen, 1940, (Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich).

Figure 1. The first page of the draft-manuscript XI, Ref. No. 177. By the courtesy of OKA Kiyoshi Collection. Nara Women's University Library, Copyright (c) 1999; All rights reserved.

Let $(X^0) \Subset (X)$ be a relatively compact cylinder subdomain and set $\Sigma_0 = \Sigma \cap (X^0)$.

Then, for a bounded holomorphic function $\varphi(x)$ in V such that $|\varphi(x)| < M$ in V, there is a holomorphic function $\Phi(x)$ in (X^0) such that at every point of Σ_0

$$\Phi(x) \equiv \varphi(x) \quad (\text{mod. } f_1, f_2, \ldots, f_p)$$

and

$$|\Phi(x)| < KM$$

on (X^0). Here K is a positive constant independent from $\varphi(x)$.

Let R be a domain in the space of n complex variables $x_1, x_2, \ldots, x_n$ (without ramification point in the interior, and finite) or a countable union of mutually disjoint such domains. We consider an analytic polyhedron (a point set) Δ in R satisfying the *following three conditions*:

1° $\Delta \Subset R$. (Therefore, Δ is contained in a finite union of connected components of R, bounded and finitely sheeted.)

2° Δ is defined as follows:

(Δ)
$$P \in R,\ x_i \in X_i,\ f_j(P) \in Y_j \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, \nu),$$

where (x) is the coordinate system of the point P, X_i and Y_j are *univalent* domains of (finite) planes, and $f_j(P)$ are holomorphic functions in R (in the sense of *one-valued* analytic functions in every connected component of R; same in what follows).

3° The vectors $[x_1, x_2, \ldots, x_n, f_1(P), f_2(P), \ldots, f_\nu(P)]$ have *distinct values* for *distinct points* of Δ.

We introduce new variables, $y_1, y_2, \ldots, y_\nu$ and consider (x,y)-space. We then consider a cylinder domain, (X,Y) with $x_i \in X_i, y_j \in Y_j$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, \nu)$ together with an analytic subset

$$(\Sigma) \qquad y_j = f_j(P),\ P \in \Delta \qquad (j = 1, 2, \ldots, \nu).$$

We map a point P of Δ with coordinate (x) to a point M of Σ with coordinate $[x, f(P)]$. By Condition 3° distinct two points P_1, P_2 of Δ are mapped always to distinct two points M_1, M_2 of Σ, and hence the map is injective. All points of Σ is contained in (X,Y) and its boundary points are all lying on the boundary of (X,Y). (If $f_j(P)$ $(j = 1, 2, \ldots, \nu)$ are simply assumed to be holomorphic functions in Δ, then the first half holds, but not the second half.) Let X_i^0, Y_j^0 $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, \nu)$ be domains of complex plane such that $X_i^0 \Subset X_i$, $Y_j^0 \Subset Y_j$, and let Δ_0 denote the corresponding part of Δ. Then, $\Delta_0 \Subset \Delta$. Now, let $\Delta_0 \Subset \Delta$ be an arbitrary subset. If P_1, P_2 both belong to Δ_0 and have the same coordinate, then the distance between M_1, M_2 carries a lower bound away from 0.

Let $\varphi(P)$ be an arbitrary holomorphic function in Δ. With a point P of Δ mapped to a point M of Σ, we consider a function $\varphi(M)$ on Σ by setting

$$\varphi(M) = \varphi(P).$$

As seen above, we may think a holomorphic function in (x,y) defined in a univalent open set containing Σ, which agrees with $\varphi(M)$ on Σ, and locally independent from (y). Therefore, Lemma I is modified to the following form:

Lemma I′. *Let the notation be as above. Let (X^0, Y^0) be a cylinder domain such that $(X^0, Y^0) \Subset (X, Y)$. Then, for a given bounded holomorphic function $\varphi(P)$ on Δ, we may find a holomorphic function $\Phi(x,y)$ in (X^0, Y^0) so that if $|\varphi(P)| < N$ in Δ, $|\Phi(x,y)| < KN$ in (X^0, Y^0), and $\Phi(x, f(P)) = \varphi(P)$ for all $[x, f(P)] \in (X^0, Y^0) \cap \Sigma$ with coordinate (x) of P. Here, K is a positive constant independent from $\varphi(P)$.*

We have the following relation between the analytic polyhedron Δ above and a finitely sheeted domain which is convex with respect to a family of holomorphic functions:[32]

Lemma 1. *Let $\mathfrak{D}$ be a domain of holomorphy in (x)-space, and let $\mathfrak{D}_0$ be a finitely sheeted open subset of $\mathfrak{D}$, which is holomorphically convex with respect to the set of all holomorphic functions in $\mathfrak{D}$. For any subset $E \Subset \mathfrak{D}_0$, there exist an analytic polyhedron Δ and an open subset R of $\mathfrak{D}_0$ such that $E \Subset \Delta$ and R satisfies the above three Conditions, where $f_j(P)$ $(j = 1, 2, \ldots, \nu)$ may be taken as holomorphic functions in $\mathfrak{D}$, X_i $(i = 1, 2, \ldots, n)$ taken as disks $|x_i| < r$, and Y_j taken as unit disks $|y_j| < 1$.*[33]

Proof. Let F be an arbitrary subset of $\mathfrak{D}_0$ which is bounded with respect to $\mathfrak{D}_0$. Since $\mathfrak{D}_0$ is finitely sheeted, it is immediate that

$$F \Subset \mathfrak{D}_0.$$

Conversely, if $F \Subset \mathfrak{D}_0$, then F is bounded with respect to $\mathfrak{D}_0$ (even if $\mathfrak{D}_0$ is not finitely sheeted).

Therefore, these two notions agree with each other.

We denote by $(\mathfrak{F})$ the family of all holomorphic functions in $\mathfrak{D}$. Then, $\mathfrak{D}_0$ is convex with respect to $(\mathfrak{F})$, and $E \Subset \mathfrak{D}_0$. As seen as above, we may take an open set $\mathfrak{D}_0'$ with $E \subset \mathfrak{D}_0' \Subset \mathfrak{D}_0$, so that for every point P_0 of $\mathfrak{D}_0$, not belonging to $\mathfrak{D}_0'$, there is at least one function $\varphi(P)$ of $(\mathfrak{F})$ satisfying

$$|\varphi(P_0)| > \max |\varphi(E)|.$$

32 Cf. the previous Report, §1 for the definition of the convexity.

33 Cf. the first two of the three papers of H. Behnke–K. Stein cited above.

(Here, the right-hand side stands for the supremum of $|\varphi(P)|$ on E.)

Let ρ denote the minimum distance of $\mathfrak{D}_0'$ with respect to $\mathfrak{D}_0$, and let r be a positive constant such that any point $P(x)$ of E satisfies $|x_i| < r$ $(i = 1,2,\dots,n)$. We consider those points of $\mathfrak{D}_0$ such that the distance to the boundary of $\mathfrak{D}_0$ is $\frac{1}{2}\rho$, and denote by Γ the part of them over the closed polydisk $|x_i| \le 2r$. As seen above, Γ is a closed set. It is clear that for an arbitrary point M of Γ, there are a small polydisk (γ) with center M contained in $\mathfrak{D}$, and a function $f(P)$ of $(\mathfrak{F})$ satisfying

$$\max|f[(\gamma)]| > 1, \quad \max|f(E)| < 1.$$

Therefore by the Borel–Lebesgue Lemma, Γ is covered by finitely many such (γ). Let $f_1(P)$, $f_2(P)$, $\dots$, $f_\lambda(P)$ be those functions associated with them. Set $R = \mathfrak{D}_0^{(\frac{\rho}{2})}$ (the set of points of $\mathfrak{D}_0$ whose distance to the boundary of $\mathfrak{D}_0$ is greater than $\frac{\rho}{2}$. We consider the following analytic polyhedron Δ:

$$(\Delta) \;\; P\in R,\; |x_i|<r,\; |f_j(P)|<1 \;\; (i=1,2,\dots,n; j=1,2,\dots,\lambda).$$

Clearly, $E \subset \Delta$ and $\Delta \Subset R$. (The condition of Lemma requires $E \Subset \Delta$, but this is the same.)

We check Condition 3°. Since $\mathfrak{D}$ is a domain of holomorphy, there is a holomorphic function whose domain of existence is $\mathfrak{D}$. Let $F(P)$ be such one. Then, by the definition of domain of holomorphy,[34] for mutually overlapped (the coordinates are the same) two points P_1 and P_2[35] of $\mathfrak{D}$, the elements[36] of $F(P)$ at P_1 and P_2 are necessarily different. Therefore, there exists a partial derivative of $F(P)$ with respect to x_i $(i = 1,2,\dots,n)$ which takes distinct values at P_1 and P_2, and the partial derivative is necessarily a holomorphic function in $\mathfrak{D}$. Let $\bar{\Delta}$ denote the union of Δ and its boundary. Since $\Delta \Subset \mathfrak{D}_0$, $\bar{\Delta}$ is a closed set. Hence by the Borel–Lebesgue Lemma, there are finitely many holomorphic functions consisting of $F(P)$ and its partial derivatives,

$$\varphi_1(P),\; \varphi_2(P), \dots,\; \varphi_\mu(P)$$

such that the vector-valued function $[\varphi_1(P), \varphi_2(P), \dots, \varphi_\mu(P)]$ takes distinct vector-values at any two distinct points of $\bar{\Delta}$. These functions are bounded in Δ. We set

$$\max|\varphi_k(\Delta)| < N, \quad f_{\lambda+k}(P) = \frac{1}{N}\varphi_k(P) \qquad (k = 1,2,\dots,\mu).$$

Then we see that the set of points of $\mathfrak{D}$ satisfying three conditions, $P \in R$, $|x_i| < r$, $|f_j(P)| < 1$ $(i = 1,2,\dots,n;\ j=1,2,\dots,\nu;\ \nu=\lambda+\mu)$ agrees with Δ. The expression of Δ of this type satisfies all Conditions 1°, 2° and 3°. C.Q.F.D.

Recall that a domain of holomorphy carries the following property:

The First Theorem of H. Cartan–P. Thullen. *A finite domain of holomorphy is convex with respect to the whole of functions holomorphic there.*

This theorem is an immediate consequence of the *Fundamental Theorem of H. Cartan–P. Thullen*[37] *on the simultaneous analytic continuation.*[38]

§2

We study the expansions of functions.[39]

We consider Δ in Lemma 1: Here we also assume that Δ satisfies the conditions added at the end of the lemma. Then, Δ is of the form:

$$(\Delta) \;\; P\in R,\; |x_i|<r,\; |f_j(P)|<1 \quad (i=1,2,\dots,n; j=1,2,\dots,\nu)$$

We introduce complex variables $y_1, y_2, \dots, y_\nu$ and in (x,y)-space we consider a polydisk

$$(C) \quad |x_i| < r, \quad |y_j| < 1 \qquad (i = 1,2,\dots,n; j = 1,2,\dots,\nu)$$

and an analytic subset defined by

$$(\Sigma) \qquad y_j = f_j(P), \quad P \in \Delta \qquad (j = 1,2,\dots,\nu).$$

Let r_0 and ρ_0 be positive numbers with $r_0 < r$ and $\rho_0 < 1$, and let $\Delta_0, (C_0), \Sigma_0$ respectively denote those defined as $\Delta, (C), \Sigma$ with $(r,1)$ replaced by (r_0,ρ_0).

Let $\varphi(P)$ be an arbitrary holomorphic function in Δ. By Lemma I′ one can construct a holomorphic function $\Phi(x,y)$ in (C_0) such that $\Phi(x, f(P)) = \varphi(P)$ for all $[x,P] \in \Sigma_0$. We expand this $\Phi(x,y)$ to a Taylor series with center at the origin of (C_0). Then the convergence is locally uniform at every point of (C_0). With substituting $y_j = f_j(P)$ $(j = 1,2,\dots,\nu)$ in that expansion, we obtain an expansion of $\varphi(P)$ in Δ_0, whose terms are all holomorphic functions in $\mathfrak{D}$; the convergence is locally uniform at every point of Δ_0.

[34] Cf. Behnke–Thullen's Monograph, p. 16.

[35] (Note by the translator.) This means that P_1 and P_2 are distinct and their projections to $\mathbf{C}^n$ (the base points) are the same.

[36] (Note by the translator.) That is, function elements or germs in the present terms.

[37] Cf. Behnke–Thullen's Monograph, Chap. 6, §1 and the following paper by H. Cartan–P. Thullen: Regularitäts-und Konvergenzbereiche, 1932 (Math. Annalen).

[38] In this way we use the Fundamental Theorem of Cartan–Thullen. However, this theorem no longer holds if ramification points or points of infinity are allowed to come in. Therefore there remains a problem how to deal with these difficulties in future, but in the present paper this theorem is not necessarily needed in fact; cf. the footnote of Theorem I. Although there do not arise no other problems of this kind, the author thinks that the one mentioned above is the most noticeable.

[39] Cf. Report I, §4.

Since $\mathfrak{D}_0$ is the limit of the monotone increasing sequence of subsets of $\mathfrak{D}_0$ satisfying the same property as Δ, we have the following theorem:

Theorem 1. *Let $\mathfrak{D}$ be a domain of holomorphy in (x)-space, and let $\mathfrak{D}_0$ be an open subset of $\mathfrak{D}$ which is finitely sheeted and convex with respect to the whole family $(\mathfrak{F})$ of holomorphic functions in $\mathfrak{D}$. Then, every holomorphic function in $\mathfrak{D}_0$ is expanded to a series of functions of $(\mathfrak{F})$, which converges locally uniformly at every point of $\mathfrak{D}_0$.*

§3

We next discuss *Cousin* I Problem.[40] We begin with a lemma.

Lemma 2. *Let Δ be as in Lemma I′, let L be a real hyperplane passing through a base point of Δ, and let S denote the part of Δ over L. Let $\Delta_0 \Subset \Delta$ be an open subset and let Δ_0' be the part of Δ_0 in one side of L and let Δ_0'' be the one in another side. Then, for a given function $\varphi(P)$ holomorphic in a neighborhood of S in R, one can find a holomorphic function $\varphi_1(P)$ (resp. $\varphi_2(P)$) in Δ_0' (resp. Δ_0'') such that both are also holomorphic at every point of S in Δ_0, and there satisfy identically*

$$\varphi_1(P) - \varphi_2(P) = \varphi(P).$$

Proof. We write $x_1 = \xi + i\eta$ with real and imaginary parts (i for the imaginary unit) and may assume that L is defined by

$$(L) \qquad \xi = 0.$$

For L is reduced to the above form by a linear transform of (x). Recall Δ to be of the following form:

$$(\Delta) \quad P \in R,\ x_j \in X_j,\ f_k(P) \in Y_k \quad (j=1,2,\dots,n; k=1,2,\dots,\nu).$$

Associated with this we consider the cylinder domain (X,Y) in (x,y)-space as done repeatedly in above, and the analytic subset Σ. Let $X_j^0, X_j^1, Y_k^0, Y_k^1$ be domains in the plane such that

$$X_j^0 \Subset X_j^1 \Subset X_j,\ Y_k^0 \Subset Y_k^1 \Subset Y_k \quad (j=1,2,\dots,n; k=1,2,\dots,\nu).$$

Let Δ_0 be the part of Δ, where (X,Y) is replaced by (X^0,Y^0). Then, one may assume Δ_0 in the lemma to be of this form.

Let A be an open subset of X_1 in x_1-plane which contains the part of the line $\xi = 0$ in X_1. Here we take A sufficiently close to this line so that $\varphi(P)$ is holomorphic in the part of Δ over $x_1 \in A$. Let $A_1 \Subset A$ be an open subset which is in the same relation with respect to X_1^1 as A to X_1.

By Lemma I′ there is a holomorphic function $\Phi(x,y)$ in the cylinder domain with $x_1 \in A_1$ and $(x,y) \in (X^1,Y^1)$, which takes the value $\varphi(P)$ at every point $[x, f(P)]$ of Σ in this cylinder domain. Taking a line segment or a finite union of them (closed set) l in the imaginary axis of x_1-plane, contained in A_1 and containing the part of the imaginary axis inside X_1^0, we consider *Cousin's integral*

$$\Psi(x,y) = \frac{1}{2\pi i}\int_l \frac{\Phi(t,x_2,\dots,x_n,y)}{t-x_1}\,dt.$$

Here the left part ($\xi < 0$) of L in Δ_0 is denoted by Δ_0', the right part by Δ_0'', and the orientation of the integration is the positive direction of the imaginary axis. Let (C') be the part $\xi < 0$ of (X^0,Y^0), and let (C'') be that of $\xi > 0$. Then, $\Psi(x,y)$ is holomorphic in (C') and in (C''). We distinguish Ψ as Ψ_1 in (C') and that as Ψ_2 in (C''). Then both of Ψ_1 and Ψ_2 are holomorphic also at every point of $\xi = 0$ inside (X^0,Y^0), and satisfy the following relation:

$$\Psi_1(x,y) - \Psi_2(x,y) = \Phi(x,y).$$

Therefore, we obtain the required functions

$$\varphi_1(P) = \Psi_1[x, f(P)], \quad \varphi_2(P) = \Psi_2[x, f(P)],$$

where (x) is the coordinate of a point P of R. C.Q.F.D.

Let $\mathfrak{D}$ be a domain in (x)-space. Assume that for each point P of $\mathfrak{D}$ there are a polydisk (γ) with center at P in $\mathfrak{D}$ and a meromorphic function $g(P)$ in (γ), and that the whole of them satisfies the following congruence condition: For every pair $(\gamma_1),(\gamma_2)$ of such (γ) with the non-empty intersection (δ), the corresponding $g_1(P)$ and $g_2(P)$ are congruent in (δ); i.e., precisely, $g_1(P) - g_2(P)$ is holomorphic in (δ). In this way, the poles were defined in $\mathfrak{D}$. Then, it is the *Cousin I Problem* to construct a meromorphic function $G(P)$ in $\mathfrak{D}$ with the given poles; in other words, it is congruent to $g(P)$ in every (γ).

Let $\mathfrak{D}$ be a finitely sheeted domain of holomorphy. By the First Theorem of *Cartan–Thullen*, $\mathfrak{D}$ is convex with respect to the family $(\mathfrak{F})$ of all holomorphic functions in $\mathfrak{D}$. Therefore we may take $\mathfrak{D} = \mathfrak{D}_0$ in Lemma 1, and hence there is a Δ in $\mathfrak{D}$ stated in the lemma. Here, however it is convenient to take a *closed analytic polyhedron* Δ with closed bounded domains X_i and Y_j $(i=1,2,\dots,n; j=1,2,\dots,\nu)$. (Naturally, $f_j(P)$ are chosen from $(\mathfrak{F})$.) Thus, $\mathfrak{D}$ is a limit of a sequence of closed analytic polyhedra,

$$\Delta_1, \Delta_2, \dots, \Delta_p, \dots,$$

where Δ_p are such ones as Δ above, and $\Delta_p \Subset E$ with the set E of all interior points of Δ_{p+1}.

[40] Cf. Report I, §3, and the proof of Theorem I in §5 in Report I.

Now, we take a Δ_p and divide it into (A) as stated in §3 of the previous Report:[41] Here, we choose $2n$-dimensional closed cubes for (A) and its base domain (α). We also allow some of (A) to be of incomplete form, and take (A) sufficiently small so that $(A) \Subset (\gamma)$ for every (A) with one of (γ) above. Choosing arbitrarily such (γ), we associate $g(P)$ with (γ), and then $g(P)$ with this (A).

Let $(A)_1, (A)_2$ be a pair of (A) adjoining by a face (a $(2n-1)$-dimensional closed cube). The meromorphic functions $g_1(P)$ and $g_2(P)$ associated with them are congruent in a neighborhood of the common face (a neighborhood in $\mathfrak{D}$, same in below). It follows from Lemma 2 that there is a meromorphic function with the given poles in a neighborhood of the union $(A)_1 \cup (A)_2$. It is the same for a union of (A) such as, e.g.,

$$\left(\alpha_{j,q}^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n)}\right),$$

where α are closed squares, q and $\alpha^{(2)}, \ldots, \alpha^{(n)}$ are given ones, and j is arbitrary. Here (A) may be disconnected. Repeating this procedure, we obtain a meromorphic function $G(P)$ in a neighborhood of Δ_p with the given poles.

Thus, we have

$$G_1(P), G_2(P), \ldots, G_p(P), \ldots.$$

We consider

$$H(P) = G_{p+1}(P) - G_p(P).$$

Then, $H(P)$ is a holomorphic function in a neighborhood of Δ_p. Hence by Theorem 1, H_p is expanded to a series of functions of $(\mathfrak{F})$ which converges uniformly in a neighborhood of Δ_p. By this we immediately see the existence of a meromorphic function $G(P)$ in $\mathfrak{D}$ with the given poles. (The method of the proof is exactly the same as in the case of univalent cylinder domains.) Thus we obtain the following theorem.

Theorem 2. *In a finitely sheeted domain of holomorphy, the Cousin I Problem is always solvable.*

II – The Main Problem

§4

In this chapter we solve the main part of the problem abstracted from the series of those discussed at the beginning by virtue of the First Fundamental Lemma.[42]

[41] (Note by the translator.) The "previous Report" is "Report X"; there in §3, small closed cubes are defined so that their sides are parallel to real and imaginary axes of the complex coordinates of the base space $\mathbf{C}^n$.

[42] Except for the use of this lemma, the content is essentially the same as in Report VI, Chap. 1.

We begin with explaining the problem. Let $\mathfrak{D}$ be a *bounded finitely sheeted domain* in (x)-space. We consider a real hyperplane with non-empty intersection with the base domain of $\mathfrak{D}$. We write x_1 as

$$x_1 = \xi + i\eta.$$

For the sake of simplicity we assume that this hyperplane is given by $\xi = 0$. Let a_1, a_2 be real numbers such that

$$a_2 < 0 < a_1,$$

and the hyperplanes $\xi = a_1,\ \xi = a_2$ have both non-empty intersections with the base domain of $\mathfrak{D}$. Let $\mathfrak{D}_1$ (resp. $\mathfrak{D}_2$) denote the part of $\xi < a_1$ (resp. $\xi > a_2$) in $\mathfrak{D}$, and let $\mathfrak{D}_3$ be the part of $a_2 < \xi < a_1$ in $\mathfrak{D}$. We assume that *every connected component of* $\mathfrak{D}_1$ *and* $\mathfrak{D}_2$ *is a domain of holomorphy.* Then, necessarily so is every component of $\mathfrak{D}_3$.

Let $f_j(P)$ $(j = 1, 2, \ldots, \nu)$ be *holomorphic functions in* $\mathfrak{D}_3$. We consider a subset E of $\mathfrak{D}$ such that $E \supset \mathfrak{D} \setminus \mathfrak{D}_3$ and the following holds: A point P of $\mathfrak{D}_3$ belongs to E if and only if

$$|f_j(P)| < 1 \qquad (j = 1, 2, \ldots, \nu).$$

We assume that E has connected components which extend over the part $\xi < a_2$ and over $\xi > a_1$. Let Δ be such one of them.

We assume the following three conditions for this Δ*:*

1° *Let* δ_1 *be a real number such that* $0 < \delta_1 < \min\{a_1, -a_2\}$. *Let* A *denote the set of point* $P(x)$ *of* Δ *with* $|\xi| < \delta_1$. *Then,*

$$A \Subset \mathfrak{D}.$$

2° *Let* δ_2 *be a positive number and let* ε_0 *be a positive number less than* 1. *For every* p *of* $1, 2, \ldots, \nu$, *any point* P *of* $\mathfrak{D}_3$ *satisfying*

$$|f_p(P)| \geq 1 - \varepsilon_0$$

does not lie over

$$|\xi - a_1| < \delta_2 \quad \textit{or} \quad |\xi - a_2| < \delta_2.$$

3° *The vector-values*

$$[f_1(P), f_2(P), \ldots, f_\nu(P)]$$

are never identical for mutually overlapped two points of A.

By the second Condition, Δ is a domain. Let ρ_0 be a real number such that $1 - \varepsilon_0 < \rho_0 < 1$, and consider a subset Δ_0 of Δ such that $\Delta_0 \supset \Delta \setminus \mathfrak{D}_3$ and for a point of $\mathfrak{D}_3 \cap \Delta$ it belongs to Δ_0 if and only if

$$|f_j(P)| < \rho_0 \qquad (j = 1, 2, \ldots, \nu).$$

By Condition 2°, Δ_0 is an open set. Denote by Δ_0' (resp. Δ_0'') the part of $\xi < 0$ (resp. $\xi > 0$) in Δ_0.

The theme of the present chapter is the following problem.

Let the notation be as above. Let $\varphi(P)$ be a given holomorphic function in A. Then, construct holomorphic functions, $\varphi_1(P)$ in Δ_0' and $\varphi_2(P)$ in Δ_0'', which are holomorphic in the part of Δ_0 over $\xi = 0$, and identically satisfy

$$\varphi_1(P) - \varphi_2(P) = \varphi(P).$$

§5

By making use of the method of Lemma 2 we first solve a part of the problem related to $\mathfrak{D}_3$. Let $y_1, y_2, \dots, y_\nu$ be complex variables, and consider in (x,y)-space the analytic subset

$$(\Sigma) \qquad y_k = f_k(P), \quad P \in \mathfrak{D}_3 \qquad (k = 1,2,\dots,\nu).$$

Let r and r_0 be positive numbers with $r_0 < r$, and let r_0 be taken sufficiently large so that the bounded domain $\mathfrak{D}$ is contained in the polydisk of radius r_0 with center at the origin. Let ρ be a number with $\rho_0 < \rho < 1$, and consider polydisks

$$(C) \quad |x_j| < r, |y_k| < \rho \quad (j = 1,2,\dots,n; k = 1,2,\dots,\nu),$$

and

$$(C_0) \quad |x_j| < r_0, |y_k| < \rho_0 \quad (j = 1,2,\dots,n; k = 1,2,\dots,\nu).$$

Let δ be a positive number with $\delta < \delta_1$, and consider a set

$$(A') \quad P \in A, \quad |\xi| < \delta, \quad |f_k(P)| < \rho \qquad (k = 1,2,\dots,\nu).$$

Since $\varphi(P)$ is holomorphic in A, by Lemma I′ we can construct a holomorphic function $\Phi(x,y)$ in the intersection of (C) and $|\xi| < \delta$ such that $\Phi[x, f(P)] = \varphi(P)$ for $[x, f(P)] \in \Sigma$ with $P \in A'$ and the coordinate x of P. We take a line segment l (connected and closed) in the imaginary axis of x_1-plane, so that it is contained in the disk $|x_1| < r$ and the both ends are out of the disk $|x_1| < r_0$. We then consider the *Cousin integral*

$$\Psi(x,y) = \frac{1}{2\pi i} \int_l \frac{\Phi(t, x_2, \dots, x_n, y_1, \dots, y_\nu)}{t - x_1}\, dt,$$

where the orientation is in the positive direction of the imaginary axis.

Substituting $y_k = f_k(P)$ in $\Psi(x,y)$, we get

$$\psi(P) = \frac{1}{2\pi i} \int_l \frac{\Phi[t, x_2, \dots, x_n, f_1(P), \dots, f_\nu(P)]}{t - x_1}\, dt.$$

The function $\psi(P)$ represents respectively a holomorphic function $\psi_1(P)$ in $\Delta_0' \cap \mathfrak{D}_3$ and $\psi_2(P)$ in $\Delta_0'' \cap \mathfrak{D}_3$. These are also holomorphic at every point of Δ_0 over $\xi = 0$, and satisfy the relation: $\psi_1(P) - \psi_2(P) = \varphi(P)$.

We modify a little the expression of this solution. We draw a circle Γ of radius ρ_0 with center at the origin in the complex plane. It follows from *Cauchy* that for $|\xi| < \delta$, $|x_j| < r$ and $|y_k| < \rho_0$ $(j = 1,2,\dots,n; k = 1,2,\dots,\nu)$

$$\Phi(x,y) = \frac{1}{(2\pi i)^\nu} \int_\Gamma \int_\Gamma \cdots \int_\Gamma \frac{\Phi(x_1, \dots, x_n, u_1, \dots, u_\nu)}{(u_1 - y_1) \cdots (u_\nu - y_\nu)}\, du_1 du_2 \cdots du_\nu,$$

where the integral is taken on Γ with the positive orientation. We write this simply as follows:

$$\Phi(x,y) = \frac{1}{(2\pi i)^\nu} \int_{(\Gamma)} \frac{\Phi(x,u)}{(u_1 - y_1) \cdots (u_\nu - y_\nu)}\, du.$$

We substitute $y_k = f_k(P)$ $(k = 1,2,\dots,\nu)$ in this integral expression of $\Phi(x,y)$, change x_1 with t, and substitute them in the integral expression of $\psi(P)$ above. Then, with $t = u_0$ we obtain

$$(1) \qquad \psi(P) = \int_{(l,\Gamma)} \chi(u,P) \Phi(x',u)\, du,$$

$$\chi(u,P) = \frac{1}{(2\pi i)^{\nu+1} (u_0 - x_1)[u_1 - f_1(P)] \cdots [u_\nu - f_\nu(P)]}.$$

Here we simply write $\Phi(x',u)$ for $\Phi(u_0, x_2, \dots, x_n, u_1, \dots, u_\nu)$, and use the same simplification for the integral symbol as above: It will be clear without further explanation. Then we can use this (1) in $\Delta_0 \cap \mathfrak{D}_3$ for the integral expression of $\psi(P)$ above.

§6

There are univalent domains of holomorphy in (u)-space, which contain the closed cylinder set (l,Γ) with $u_0 \in l$, $u_k \in \Gamma$ $(k = 1,2,\dots,\nu)$, and are arbitrarily close to (l,Γ). Let V be such one of them. We shall take V sufficiently close to (l,Γ), as we will explain at each step in below.

Firstly, we would like to *construct a meromorphic function $\chi_1(u,P)$ in $(V, \mathfrak{D}_1)$ $((u) \in V,\ P(x) \in \mathfrak{D}_1)$, with the same poles as $\chi(u,P)$ of* (1) *in $(V, \mathfrak{D}_3)$ and without other poles.*

This is possible by Theorem 2, because $(V, \mathfrak{D}_1)$ is a finitely sheeted domain of holomorphy, and for the pole distribution the congruent condition is satisfied with V sufficiently close to (l,Γ) by Condition 2° on Δ.

Note that $\chi - \chi_1$ is holomorphic in $(V, \mathfrak{D}_3)$. By the First Theorem of *Cartan–Thullen* $(V, \mathfrak{D}_3)$ is convex with respect to the family of all holomorphic functions in $(V, \mathfrak{D}_1)$. By Theorem 1, $\chi - \chi_1$ is hence expanded to a series of holomorphic functions in $(V, \mathfrak{D}_1)$, convergent locally uniformly at every point of $(V, \mathfrak{D}_3)$. Therefore, taking V closer to (l,Γ), we have the following function $F_1(u,P)$ for a positive number

ε: *$F_1(u,P)$ is holomorphic in $(V,\mathfrak{D}_1)$ and for the analytic polyhedron A given in §4,*

$$|\chi - \chi_1 - F_1| < \varepsilon \quad \text{in } (V,A).$$

Put

$$K_1(u,P) = \chi - \chi_1 - F_1.$$

The function $K_1(u,P)$ is holomorphic in $(V,\mathfrak{D}_3)$, and $|K_1| < \varepsilon$ in (V,A). For $\mathfrak{D}_2$, we construct $K_2(u,P)$, similarly. With these preparations we change the integration (1) as follows:

$$(2) \qquad I_1(P) = \int_{(l,\Gamma)} [\chi(u,P) - K_1(u,P)]\Phi(x',u)\,du,$$

$$I_2(P) = \int_{(l,\Gamma)} [\chi(u,P) - K_2(u,P)]\Phi(x',u)\,du.$$

If $(u) \in (l,\Gamma)$, then $\chi - K_1$ is equal to $\chi_1 + F_1$, so that it is meromorphic in $P(x) \in \mathfrak{D}_1$, and in particular, it is holomorphic in Δ_0'. Therefore, *$I_1(P)$ is holomorphic in* Δ_0'; similarly, *$I_2(P)$ is holomorphic in Δ_0''.*

The analytic functions $I_1(P)$ and $I_2(P)$ are holomorphic at every point of Δ_0 over $\xi = 0$: For $\psi(P)$ in (1) has this property and the both of K_1 and K_2 are holomorphic functions. By the property of $\psi(P)$, the functions $I_1(P)$ and $I_2(P)$ satisfy the *following relation*:

$$(3) \qquad I_1(P) - I_2(P) = \varphi(P) - \int_{(l,\Gamma)} [K_1(u,P) - K_2(u,P)]\Phi(x',u)\,du.$$

We write

$$K(u,P) = K_1(u,P) - K_2(u,P).$$

Observing this identity again, we see that $\varphi(P)$ is a holomorphic function in $P \in A$, K is a holomorphic function in $(u) \in V$ and $P \in \mathfrak{D}_3$, and $\Phi(x,y)$ is a holomorphic function in $(x,y) \in (C)$ with $|\xi| < \delta$. Therefore, the right-hand side is *a holomorphic function in $P(x) \in A$*; hence, *it is the same for the left-hand side as above.* Put

$$\varphi_0(P) = I_1(P) - I_2(P).$$

Let φ_0 and K be given functions, and let φ, Φ be a pair of unknown functions satisfying the relations described next below.[43] We consider a *functional equation*

$$(4) \qquad \varphi(P) = \int_{(l,\Gamma)} K(u,P)\Phi(x',u)\,du + \varphi_0(P).$$

Here, $\Phi(x',u)$ stands for $\Phi(u_0,x_2,\dots,x_n,u_1,\dots,u_\nu)$, $\varphi_0(P)$ is a holomorphic function in A, and $K(u,P)$ is a holomorphic function in $(V,\mathfrak{D}_3)$. In (V,A), $|K(u,P)| < 2\varepsilon$. For the unknown functions $\varphi(P)$ and $\Phi(x,y)$, the following condition is imposed besides (4): $\varphi(P)$ is a holomorphic function in $P \in A$, $\Phi(x,y)$ is a holomorphic function in $(x,y) \in (C)$ with $|\xi| < \delta$, and for every point $[x,f(P)]$ of Σ with $P \in A'$, $\Phi(x,y) = \varphi(P)$.

Since these conditions are imposed, this functional equation is not so different from the definite integral equation.

We are going to show that this equation has necessarily a solution for a sufficiently small ε. Before it we confirm that it suffices for our end. Suppose that there exist functions $\varphi(P)$ and $\Phi(x,y)$ as above. Substitute $\Phi(x',u)$ to (2). The function $I_1(P)$ thus obtained is clearly holomorphic in Δ_0'. Similarly, $I_2(P)$ is holomorphic in Δ_0''. It is clear that these analytic functions are also holomorphic at every point of Δ_0 over $\xi = 0$. One easily sees relation (3) among them. (The argument above is just a repetition of a deduction once done with clarifying the conditions.) Thus, these $I_1(P)$ and $I_2(P)$ are the solutions of the problem described in §4. As seen above, it suffices to solve equation (4); here one may take ε as small as necessary.

Now, we solve equation (4). Recall that the analytic polyhedron A is of the following form:

$$(A) \qquad P \in \Delta, \quad |\xi| < \delta_1, \quad |f_k(P)| < 1 \qquad (k = 1,2,\dots,\nu).$$

Moreover, the analytic polyhedron A' is obtained by replacing $(\delta_1, 1)$ of A by (δ, ρ) with $0 < \delta < \delta_1$ and $\rho_0 < \rho < 1$. Taking (δ', ρ') with $\rho < \rho' < 1$ and $\delta < \delta' < \delta_1$, we define an analytic polyhedron A'', replacing (δ, ρ) by this (δ', ρ') in the definition of A'. We have the following relation among them:

$$A' \Subset A'' \Subset A.$$

The function $\varphi_0(P)$ is holomorphic in A, and hence bounded on A''. Suppose that

$$|\varphi_0(P)| < M_0 \quad \text{on } A''.$$

We denote by (C') the cylinder domain given by $(x,y) \in (C)$ and $|\xi| < \delta$. By *Lemma I′* we can take a holomorphic function $\Phi_0(x,y)$ in (C') so that it has values $\varphi_0(P)$ at points $[x,f(P)]$ of Σ with $P \in A'$, and

$$|\Phi_0(x,y)| < NM_0 \quad \text{on } (C'),$$

where N is a positive constant independent of $\varphi_0(P)$ (also independent of M_0, and of $\varphi_0(P)$ being holomorphic in A). Applying the operator $K(\Phi_0)$ for $\Phi_0(x,y)$ defined by

$$\varphi_1(P) = K(\Phi_0) = \int_{(l,\Gamma)} K(u,P)\Phi_0(x',u)\,du,$$

we construct a function $\varphi_1(P)$. For $(u) \in (l,\Gamma)$, $K(u,P)$ is holomorphic in $P(x) \in \mathfrak{D}_3$, and $\Phi_0(x',u)$ is holomorphic

[43] (Note by the translator.) Here is a point of the arguments of the proof, but one must be careful of the notational confusion with φ, φ_0 and Φ discussed already.

in $|x_j| < r$ $(j = 2, 3, \ldots, n)$, and so is in (C). Hence, $\varphi_1(P)$ *is a holomorphic function in* $\mathfrak{D}_3$.

We next estimate $\varphi_1(P)$. For $(u) \in (l, \Gamma)$ and $P \in A$, $|K(u, P)| < 2\varepsilon$, and $|\Phi_0(x, y)| < NM_0$ in (C'). Therefore, we have in A,

$$|\varphi_1(P)| < 2\varepsilon NN_1M_0, \quad N_1 = 2r(2\pi\rho_0)^{\nu}.$$

Therefore in first we take ε so that

$$2\varepsilon NN_1 = \lambda < 1.$$

Thus, $\varphi_1(P)$ *is a bounded holomorphic function in* A*, and necessarily so is in* A''. As we choose a function $\Phi_0(x, y)$ for $\varphi_0(P)$, we choose a function $\Phi_1(x, y)$ for $\varphi_1(P)$, and by setting $\varphi_2(P) = K(\Phi_1)$, we construct $\varphi_2(P)$. Inductively, we obtain $\varphi_p(P)$ and $\Phi_p(x, y)$ $(p = 0, 1, 2, \ldots)$. Then we consider the following function series:

$$\varphi_0(P) + \varphi_1(P) + \cdots + \varphi_p(P) + \cdots, \tag{5}$$

$$\Phi_0(x, y) + \Phi_1(x, y) + \cdots + \Phi_p(x, y) + \cdots. \tag{6}$$

It follows that $\varphi_p(P)$ is holomorphic in $\mathfrak{D}_3$, and $\Phi_p(x, y)$ is holomorphic in (C'). In A,

$$|\varphi_p(P)| < \lambda^p M_0 \quad (p > 0),$$

and in (C'),

$$|\Phi_p(x, y)| < \lambda^p NM_0.$$

Therefore, (5) (resp. (6)) converges uniformly in A (resp. (C')). We denote the limits by $\varphi(P)$ and $\Phi(x, y)$, respectively. We see that $\varphi(P)$ (resp. $\Phi(x, y)$) is holomorphic in A (resp. (C')). Since $\Phi_p(x, y)$ $(p = 0, 1, \ldots)$ take values $\varphi_p(P)$ at points $[x, f(P)]$ of Σ with $P \in A'$, $\Phi(x, y)$ there takes values $\varphi(P)$. Therefore, it suffices to show that $\varphi(P)$ and $\Phi(x', u)$ satisfy functional equation (4) in $P \in A$. Now for $P \in A$ we have

$$\varphi_0 = \varphi_0, \ \varphi_1 = K(\Phi_0), \ \varphi_2 = K(\Phi_1), \ldots, \varphi_{p+1} = K(\Phi_p), \ldots,$$

so that

$$\varphi = K(\Phi) + \varphi_0.$$

Thus, the problem stated at the end of §4 is always solvable.

III – Pseudoconvex Domains and Domains of Holomorphy, Theorems on Domains of Holomorphy

§7

Apart from the theme we prepare some lemmata for a moment (§§7–9).

We begin with reformulating the Second Fundamental Lemma.

Lemma II. *Let* $\mathfrak{D}$ *be a finite unramified pseudoconvex domain over* (x)*-space. Then there necessarily exists a real-valued continuous function* $\varphi_0(P)$*, satisfying the following two conditions:*

1° *For every real number* α*,* $\mathfrak{D}_\alpha \Subset \mathfrak{D}$*, where* $\mathfrak{D}_\alpha$ *denotes the set of all points* $P \in \mathfrak{D}$ *with* $\varphi_0(P) < \alpha$*.*

2° *In a neighborhood* U *of every point* P_0 *of* $\mathfrak{D}$*, there is a hypersurface* $\Sigma \subset U$*, passing through* P_0 *such that* $\varphi_0(P) > \varphi_0(P_0)$ *for* $P \in \Sigma \setminus P_0$*.*

Proof. As a consequence of the former Report we know that there is a pseudoconvex function in $\mathfrak{D}$ satisfying Condition 1° and Condition 2° outside of an exceptional discrete subset without accumulation point in $\mathfrak{D}$. Let $\varphi(P)$ be a such function, and let E_0 denote the exceptional discrete subset, provided that it exists. If there is a point of E_0 on $\varphi(P) = \lambda$ for $\lambda \in \mathbf{R}$, we then call λ an *exceptional value* of $\varphi(P)$. For an arbitrary real number α, we denote by $\mathfrak{D}_\alpha$ the set of all points $P \in \mathfrak{D}$ with $\varphi(P) < \alpha$. Since $\mathfrak{D}_\alpha \Subset \mathfrak{D}$ by Condition 1°, $\mathfrak{D}_\alpha$ is bounded and finitely sheeted. This remains valid for a little bit larger α, and so there are only finitely many points of E_0 in $\mathfrak{D}_\alpha$. Since $\lim_{\alpha\to\infty} \mathfrak{D}_\alpha = \mathfrak{D}$, the set of the exceptional values is countable. Let the exceptional values be

$$\lambda_1, \lambda_2, \ldots, \lambda_p, \ldots, \qquad \lambda_p < \lambda_{p+1}.$$

Let α_0 be a non-exceptional value and set $\mathfrak{D}_{\alpha_0} = \Delta$. In Δ we consider

$$\psi(P) = -\log d(P).$$

Here $d(P)$ denotes the Euclidean boundary distance function with respect to Δ, and the logarithm symbol stands for the real branch. Since Δ is bounded, $\psi(P)$ is a continuous function. For any real number α, we denote by Δ_α the set of all points P of Δ with $\psi(P) < \alpha$. Then, $\Delta_\alpha \Subset \Delta$. Thus, $\psi(P)$ satisfies Condition 1° in Δ. We next check Condition 2°. Let P_0 be an arbitrary point of Δ, and set $\psi(P_0) = \beta$. We draw a $2n$-dimensional ball S of radius $e^{-\beta}$ with center P_0 in $\mathfrak{D}$. Then, $S \subset \Delta$ and there is a point M on the boundary of S, satisfying $\varphi(M) = \alpha_0$. Since $\varphi(P)$ satisfies Condition 2° in a neighborhood of $\varphi(P) = \alpha_0$, there is a complex hypersurface σ in a neighborhood of M, passing through M, such that $\varphi(P) > \alpha_0$ for $P \in \sigma \setminus \{M\}$. By a parallel translation

$$x'_i = x_i + a_i \qquad (i = 1, 2, \ldots, n), \tag{T}$$

we move M to P_0, and σ to σ'. Then, σ' is defined in a neighborhood of P_0. Let P' be a point of σ' different to P_0. Then the corresponding point P of σ lies in $\varphi(P) > \alpha_0$, and the (Euclidean) distance between P and P' is $e^{-\beta}$, so that if P' belongs to Δ, P' lies in the part of

$\psi(P) > \alpha$.[44] Therefore, $\psi(P)$ *is a continuous function in* Δ, *satisfying Conditions* 1° *and* 2°.

We take a sequence of real numbers, $\mu_1, \mu_2, \ldots, \mu_p, \ldots$ such that

$$\mu_1 < \lambda_1, \quad \lambda_p < \mu_{p+1} < \lambda_{p+1}.$$

Taking α_0 with

$$\lambda_1 < \alpha_0 < \mu_2,$$

we consider $\psi(P)$ above. Choosing α_0 sufficiently close to λ_1, we may take β for this $\psi(P)$, satisfying

$$\mathfrak{D}_{\mu_1} \Subset \Delta_\beta \Subset \mathfrak{D}_{\lambda_1}.$$

Modifying $\varphi(P)$ by making use of $\psi(P)$ thus obtained (similarly to the last part of the previous Report), we construct $\varphi_1(P)$: We explain it in below.

Let β_1, β_2 be real numbers with the same property as β above such that

$$\beta_1 < \beta_2.$$

Let γ_1, γ_2 be real numbers with

$$\lambda_1 < \gamma_1 < \gamma_2 < \alpha_0.$$

We divide $\mathfrak{D}$ into five parts $\mathfrak{D}_j$ $(j = 1, 2, \ldots, 5)$ defined by

$$\mathfrak{D}_1 = \Delta_{\beta_1}, \quad \Delta_{\beta_1} \cup \mathfrak{D}_2 = \Delta_{\beta_2}, \quad \Delta_{\beta_2} \cup \mathfrak{D}_3 = \mathfrak{D}_{\gamma_1},$$
$$\mathfrak{D}_{\gamma_1} \cup \mathfrak{D}_4 = \mathfrak{D}_{\gamma_2}, \quad \mathfrak{D}_{\gamma_2} \cup \mathfrak{D}_5 = \mathfrak{D}.$$

By taking a suitable B and a sufficiently large positive A, we have

$$\Psi(P) = A[\psi(P) - B]$$

satisfying

$$\begin{aligned} \varphi(P) &> \Psi(P) && \text{in } \mathfrak{D}_1, \\ \varphi(P) &< \Psi(P) && \text{in } \mathfrak{D}_3 \cup \mathfrak{D}_4. \end{aligned}$$

Also by taking a suitable real number B' and a sufficiently large positive number A', we have

$$\Phi(P) = A'[\psi(P) - B']$$

satisfying

$$\begin{aligned} \Psi(P) &> \Phi(P) && \text{in } \mathfrak{D}_3, \\ \Psi(P) &< \Phi(P) && \text{in } \mathfrak{D}_5', \\ \varphi(P) &< \Phi(P) && \text{in } \mathfrak{D}_5, \end{aligned}$$

where $\mathfrak{D}_5'$ is the part of $\mathfrak{D}_5$ (a neighborhood) containing the point set, $\varphi(P) = \gamma_2$. We define $\varphi_1(P)$ as follows:

$$\begin{aligned} \varphi_1(P) &= \varphi(P) && \text{in } \mathfrak{D}_1, \\ \varphi_1(P) &= \max[\varphi(P), \Psi(P)] && \text{in } \mathfrak{D}_2, \\ \varphi_1(P) &= \Psi(P) && \text{in } \mathfrak{D}_3, \\ \varphi_1(P) &= \max[\Psi(P), \Phi(P)] && \text{in } \mathfrak{D}_4, \\ \varphi_1(P) &= \Phi(P) && \text{in } \mathfrak{D}_5. \end{aligned}$$

We examine $\varphi_1(P)$ thus defined. It follows that $\varphi_1(P)$ is a real one-valued function in $\mathfrak{D}$, which is clearly continuous. Since $\psi(P)$ satisfies Condition 2°, and $\varphi(P)$ satisfies Condition 2° outside a set of exceptional points without accumulation point in $\mathfrak{D}$, $\varphi_1(P)$ satisfies the same condition as $\varphi(P)$. We check up the exceptional value of $\varphi_1(P)$. Since $\varphi_1 = \Psi$ in $\mathfrak{D}_3$, and $\varphi_1 = \Phi$ in $\mathfrak{D}_5$, we have for the exceptional values of $\varphi_1(P)$

$$\lambda_2', \lambda_3', \ldots, \lambda_p', \ldots,$$

where the point set of $\varphi_1(P) = \lambda_p'$ is the same as the point set of $\varphi(P) = \lambda_p$. Comparing $\varphi_1(P)$ with the original $\varphi(P)$, we easily see that $\varphi_1(P) = \varphi(P)$ in $\mathfrak{D}_{\mu_1}$, and $\varphi_1(P) \geq \varphi(P)$ in $\mathfrak{D}$. Since $\varphi_1 \geq \varphi$, φ_1 satisfies Condition 1°. This $\varphi_1(P)$ is a function satisfying almost the same property as $\varphi(P)$. Although they differ only in the property of pseudoconvexity, the above operation does not involve this property of $\varphi(P)$. Therefore, in the same way as to produce $\varphi_1(P)$ from $\varphi(P)$, we may construct $\varphi_2(P)$ from $\varphi_1(P)$. We repeat this operation as far as the exceptional values remain, and thus obtain

$$\varphi_1(P), \varphi_2(P), \ldots, \varphi_p(P), \ldots.$$

The part of properties of $\varphi_p(P)$ $(p > 1)$ which varies with p is as follows: The exceptional values of $\varphi_p(P)$ are

$$\lambda_{p+1}^{(p)}, \lambda_{p+2}^{(p)}, \ldots, \lambda_{p+q}^{(p)}, \ldots,$$

where $\varphi_p(P) = \lambda_{p+q}^{(p)}$ and $\varphi(P) = \lambda_{p+q}$ are the same point set, and in $\mathfrak{D}_{\mu_p}$, $\varphi_p(P) = \varphi_{p-1}(P)$, and in $\mathfrak{D}$, $\varphi_p(P) \geq \varphi_{p-1}(P)$ (note that in $\mathfrak{D}_5$, $\varphi_1 = \Phi$). We can thus choose such $\varphi_p(P)$. Let $\varphi_0(P)$ be the limit function of them, or the last function in case the sequence is finite. Then $\varphi_0(P)$ is clearly the required function. C.Q.F.D.

The function $\varphi_0(P)$ thus obtained is in fact a pseudoconvex function.[45]

§8

At the beginning of the second Report[46] we explained the outer-convex "Hülle" with respect to poly-

[44] (Note by the translator.) Here P is used in a different sense from the one just before in the same sentence, and α is a typo of β. They should be read as "$\psi > \beta$"

[45] For this, the pseudoconvexity of $\mathfrak{D}_{\alpha_0} = \Delta$ suffices (Theorem 3 of the 9th Report). Cf. §9.

[46] (Note by the translator.) This is the published second paper of the series in J. Sci. Hiroshima Univ. Ser. A 7 (1937), 115–130.

nomials. We generalize it a bit more to supplement the fundamental lemma of the previous section, but here we consider the (inner) convexity for convenience.

Lemma 3. *Let $\mathfrak{D}$ be a finitely sheeted domain of holomorphy over (x)-space. Let $E_0 \Subset \mathfrak{D}$ be an open subset. Then, we have:*

1° *There exists a smallest open subset H among the open subsets of $\mathfrak{D}$, containing E_0, which are convex with respect to the family of all holomorphic functions in $\mathfrak{D}$, and so $H \Subset \mathfrak{D}$.*

2° *There is no locally defined hypersurface σ satisfying the following properties: σ passes through a boundary point of H, but not through any point of H, E_0 or the boundary of E_0, and the boundary points of σ do not lie in H nor on its boundary, and σ is defined in a form as follows:*

$$\varphi(P) = 0, \quad P \in V,$$

where V is a domain with $V \Subset \mathfrak{D}$, and $\varphi(P)$ is a holomorphic function in a neighborhood of V over $\mathfrak{D}$.[47]

Proof. 1°. We first show the existence of the Hülle H, for which we make some preparations.

Since $\mathfrak{D}$ is *finitely sheeted*, a subset $\mathfrak{D}'$ of $\mathfrak{D}$ is bounded with respect to $\mathfrak{D}$ if and only if $\mathfrak{D}' \Subset \mathfrak{D}$. Let $(\mathfrak{F})$ be the set of all holomorphic functions in $\mathfrak{D}$. Since $\mathfrak{D}$ is a domain of holomorphy, the First Theorem of *Cartan–Thullen* implies that $\mathfrak{D}$ is convex with respect to $(\mathfrak{F})$. Therefore, regarding $\mathfrak{D} = \mathfrak{D}_0$ in Lemma 1, we can construct an analytic polyhedron Δ of this lemma, which is of the form:

$$(\Delta) \quad P \in R,\ |x_i| < r,\ |f_j(P)| < 1 \quad (i = 1, 2, \dots, n; j = 1, 2, \dots, \nu).$$

Here, $f_j(P) \in (\mathfrak{F})$ and R is an open subset of $\mathfrak{D}$ with $R \Supset \Delta$. Further, note that for any given subset $E \Subset \mathfrak{D}$, one may choose $\Delta \Supset E$.

Let ρ be an arbitrary positive number, and let $d(P)$ denote the Euclidean boundary distance of $\mathfrak{D}$. Let $\mathfrak{D}_\rho$ be the set of all points $P \in \mathfrak{D}$ with $d(P) > \rho$. (Here, ρ is chosen so that $\mathfrak{D}_\rho$ is not empty.) If $\mathfrak{D}$ coincides with the finite (x)-space, then $\mathfrak{D}_\rho = \mathfrak{D}$. By a parallel translation

$$(T) \qquad x_i' = x_i + a_i, \quad \sum |a_i|^2 \le \rho^2 \quad (i = 1, 2, \dots, n),$$

we move a point P of $\mathfrak{D}_\rho$ to P' of $\mathfrak{D}$. If P is given, P' is uniquely determined. For a function $f(P)$ of $(\mathfrak{F})$, we set

$$F(P) = f(P').$$

Then, $F(P)$ is a holomorphic function in $\mathfrak{D}_\rho$. Let (T) be any of the parallel translation within the restriction mentioned above, and let $(\mathfrak{F}_\rho)$ be the set of all functions $F(P)$ induced from functions $f(P)$ of $(\mathfrak{F})$.

Let $A \Subset \mathfrak{D}$ be an open subset. Assume that A is convex with respect to $(\mathfrak{F})$.

> *Let $A_0 \Subset A$ be an arbitrary open subset. For a boundary point M of A, there is a point P_0 arbitrarily close to M such that there is at least one function $f(P)$ of $(\mathfrak{F})$ with $|f(P_0)| > \max |f(A_0)|$.*

We call this *Property* (α) for a moment. Conversely, we prove that if A carries Property (α), *A is convex with respect to* $(\mathfrak{F})$. Since $A \Subset \mathfrak{D}$, an analytic polyhedron Δ above mentioned is taken, so that $A \Subset \Delta$. Let ρ be a sufficiently small positive number such that $\Delta \subset \mathfrak{D}_\rho$. Since A satisfies Property (α), it is clear that A is convex with respect to $(\mathfrak{F}_\rho)$. Now, since every function of $(\mathfrak{F}_\rho)$ is holomorphic in Δ, it follows from Theorem 1 that it can be expanded to a series of functions of $(\mathfrak{F})$, converging locally uniformly in Δ. Therefore, it is clear that A is convex with respect to $(\mathfrak{F})$.

Now, let A be an open subset of $\mathfrak{D}$, containing E_0 and convex with respect to $(\mathfrak{F})$. Let H be the subset of $\mathfrak{D}$ consisting of all interior points of the intersection of all such A's.

Since E_0 is open, $E_0 \subset H$. For Δ above, we may take $E = E_0$, and hence $H \Subset \mathfrak{D}$. It is clear that H carries Property (α). Therefore, H is convex with respect to $(\mathfrak{F})$. Thus, H is the smallest open subset of $\mathfrak{D}$ which contains E_0 and is convex with respect to $(\mathfrak{F})$, and $H \Subset \mathfrak{D}$.

2°. We assume the existence of a hypersurface σ with the properties stated in the lemma. It suffices to deduce a contradiction. Let $\varphi(P)$ be holomorphic in V' such that $V \Subset V' \subset \mathfrak{D}$. Let $d(P)$ denote the Euclidean boundary distance with respect to V'. We choose a positive number ρ such that $\min d(V) > \rho$ (the left-hand side of the inequality stands for the infimum of $d(P)$ in V). Through the parallel translation

$$x_i' = x_i + z_i, \quad \sum |z_i|^2 \le \rho^2 \qquad (i = 1, 2, \dots, n),$$

we move a point P of V to a point P' of V'. Regarding (z) as complex parameters, we set

$$\psi(P, z) = \varphi(P'),$$

and consider a family of hypersurface pieces,

$$(\mathfrak{G}) \ : \ \psi(P, z) = 0, P \in V.$$

We take ρ small enough, so that the boundary of any hypersurface piece of $(\mathfrak{G})$ never intersects H.

Let H_0 be the set of all points P of H such that P does not belong to any hypersurface piece of $(\mathfrak{G})$. Let A_0 be an open subset with $A_0 \Subset H_0$. As seen above, there is a minimal open subset A of $\mathfrak{D}$, containing A_0,

[47] Part 1° above immediately follows from the existence theorem of K-Konvexe Hülle due to H. Cartan–P.Thullen. Here, the original proof is based on a fundamental theorem of simultaneous analytic continuation. Cf. the paper of Cartan–Thullen mentioned above. (See also the footnote of Theorem I.)

which is convex with respect to $(\mathfrak{F})$. Since H is convex with respect to $(\mathfrak{F})$, similarly to the case of H above, we have $A \Subset H$ by Lemma 1. Now, we show that $A \subset H_0$.

We describe a $2n$-dimensional ball S with radius ρ and center at the origin in (z)-space. The open subset (H,S) $((x) \in H, (z) \in S)$ in (x,z)-space is convex with respect to the set of all holomorphic functions in the domain $(x) \in \mathfrak{D}$. Therefore by Theorem 2 there is a meromorphic function $G(P,Z)$ in (H,S) such that it is congruent to

$$1/\psi(P,z)$$

in the intersection of (H,S) and (V,S), and it has no pole elsewhere. (Theorem 2 is stated for finitely sheeted domains of holomorphy, but in fact, it needs only the properties which are endowed with $\mathfrak{D}_0$ in Lemma 1.)

Suppose that A is not contained in H_0. Then, A, which is an open set, contains a point outside H_0. We may take a point (z^0) in S such that a point P_0 of A lies on $\psi(P,z^0)=0$. With a complex variable t, we consider a function

$$G(P,tz^0).$$ [48]

Then this is meromorphic when P is in H and t is in a neighborhood of the line segment $(0,1)$, has poles at $P=P_0, t=1$, and $G(P,0)$ has no pole in a neighborhood of A (over $\mathfrak{D}$). As t moves over the line segment $(0,1)$ from 1 to 0, we denote by t_0 the last t such that $G(P,tz^0)$ carries a pole in A or its boundary. Then, $G(P,t_0z^0)$ has to carry a pole on the boundary of A and to be holomorphic in A. et M be one of such poles. Let P_1 be a point of A, sufficiently close to M. Since $A_0 \Subset H_0$ and M is not a point of indeterminacy locus, we have

$$|G(P_1,t_0z^0)| > \max|G(A_0,t_0z^0)|.$$

By Theorem 1, $G(P,t_0z^0)$ is expanded to a series of functions of $(\mathfrak{F})$, locally uniformly convergent in A: This clearly contradicts the minimality of A. Thus, "$A \subset H_0$" holds.

Since A_0 is an arbitrary open subset with $A_0 \Subset H_0$, the above consequence implies that the open set H_0 satisfies Property (α). Therefore, H_0 is convex with respect to $(\mathfrak{F})$; this conclusion holds no matter how ρ is small. Now, for sufficiently small ρ, $E_0 \subset H_0$: This again contradicts the minimality of H. C.Q.F.D.

§9

The following two lemmata are easily deduced from Lemma 3.

Lemma 4. *Let Δ be a univalent domain of (x)-space which is convex with respect to polynomials, and let $\varphi(x)$ be a real-valued continuous function in a neighborhood of Δ, satisfying Condition 2° stated in Lemma II. If $\Delta_\alpha = \{x \in \Delta : \varphi(x) < \alpha\}$ for an arbitrarily given real number α, then Δ_α is convex with respect to polynomials, provided that it exists.*

Proof. It follows from Lemma 3 that there is a univalent minimal open subset H containing Δ_α, which is convex with respect to polynomials. Clearly, $H \subset \Delta$. Therefore, $\varphi(x)$ is defined in a neighborhood of H. Let $\bar{H}$ be the closure of H, and let β be the maximum value of $\varphi(x)$ on $\bar{H}$. There are points of $\bar{H}$ with $\varphi(x)=\beta$. Let M be one of them. Since $\varphi(x)$ satisfies Condition 2°, M lies on the boundary of H. Furthermore, by the same property, there is a hypersurface in a neighborhood of M, passing through M and no other point of $\bar{H}$. By Lemma 3, M must be a boundary point of Δ_α. It follows that $\beta=\alpha$, and so $H=\Delta_\alpha$. Therefore, Δ_α is convex with respect to polynomials. C.Q.F.D.

Lemma 5. *Let $\varphi(P)$ be a real-valued continuous function in a domain $\mathfrak{D}$ of (x)-space, satisfying Condition 2° in Lemma II. Let Δ be a domain of holomorphy such that $\Delta \Subset \mathfrak{D}$. Put $\mathfrak{D}_\alpha = \{P \in \mathfrak{D} : \varphi(P) < \alpha\}$ for a real number α. If $\mathfrak{D}_\alpha \Subset \Delta$, then $\mathfrak{D}_\alpha$ is convex with respect to all holomorphic functions in Δ.*

Since $\Delta \Subset \mathfrak{D}$, Δ is finitely sheeted. Thus, Δ is a finitely sheeted domain of holomorphy, and $\mathfrak{D}_\alpha \Subset \Delta$. Hence, Lemma 3 can be applied for Δ_α, and the rest is exactly the same as above.

We next state the theorems of *H. Cartan–P. Thullen* and *H. Behnke–K. Stein*:

The Second Theorem of H. Cartan–P. Thullen. *Let $\mathfrak{D}$ be a domain of (x)-space, and let $(\mathfrak{F})$ be the family of all holomorphic functions in $\mathfrak{D}$. If the following two conditions are satisfied, then $\mathfrak{D}$ is a domain of holomorphy.*

- 1° *For an arbitrary set $\mathfrak{D}_0$ with $\mathfrak{D}_0 \Subset \mathfrak{D}$, there is an open set $\mathfrak{D}'$ with $\mathfrak{D}_0 \Subset \mathfrak{D}' \Subset \mathfrak{D}$ such that for every boundary point M of $\mathfrak{D}'$ there is a function $f(P)$ of $(\mathfrak{F})$, satisfying $|f(M)| > \max|f(\mathfrak{D}_0)|$.*
- 2° *For distinct two points P_1, P_2 of $\mathfrak{D}$, there is a function $f(P)$ of $(\mathfrak{F})$ with $f(P_1) \neq f(P_2)$.*[49]

Lemma of H. Behnke–K. Stein. *Let $\mathfrak{D}$ be a domain of (x)-space, and let*

$$\mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_p, \ldots$$

be a sequence of open subsets of $\mathfrak{D}$ such that $\mathfrak{D}_p \Subset \mathfrak{D}_{p+1}$ and the limit is $\mathfrak{D}$. We assume:

[48] (Note by the translator.) In the manuscripts of Oka Library [26], References of Part I of the present article, this is misprinted as $G(P,t,z^0)$: It is confirmed to see the 11th document, 1943, Catalogue of Dr. Kiyoshi Oka's own handwriting manuscripts in the web-site of [26].

[49] The original authors stated this Second Theorem (also, the First Theorem) in terms of K-convexity, but we stated it in the form above for convenience: The proof is fully similar and direct.

1° *Every $\mathfrak{D}_p$ is convex with respect to the family $(\mathfrak{F}_{p+1})$ of all holomorphic functions in $\mathfrak{D}_{p+1}$;*
2° *For any two distinct points P_1, P_2 of $\mathfrak{D}_p$, there is a function $f(P)$ in $(\mathfrak{F}_{p+1})$ with $f(P_1) \neq f(P_2)$.*

Then, $\mathfrak{D}_p$ has the same properties as 1° and 2° above with respect to the family $(\mathfrak{F})$ of all holomorphic functions in $\mathfrak{D}$.[50]

Proof. (Since $\mathfrak{D}_{p+1}$ is a domain of holomorphy by the Second Theorem of *Cartan–Thullen*), it follows from Theorem 1 that every holomorphic function $\varphi(P)$ in $\mathfrak{D}_p$ is expanded to a series of functions of $(\mathfrak{F}_{p+1})$, locally uniformly convergent in $\mathfrak{D}_p$. This holds for $p+1, p+2, \ldots$, as well, and so $\varphi(P)$ may be similarly expanded to a series of functions of $(\mathfrak{F})$. Therefore, $\mathfrak{D}_p$ clearly has properties 1° and 2° with respect to $(\mathfrak{F})$.
C.Q.F.D.

Theorem of H. Behnke–K. Stein. *Let $\mathfrak{D}$ be a domain of (x)-space. Assume that for an arbitrary subset $\mathfrak{D}_0$ with $\mathfrak{D}_0 \Subset \mathfrak{D}$, there is a domain of holomorphy $\mathfrak{D}'$ with $\mathfrak{D}_0 \subset \mathfrak{D}' \Subset \mathfrak{D}$. Then, $\mathfrak{D}$ is a domain of holomorphy.*[51]

Proof. Since $\mathfrak{D}'$ is a domain of holomorphy, it is pseudoconvex by *F. Hartogs*. Therefore it is inferred from Corollary 2 of Theorem 2 in the IX-th Report[52] that $\mathfrak{D}$ is pseudoconvex. Thus, there is a function $\varphi_0(P)$ given in Lemma II for $\mathfrak{D}$. By Lemma 5, $\mathfrak{D}_\alpha$ $(\varphi_0(P) < \alpha, P \in \mathfrak{D})$ is convex with respect to all of holomorphic functions in a domain of holomorphy $\mathfrak{D}'$ with $\mathfrak{D}_\alpha \Subset \mathfrak{D}'$. Therefore, if α, β are arbitrary real numbers with $\alpha < \beta$, $\mathfrak{D}_\alpha$ satisfies the two conditions stated in Lemma of *Behnke–Stein* with respect to all of holomorphic functions in $\mathfrak{D}_\beta$, and hence $\mathfrak{D}_\alpha$ satisfies the same with respect to all of holomorphic functions in $\mathfrak{D}$. Therefore by the Second Theorem of *Cartan–Thullen*, $\mathfrak{D}$ is a domain of holomorphy. C.Q.F.D.

We here generalize a bit more some parts of Lemmata 4 and 5.

Lemma 6. *Let $\mathfrak{D}$ be a finitely sheeted domain of holomorphy over (x)-space, and let $\varphi(P)$ be a real-valued continuous function in $\mathfrak{D}$, satisfying Condition 2° in Lemma II. If $\mathfrak{D}_\alpha = \{P \in \mathfrak{D} : \varphi(P) < \alpha\}$ for an arbitrarily given real number α, then, every connected component of $\mathfrak{D}_\alpha$ is a domain of holomorphy (provided that $\mathfrak{D}_\alpha$ is not empty).*[53]

Proof. Suppose that $\mathfrak{D}_\alpha$ exists. Since $\mathfrak{D}$ is a domain of holomorphy, thanks to *F. Hartogs*, $\mathfrak{D}$ is pseudoconvex, so that there is a real-valued function $\psi(P)$ in $\mathfrak{D}$, stated in Lemma II. Let β be a real number with $\beta < \alpha$, and let γ be an arbitrary number. We consider an open set defined by

$$(\mathfrak{D}_{\beta\gamma}) \qquad P \in \mathfrak{D}, \quad \varphi(P) < \beta, \quad \psi(P) < \gamma.$$

Since $\mathfrak{D}$ is a finitely sheeted domain of holomorphy and $\mathfrak{D}_{\beta\gamma} \Subset \mathfrak{D}$, we can apply Lemma 3 with $E_0 = \mathfrak{D}_{\beta\gamma}$. Hereafter, fully in the same way as the case of Lemma 4, we easily see that $\mathfrak{D}_{\beta\gamma}$ is convex with respect to all of holomorphic functions in $\mathfrak{D}$. Therefore, $\mathfrak{D}_{\beta\gamma} \subset \mathfrak{D}$, so that by the Second Theorem of *Cartan–Thullen*, each connected component of $\mathfrak{D}_{\beta\gamma}$ is a domain of holomorphy. Note that $\mathfrak{D}_{\beta\gamma} \Subset \mathfrak{D}_\alpha$, and $\mathfrak{D}_{\beta\gamma}$ can be chosen arbitrarily close to $\mathfrak{D}_\alpha$. It follows from Theorem of *Behnke–Stein* that each connected component of $\mathfrak{D}_\alpha$ is a domain of holomorphy.[54] C.Q.F.D.

§10

We return to our theme. In first, we claim that a pseudoconvex domain is a domain of holomorphy.

We consider a *finitely sheeted* domain $\mathfrak{D}$ in (x)-space. We write

$$x_1 = \xi + i\eta$$

with real and imaginary parts. Let a_1 and a_2 be real numbers such that

$$a_2 < 0 < a_1,$$

and denote by $\mathfrak{D}_1$ the part of $\mathfrak{D}$ with $\xi < a_1$, by $\mathfrak{D}_2$ the part of $\mathfrak{D}$ with $\xi > a_2$, and set $\mathfrak{D}_3 = \mathfrak{D}_1 \cap \mathfrak{D}_2$. Assuming that the parts of $\mathfrak{D}$ with $\xi < a_2$ and $\xi > a_1$ are not empty, we take points Q_1, Q_2 therein respectively. Assume that *every connected component of $\mathfrak{D}_1$ and $\mathfrak{D}_2$ is a domain of holomorphy*. Then, necessarily so is $\mathfrak{D}_3$.

Since a domain of holomorphy is pseudoconvex by *F. Hartogs*, $\mathfrak{D}$ is pseudoconvex. We may consider a real-valued function $\varphi_0(P)$, stated in Lemma II for this $\mathfrak{D}$. With a real number α, we consider a subset $\mathfrak{D}_\alpha$ of $\mathfrak{D}$ such that $\varphi_0(P) < \alpha$. For a large α, $\mathfrak{D}_\alpha$ contains the fixed points Q_1 and Q_2 in one connected component denoted by A. It is noted that A is bounded and finitely sheeted. We denote respectively by A_1, A_2, A_3 the parts of A with $\xi < a_1$, $\xi > a_2$, and $a_2 < \xi < a_1$. It follows from

[50] H. Behnke–K. Stein: Konvergente Folgen von Regularitätsbereichen und die Meromorphiekonvexität, 1938 (Math. Annalen).
[51] The same as 50.
[52] (Note by the translator.) This is the IX-th Report of the present series VII–XI, 1943.
[53] In fact, $\mathfrak{D}_\alpha$ is convex for the family of all holomorphic functions in $\mathfrak{D}$.

[54] By F. Hartogs, domains of holomorphy are pseudoconvex, so that we easily see the property of pseudoconvex domains by Lemma 4 together with the theorems of the present section and those of the IX-th Report: *Let $\varphi(x)$ be a pseudoconvex function in a neighborhood of a $2n$-dimensional ball S, and let S_α denote the sets of all points x of S with $\varphi(x) < \alpha$ (α is an arbitrary real number). Then, S_α, if exists, is pseudoconvex.*

Lemma 6 that every connected component of A_1, A_2 and A_3 is a domain of holomorphy.

We denote by Γ the boundary of A over $\xi=0$. Let M be any point of Γ. Then, there is a hypersurface piece σ defined locally in a neighborhood of M and passing through M such that $\sigma_0 \setminus \{M\}$ lies only in such a part of a neighborhood of σ in $\mathfrak{D}$ that $\varphi_0(P)>\alpha$. Let $\psi(P)=0$ ($\psi(P)$ is a holomorphic function) be a defining equation of σ. Choose β with $\alpha<\beta$, sufficiently close to α. Then σ does not have the boundary point in $\mathfrak{D}_\beta$ ($\varphi_0(P)<\beta$). (Here, if necessary, we take out a neighborhood of the boundary of σ.) Let B denote the part of $\mathfrak{D}_\beta$ with $a_2<\xi<a_1$. Then, B is finitely sheeted, and every connected component of B is a domain of holomorphy. Therefore, by Theorem 2 there is a function $G(P)$, meromorphic in B with poles $1/\psi(P)$ only on σ and no other poles. In A_3, $G(P)$ is holomorphic. For every point M of Γ, there is such a function $G(P)$ associated. Also, every connected component of A_3 is a domain of holomorphy (cf. the method of the proof of Lemma 1). Therefore, if positive δ_0 and ε_0 are chosen sufficiently small, by the standard arguments we easily deduce the existence of holomorphic functions $f_j(P)$ ($j=1,2,\dots,\nu$) in A_3 satisfying the following three conditions:

1° Let A_0 denote the set of all points of A with $|\xi|<\delta_0$, $|f_j(P)|<1$ ($j=1,2,\dots,\nu$). Then, $A_0 \Subset A$.
2° Let p be anyone of $1,2,\dots,\nu$. Then there is no point of $\mathfrak{D}_3$ with $|f_p(P)|\geq 1-\varepsilon_0$, lying over $|\xi-a_1|<\delta_0$, or over $|\xi-a_2|<\delta_0$.
3° The vector-valued function $[f_1(P),f_2(P),\dots,f_\nu(P)]$ never takes the same vector-value for mutually overlapped two points of A_0.

Further, letting A_4 be the set of points of A_3 satisfying $|f_j(P)|<1$ ($j=1,2,\dots,\nu$), we see that A_4 can be chosen arbitrarily close to A_3. The union of A_4 and the part of A satisfying $\xi\leq a_2$ or $\xi\geq a_1$ is an open set. Choose $f_j(P)$ ($j=1,2,\dots,\nu$) so that A_4 is sufficiently close to A_3. Then that open set contains the fixed points Q_1 and Q_2 in the same connected component, which is denoted by Δ. *The domain Δ satisfies the conditions given in §4.*

If α is chosen to be larger than a certain number α_0, we may consider A as a connected component of $\mathfrak{D}_\alpha$, which contains Q_1 and Q_2. Choose α' with $\alpha_0<\alpha'<\alpha$. In the same way as we associate α with A, we associate α' with A'. Needless to say, $A'\Subset A$. Let A_1' (resp. A_2') denote the part of A' with $\xi<0$ (resp. $\xi>0$). Since Δ can be chosen arbitrarily close to A, we immediately obtain the following consequence from the result of the previous chapter: *For a given holomorphic function $\Phi(P)$ in the open set, $P\in A$ with $|\xi|<\delta_0$ (here, δ_0 can be arbitrarily small), we can construct a holomorphic function $\Phi_1(P)$ (resp. $\Phi_2(P)$) in A_1' (resp. A_2'), which is holomorphic in the part of A' with $\xi=0$, such that $\Phi_1(P)-\Phi_2(P)=\Phi(P)$ holds there identically.*

Suppose that a pole $(\wp)$ is given in A. By Theorem 2 we may construct a meromorphic function $G_1(P)$ in A_1 with pole $(\wp)$. It is the same in A_2, and so the meromorphic function is denoted by $G_2(P)$. The difference $G_1(P)-G_2(P)$ is holomorphic in A_3. By the result above we see the following: *For a Cousin I Problem given in A we can solve it in A'.*

We come back to A: A is a connected component of $\mathfrak{D}_\alpha$ ($\alpha_0<\alpha$), containing Q_1 and Q_2. Let M be any boundary point of A. Let (γ) be the polydisk described over $\mathfrak{D}$ with center M. For sufficiently small (γ), there is a hypersurface piece σ defined in (γ), passing through M, which lies in $\varphi_0(P)>\alpha$ except for M. Let σ be defined by

$$\psi(P)=0, \quad P\in(\gamma),$$

where $\psi(P)$ is a holomorphic function in (γ). If necessary, (γ) is chosen a little smaller, there is α'' close to α with $\alpha<\alpha''$, and the associated domain A'' contains no boundary point of σ. Therefore, by the arguments as above, choosing α'' even closer to α, we may obtain a meromorphic function $G(P)$ in A'' such that it has poles $1/\psi(P)$ over σ, and has no other pole. Here M is an arbitrary boundary point of A.

We examine the two conditions of the Second Theorem of *Cartan–Thullen* for A. Let $(\mathfrak{F})$ denote the set of all holomorphic functions in A. Clearly by what we have seen above, 1° A is convex with respect to $(\mathfrak{F})$.

Let P_1,P_2 be an arbitrary pair of mutually overlapped points of A and denote the common base point by $\underline{P}$. We describe a half-line $\underline{L}$ with one end at $\underline{P}$ in (x)-space. We describe a half-line on A starting from P_1 over $\underline{L}$. Since A is bounded, this half-line necessarily intersects the boundary of A. Let M_1 be such a point, and let L_1 be the line segment (P_1,M_1). Similarly, we describe a half-line L_2 starting from P_2. Suppose that the length of L_1 does not exceed that of L_2. (Clearly, this assumption does not lose generality.) We denote by $G_0(P)$ the function $G(P)$ associated with $M=M_1$; $G_0(P)$ is holomorphic in A, holomorphic at every boundary point of A except for M_1, and has a pole at M_1. Therefore, $G_0(P)$ has to have different function elements at P_1 and P_2. Thus we have 2°: For any distinct two points of A, there is necessarily a function of $(\mathfrak{F})$ having different values at those points.

Thus, Conditions 1° and 2° are satisfied, and so by the Second Theorem of *Cartan–Thullen*, A is a domain of holomorphy. Since $\mathfrak{D}$ is a finitely sheeted domain, and A can be chosen arbitrarily close to it, *Theorem of Behnke–Stein* implies that $\mathfrak{D}$ *is a domain of holomorphy.*

Now, we assume that $\mathfrak{D}$ is a *pseudoconvex domain* in (x)-space. For this $\mathfrak{D}$ we take a function $\varphi_0(P)$ given in Lemma II, and consider $\mathfrak{D}_\alpha$ ($\varphi_0(P)<\alpha$) with an arbitrary real number α. (Here we take α enough large, so that $\mathfrak{D}_\alpha$ really exists.) As in the proof of Theorem 2 (cf. §3 and the last Report, §3), we divide $\mathfrak{D}_\alpha$ into small $2n$-dimensional cubes (open sets) (A); here however, (A) are not necessarily of complete form. After sufficiently fine division, it follows from Lemma 4 that every (A) (not mentioning the case of complete form, but also in another case) is a univalent open set, convex with respect to polynomials. Therefore, by the Second Main Theorem of *Cartan–Thullen* every connected component of them is a domain of holomorphy. After taking the division sufficiently fine, it is the same for (B) ($(B)_0$ is a $2n$-dimensional cube with center $(A)_0$, consisting of 9^n number of (A) and some parts of their boundaries, which may be not of complete form). Hence, from the result obtained above we easily infer in the same way as in the case of *Cousin* I Problem that every connected component of $\mathfrak{D}_\alpha$ is a domain of holomorphy. Therefore, Theorem of *Behnke–Stein* implies $\mathfrak{D}$ being a domain of holomorphy.

Theorem I. *A finite pseudoconvex domain with no interior ramification point is a domain of holomorphy.*[55]

By this theorem, the problem to show a domain being of holomorphy is reduced to show the pseudoconvexity of the domain.[56]

§11

We extend the definition of convexity (the last Report, §1) a little, and redefine it as follows:

Definition. Let $\mathfrak{D}$ be a finite domain over (x)-space with no interior ramification point, and let $(\mathfrak{F})$ be a family of holomorphic functions in $\mathfrak{D}$. The domain $\mathfrak{D}$ is said to be convex with respect to $(\mathfrak{F})$ if for every subset $\mathfrak{D}_0 \Subset \mathfrak{D}$, there is an open set $\mathfrak{D}'$ with $\mathfrak{D}_0 \subset \mathfrak{D}' \subset \mathfrak{D}$, bounded with respect to $\mathfrak{D}$, and satisfying that for an arbitrary point $P \in \mathfrak{D} \setminus \mathfrak{D}'$ there is at least one function $f(P)$ of $(\mathfrak{F})$ with $|f(P_0)| > \max|f(\mathfrak{D}_0)|$. In the case where $\mathfrak{D}$ consists of finite or infinite number of disjoint domains satisfying the property above, we use the same terminologies as defined.

[55] To detour around the use of the First Theorem of Cartan–Thullen, it suffices just to replace "*domain of holomorphy*" by "*domain $\mathfrak{D}$ satisfying the following two conditions*": Condition 1°, with $(\mathfrak{F})$ denoting the set of all holomorphic functions in $\mathfrak{D}$, $\mathfrak{D}$ is convex with respect to $(\mathfrak{F})$.; 2°, for every pair of distinct points of $\mathfrak{D}$ there is a function in $(\mathfrak{F})$ having distinct values at the two different points. Consequently, Theorem I and the First Theorem of Cartan–Thullen are obtained simultaneously.

[56] Cf. Report VI, Introduction. As an example we frequently encounter, we consider a "Überlargerungsbereich" over a pseudoconvex domain, which is pseudoconvex, too. Therefore, for example, in the Second Theorem of Cartan–Thullen, the second condition is unnecessary.

The convexity in the sense of this definition clearly implies that of the former definition. It is convenient to consider the following convexity as well:

Definition. In the above setting, $\mathfrak{D}$ is said to be strictly convex with respect to $(\mathfrak{F})$ if for every subset $\mathfrak{D}_0 \Subset \mathfrak{D}$, there is an open set $\mathfrak{D}'$ with $\mathfrak{D}_0 \subset \mathfrak{D}' \Subset \mathfrak{D}$, satisfying the condition mentioned above.

The strict convexity clearly implies the convexity. If $\mathfrak{D}$ is finitely sheeted, these two new notions of convexity agree with the former one. When $\mathfrak{D}$ is convex (resp. strictly convex) with respect to the family of all holomorphic functions in $\mathfrak{D}$, $\mathfrak{D}$ is simply said to be *holomorphically convex* (resp. *strictly holomorphically convex*).[57]

It has been a question since the last Report if a domain of holomorphy is strictly holomorphically convex.[58] We study it, here.

Lemma 7. *In Lemma II* (§7)*, $\mathfrak{D}_\alpha$ is convex with respect to the family of all holomorphic functions in $\mathfrak{D}$.*

Proof. Note that $\mathfrak{D}_\alpha$ is pseudoconvex (due to Lemma 4, the Second Theorem of *Cartan–Thullen* and *Hartogs*' Theorem). Therefore, $\mathfrak{D}_\alpha$ is a domain of holomorphy by Theorem I. Hence, with a real number β such that $\alpha<\beta$, $\mathfrak{D}_\alpha$ is convex with respect to the family of all holomorphic functions in $\mathfrak{D}_\beta$ by Lemma 5. Therefore, it follows from Lemma of Behnke–Stein that $\mathfrak{D}_\alpha$ is convex with respect to the family of all holomorphic functions in $\mathfrak{D}$. C.Q.F.D.

Theorem II. *A finite domain of holomorphy is strictly holomorphically convex.*

Proof. Let $\mathfrak{D}$ be a (finite) domain of holomorphy over (x)-space. Let $E \Subset \mathfrak{D}$ be an arbitrary subset. We take $\mathfrak{D}_\alpha$ in Lemma II so that $E \Subset \mathfrak{D}_\alpha$. By Lemma 7 above, $\mathfrak{D}_\alpha$ is convex with respect to the family of all holomorphic functions in $\mathfrak{D}$, and then by Lemma 1, with regarding $\mathfrak{D}_0 = \mathfrak{D}_\alpha$, we can choose an analytic polyhedron Δ of the form

$$(\Delta)\quad P\in R,\ |x_i|<r,\ |f_j(P)|<1\ \ (i=1,2,\ldots,n;\, j=1,2,\ldots,\nu),$$

such that $E \Subset \Delta$. Here, $f_j(P)$ are functions of $(\mathfrak{F})$, and R is a certain open set such that $\Delta \Subset R \subset \mathfrak{D}$.

[57] H. Behnke and people of his school use "convexity" in the sense of "strict convexity". (Cf. Behnke–Thullen's monograph, the first two papers of H. Behnke–K. Stein referred at the beginning of §1, in particular the second one.) Here, as mentioned once before, the notion of global convexity with respect to a family of holomorphic functions was introduced by H. Cartan. (Cf. H. Cartan's paper referred in the footnote at the end of Report IV.)

[58] Cf. its §1. We did not leave from univalent domains until the first research project (from Report I to Report VI) was finished: The reason was at this point.

Let $P_0 \in \mathfrak{D} \setminus \Delta$ be any point. It suffices to show that for this P_0 there is a function $f(P)$ of $(\mathfrak{F})$ with $|f(P_0)| > \max|f(E)|$. We take Δ' with the same property as Δ such that $\Delta \Subset \Delta'$ and $P_0 \in \Delta'$. Let Δ' be of the form:

$$(\Delta') \quad P \in R', \ |x_i| < r', \ |F_k(P)| < 1 \quad (i = 1, 2, \ldots, n; k = 1, 2, \ldots, \mu).$$

Here, we choose r' so that $r \leq r'$. From Δ and Δ' we form

$$(\Delta'') \qquad P \in R', \quad |x_i| < r, \quad |f_j(P)| < 1, \quad |F_k(P)| < 1$$
$$(i = 1, 2, \ldots, n; j = 1, 2, \ldots, \nu; k = 1, 2, \ldots, \mu).$$

Clearly, Δ is one or a union of several connected components of Δ''. If P_0 does not belong to Δ'', there exists necessarily a function with required property among x_i, $f_j(P)$. If P_0 belongs to Δ'', we consider a function in Δ'' such that it is 0 in Δ, and 1, elsewhere. Then this function is holomorphic in Δ'', and so by Theorem 1 it is expanded to a series of functions of $(\mathfrak{F})$, locally uniformly convergent in Δ''. Therefore, there is such a required function in this case, too. C.Q.F.D.

Corollary. *Let $\mathfrak{D}$ be a finite domain of holomorphy over (x)-space, and let $\mathfrak{D}_0$ be an open subset of $\mathfrak{D}$, convex with respect to the family of all holomorphic functions in $\mathfrak{D}$. Then, $\mathfrak{D}_0$ is strictly convex with respect to $(\mathfrak{F})$.*

Proof. Since $\mathfrak{D}_0$ is convex with respect to $(\mathfrak{F})$, for any subset $E \Subset \mathfrak{D}_0$, there is an open set $\mathfrak{D}'$ in $\mathfrak{D}_0$ such that $E \subset \mathfrak{D}' \subset \mathfrak{D}_0$, $\mathfrak{D}'$ is bounded with respect to $\mathfrak{D}_0$, and $\mathfrak{D}'$ satisfies the condition stated in the definition of "convexity". On the other hand, the above Theorem II implies the existence of an open set $\mathfrak{D}''$ in $\mathfrak{D}$ with $E \subset \mathfrak{D}'' \Subset \mathfrak{D}$, which satisfies the same condition with respect to $\mathfrak{D}$, and hence naturally with respect to $\mathfrak{D}_0$. We consider $\mathfrak{D}' \cap \mathfrak{D}'' = \mathfrak{D}_1$. Then, $E \subset \mathfrak{D}_1 \subset \mathfrak{D}_0$ and satisfies this condition. Now, $\mathfrak{D}''$ is finitely sheeted and $\mathfrak{D}'$ is bounded with respect to $\mathfrak{D}_0$, so that $\mathfrak{D}_1 \Subset \mathfrak{D}_0$. Therefore, $\mathfrak{D}_0$ is strictly convex with respect to $(\mathfrak{F})$. C.Q.F.D.

From Theorem 1 and this corollary, we obtain the following consequence:

Theorem III. *Let $\mathfrak{D}$ be a finite domain of holomorphy over (x)-space, and let $\mathfrak{D}_0$ be an open subset of $\mathfrak{D}$, which is convex with respect to the family $(\mathfrak{F})$ of all holomorphic functions in $\mathfrak{D}$. Then, every holomorphic function in $\mathfrak{D}_0$ is expanded to a series of functions of $(\mathfrak{F})$, convergent locally uniformly in $\mathfrak{D}_0$.*

The following result is deduced from Theorem 2 and Theorems II and III:

Theorem IV. *In a finite domain of holomorphy, Cousin I Problem is always solvable.*

(End, Report XI, 3.12.12)

(Translated by Junjiro Noguchi (Tokyo))

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References

[1] A. Andreotti and R. Narasimhan, Oka's Heftungslemma and the Levi problem, Trans. Amer. Math. Soc. **111** (1964), 345–366.

[2] H. Behnke and P. Thullen, Theorie der Funktionen mehrerer komplexer Veränderlichen, Ergebnisse der Mathematik und ihrer Grenzgebiete Bd. 3, Springer-Verlag, Heidelberg, 1934.

[3] H. Cartan and P. Thullen, Regularitäts- und Konvergenzbereiche, Math. Ann. **106** (1932), 617–647.

[4] P. Cousin, Sur les fonctions de n variables complexes, Acta Math. **19** (1895), no. 1, 1–61.

[5] H.J. Bremermann, Über die Äquivalenz der pseudokonvexen Gebiete und der Holomorphiegebiete im Raum von n komplexen Veränderlichen, Math. Ann. **128** (1954), 63–91.

[6] J.E. Fornæss, A counterexample for the Levi problem for branched Riemann domains over $\mathbf{C}^n$, Math. Ann. **234** (1978), 275–277.

[7] R. Fujita, Domaines sans point critique intérieur sur l'espace projectif complexe, J. Math. Soc. Jpn. **15** (1963), 443–473.

[8] H. Grauert and R. Remmert, Coherent Analytic Sheaves, Grundl. der Math. Wissen. vol. 265, Springer-Verlag, Berlin, 1984.

[9] R.C. Gunning and H. Rossi, Analytic Functions of Several Complex Variables, Prentice-Hall, 1965.

[10] S. Hitotsumatsu, On Oka's Heftungs Theorem (Japanese), Sugaku **1** (4) (1949), 304–307, Math. Soc. Jpn.

[11] L. Hörmander, Introduction to Complex Analysis in Several Variables, First Edition 1966, Third Edition, North-Holland, 1989.

[12] I. Lieb, Das Levische Problem, Bonn. Math. Schr. No. 387 (2007), 1–34; Translation into French, Le problème de Levi, Gaz. Math. Soc. Math. Fr. **115** (2008), 9–34.

[13] R. Narasimhan, The Levi problem in the theory of several complex variables, ICM Stockholm (1962).

[14] T. Nishino, Function Theory in Several Complex Variables (in Japanese), The University of Tokyo Press, Tokyo, 1996; Translation into English by N. Levenberg and H. Yamaguchi, Amer. Math. Soc. Providence, R.I., 2001.

[15] J. Noguchi, Analytic Function Theory of Several Variables – Elements of Oka's Coherence, Springer, Singapore, 2016; translated from Analytic Function Theory of Several Variables (in Japanese), Asakura-Shoten, Tokyo, 2013.

[16] J. Noguchi, Analytic Function Theory of Several Variables (in Japanese, Tahensu Kaiseki Kansuron) – Oka's Coherence for Undergraduates, Second Edition, Asakura-Shoten, Tokyo, 2019.

[17] J. Noguchi, Inverse of Abelian integrals and ramified Riemann domains, Math. Ann. **367** No. 1 (2017), 229–249.

[18] J. Noguchi, A weak coherence theorem and remarks to the Oka theory, Kodai Math. J. **42** (2019), 566–586.

[19] J. Noguchi, A New Introduction to Oka Theory—Basics of Function Theory of Several Variables (in Japanese), Shokabo, Tokyo, 2021.

[20] J. Noguchi, A brief chronicle of the Levi (Hartogs' Inverse) Problem, Coherence and an open problem, Notices Intern. Cong. Chin. Math. **7** No. 2 (2019), 19–24.

[21] F. Norguet, Sur les domains d'holomorphie des fonctions uniformes de plusieurs variables complexes (Passage du local au global), Bull. Soc. Math. France **82** (1954), 137–159.

[22] K. Oka:

- Sur les fonctions analytiques de plusieurs variables – I Domaines convexes par rapport aux fonctions rationnelles, J. Sci. Hiroshima Univ. Ser. A **6** (1936), 245–255 [Rec. 1 mai 1936].[59]
- Sur les fonctions analytiques de plusieurs variables – II Domaines d'holomorphie, J. Sci. Hiroshima Univ. Ser. A **7** (1937), 115–130 [Rec. 10 déc. 1936].
- Sur les fonctions analytiques de plusieurs variables – III Deuxième problème de Cousin, J. Sci. Hiroshima Univ. **9** (1939), 7–19 [Rec. 20 jan. 1938].
- Sur les fonctions analytiques de plusieurs variables – IV Domaines d'holomorphie et domaines rationnellement convexes, Jpn. J. Math. **17** (1941), 517–521 [Rec. 27 mar. 1940].
- Sur les fonctions analytiques de plusieurs variables – V L'intégrale de Cauchy, Jpn. J. Math. **17** (1941), 523–531 [Rec. 27 mar. 1940].
- Sur les domaines pseudoconvexes, Proc. of the Imperial Academy, Tokyo **17** (1941) 7–10 [Communicated 13 jan. 1941].
- Sur les fonctions analytiques de plusieurs variables – VI Domaines pseudoconvexes, Tôhoku Math. J. **49** (1942), 15–52 [Rec. 25 oct. 1941].
- Sur les fonctions analytiques de plusieurs variables – VII Sur quelques notions arithmétiques, Bull. Soc. Math. France **78** (1950), 1–27 [Rec. 15 oct. 1948].
- Note sur les fonctions analytiques de plusieurs variables, Kōdai Math. Sem. Rep. (1949). no. **5-6**, 15–18 [Rec. 19 déc. 1949].
- Sur les fonctions analytiques de plusieurs variables – VIII Lemme fondamental, J. Math. Soc. Japan **3** (1951) No. 1, 204–214; No. 2, 259–278 [Rec. 15 mar. 1951].
- Sur les fonctions analytiques de plusieurs variables – IX Domaines finis sans point critique intérieur, Jpn. J. Math. **23** (1953), 97–155 [Rec. 20 oct. 1953].
- Sur les fonctions analytiques de plusieurs variables – X Une mode nouvelle engendrant les domaines pseudoconvexes, Jpn. J. Math. **32** (1962), 1–12 [Rec. 20 sep. 1962].

[23] K. Oka, Appendice—Sur les formes objectives et les contenus subjectifs dans les sciences mathématiques; Propos postérieur—Pourquoi le présent mémoire est publié de nouveau, 1953: URL http://www.ms.u-tokyo.ac.jp/~noguchi/oka/.

[24] K. Oka, Sur les fonctions analytiques de plusieurs variables, Iwanami Shoten, Tokyo, 1961.

[25] K. Oka, Collected Works, Translated by R. Narasimhan, Ed. R. Remmert, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1984.

[26] K. Oka, Posthumous Papers of Kiyoshi Oka, Eds. T. Nishino and A. Takeuchi, Kyoto, 1980–1983: Oka Kiyoshi Collection, Library of Nara Women's University, URL http://www.lib.nara-wu.ac.jp/oka/index_eng.html.

[27] A. Takeuchi, Domaines pseudoconvexes infinis et la métrique riemannienne dans un espace projectif, J. Math. Soc. Jpn. **16** (1964), 159–181.

[59] The most commonly cited reference for Oka's work should be [25], but all the records of the received dates of the papers were erased there by unknown reason: Here, they are listed for the sake of convenience.