The Geometry and Topology of Compact Complex Manifolds with RC-Positive Tangent Bundles

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Abstract. In this note, we give a brief exposition on the recent progress on the geometry and topology of compact complex manifolds with RC-positive tangent bundles, with particular focus on several open problems proposed by S.-T. Yau around 40 years ago.

The uniformization theorem demonstrates satisfactorily that: every simply connected Riemann surface is isomorphic to one of three Riemann surfaces: the open unit disk $\mathbb{D}$, the complex plane $\mathbb{C}$, or the Riemann sphere $\mathbb{S}^2$. However, the geometry and topology of higher dimensional complex manifolds are far more complicated, and the classifications of such manifolds are still challenging tasks in modern geometry. As analogous to the geometric properties of three model Riemann surfaces, many terminologies are developed to investigate higher dimensional complex manifolds. For instances, a variety of curvature notions are used in differential geometry, e.g., holomorphic bisectional curvature, holomorphic sectional curvature, Ricci curvature and etc. With the help of the intersection theory, algebraic geometers created many algebraic concepts to describe the geometry of algebraic manifolds, such as ampleness, nefness and so on. On the other hand, it is natural to investigate complex manifolds by analyzing model Riemann surfaces contained in them. The category of algebraic manifolds containing rational curves ($\mathbb{CP}^1 \cong \mathbb{S}^2$) plays a significant role in algebraic geometry, and complex manifolds without entire curves $\mathbb{C}$—so called hyperbolic manifolds—are fundamental in analytical geometry.

The relationships between these key concepts are of primary concern in understanding the geometry of topology of complex manifolds. The seminal works of Siu-Yau and Mori on the solutions to Frankel conjecture and Hartshorne conjecture ([SY80], [Mor79]) exhibit a wonderful correspondence along this line.

Theorem 0.1. *Let X be a compact complex manifold. The holomorphic tangent bundle T_X is ample if and only if X admits a smooth Kähler metric with positive holomorphic bisectional curvature. Moreover, X is biholomorphic to a complex projective space $\mathbb{P}^n$.*

Many remarkable generalizations of Theorem 0.1 have been established, for instances, Mok's uniformization theorem on compact Kähler manifold with non-negative holomorphic bisectional curvature ([Mok88]) and the works of Campana, Demailly, Peternell and Schneider ([CP91], [DPS94]) on the structure of projective manifolds with nef tangent bundles. For further developments on this comprehensive topic, we refer to [AW01, Yan17, FLW17, Liu19, LOY19, LOY20] and the references therein.

The holomorphic sectional curvature also carries much geometric information of complex manifolds. The classical Schwarz lemma [Che68, Lu68, Yau78a] asserts that:

Lemma 0.2. *Let X be a complex or Kähler manifold. If it has a Hermitian metric with non-positive holomorphic sectional curvature, then it contains no rational curve.*

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As we pointed out before, projective manifolds without rational curve are fundamental in algebraic geometry. Indeed, Mori established in [Mor82] that

Theorem 0.3. *Let X be a projective manifold. If it contains no rational curve, then its canonical bundle K_X is nef.*

Theorem 0.3 is a powerful tool in many transcendental methods, and in differential geometry, it is still a widely open problem to formulate a purely analytical proof of this result. In the recent breakthrough work [WY16a] of Damin Wu and Shing-Tung Yau, it has been established that

Theorem 0.4. *Let X be a projective manifold. If it admits a Kähler metric with negative holomorphic sectional curvature, then its canonical bundle K_X is ample.*

In their elegant proof of Theorem 0.4, analytical methods and algebraic methods are unified ingeniously. Based on the key ideas of [WY16a] and by passing Mori's theory, Valentino Tosatti and the author achieved in [TY17] the following result

Theorem 0.5. *Let X be a compact Kähler manifold. If it admits a Kähler metric with non-positive holomorphic sectional curvature, then its canonical bundle K_X is nef.*

It worths to point out that there are many compact Kähler manifolds which have non-positive holomorphic sectional curvature, but they are not projective manifolds. Generic tori are well-known examples. In this case, Mori's theory can not be applied. As a consequence of Theorem 0.4 and Theorem 0.5, it is established that a compact Kähler manifold with negative or quasi-negative holomorphic sectional curvature is algebraic and has ample canonical bundle ([TY17, WY16b, DT19]), which settles down a long-standing conjecture of S.-T. Yau affirmatively.

Theorem 0.6. *Let X be a compact Kähler manifold. If it admits a Kähler metric with (quasi-)negative holomorphic sectional curvature, then X is projective and K_X is ample.*

By applying similar ideas in Riemannian geometry, Bing-Long Chen and the author established in [CY18] (see also a generalization in [CY21]) that:

Theorem 0.7. *Let X be a compact Kähler manifold. If X is homotopic to a compact Riemannian manifold with negative sectional curvature, then X is a Kähler-Einstein manifold of general type.*

For more recent works on non-positive holomorphic sectional curvature, we refer to [HLW10, HLW16, HLWZ18, Nom18, Gue18, YZ19, WY20] and the references therein.

In his "Problem Section", S.-T. Yau proposed the well-known conjecture [Yau82, Problem 47] that compact Kähler manifolds with positive holomorphic sectional curvature must be projective and rationally connected, i.e. any two points of X can be connected by some rational curve. Recently, the author solved this conjecture completely in [Yan18b]:

Theorem 0.8. *Let X be a compact Kähler manifold. If it admits a Kähler metric with positive holomorphic sectional curvature, then it is projective and rationally connected. In particular, X is simply connected.*

The author introduced in [Yan18b] a new terminology of positivity, so called RC-positivity, for abstract vector bundles as natural generalizations for positive holomorphic sectional curvature:

Definition 0.9. A holomorphic vector bundle $\mathscr{E}$ over a complex manifold X is called **RC-positive**, if there exists a smooth Hermitian metric h on $\mathscr{E}$, and for each point $q \in X$ and for each nonzero vector $v \in \mathscr{E}_q$, there exists some nonzero vector $u \in T_qX$ such that

$$R^{\mathscr{E}}(u,\overline{u},v,\overline{v}) > 0.$$

Many geometric properties of RC-positive vector bundles are developed in [Yan18b, Yan18a, Yan20, Yan21]. Notably, the author established a criterion for projectivity of compact Kähler manifolds.

Theorem 0.10. *Let X be a compact Kähler manifold. If $\wedge^2 T_X$ is RC-positive, then X is a projective manifold.*

As an application of geometric properties of RC-positive vector bundles and a criterion for rational connectedness developed in [CDP15] and [GHS03], the following result is achieved in [Yan18b], and Theorem 0.8 is a consequence of it:

Theorem 0.11. *Let X be a compact Kähler manifold of complex dimension n. If $\wedge^p T_X$ are RC-positive for all $1 \le p \le n$, then X is projective and rationally connected. In particular, X is simply connected.*

As we mentioned before, it is a major topic to investigate the correspondences between differential geometric and algebraic geometric objects. Kodaira's embedding theorem is a pioneer work in this direction, which asserts that a line bundle is ample if and only if it has a positive metric. Moreover, Yau's solution to the Calabi conjecture [Yau78b] establishes a fundamental characterization of Fano manifolds:

Theorem 0.12. *Let X be a projective manifold. Then X is Fano if and only if it has a Kähler metric ω with positive Ricci curvature.*

We also proved in [Yan18b] that if a compact Kähler manifold admits a Hermitian metric with positive second Chern-Ricci curvature $\mathrm{Ric}^{(2)}(\omega)$, then X is projective and rationally connected. This is also a generalization of the classical result of Campana [Cam92] and Kollár-Miyaoka-Mori [KMM92] that Fano manifolds are rationally connected. As an analog of Theorem 0.12, the author obtained in [Yan19] a differential geometric characterization of uniruled manifolds which was also conjectured by S.-T. Yau.

Theorem 0.13. *Let X be a projective manifold. Then X is uniruled if and only if it has a Kähler metric ω with RC-positive Ricci curvature, i.e. $\mathrm{Ric}(\omega)$ has at least one positive eigenvalue everywhere.*

It is well-known that the category of rationally connected manifolds is larger than that of Fano manifolds, and is contained in the class of uniruled manifolds. As inspired by Theorem 0.12 and Theorem 0.13, the following differential geometric concept is proposed to characterize rationally connected manifolds.

Definition 0.14. A holomorphic vector bundle $\mathscr{E}$ over a complex manifold X is called **uniformly RC-positive** if there exists a smooth Hermitian metric h on $\mathscr{E}$, and at each point $q \in X$, there exists some nonzero vector $u \in T_qX$ such that for every nonzero vector $v \in \mathscr{E}_q$,

$$R^{\mathscr{E}}(u,\overline{u},v,\overline{v}) > 0.$$

It is proved in [Yan18b] that Kähler manifolds with positive holomorphic sectional curvature have uniformly RC-positive tangent bundle. The first significant property of the uniform RC-positivity is the following differential geometric criterion for rational connectedness.

Theorem 0.15. *Let X be a compact Kähler manifold. If T_X is uniformly RC-positive, then X is projective and rationally connected. In particular, X is simply connected.*

As converse to Theorem 0.15, it is proposed in [Yan20] that the holomorphic tangent bundles of rationally connected projective manifolds should be RC-positive or uniformly RC-positive. A partial answer to this problem is obtained in [Yan20].

Theorem 0.16. *Let X be a rationally connected manifold. Then there exists a uniformly RC-positive Finsler metric on X.*

By using the ideas of RC-positivity and some analytical techniques in algebraic geometry, Shinichi Matsumura established in [Mat18a, Mat18b, Mat18c] a structure theorem for projective manifolds with non-negative holomorphic sectional curvature, which is analogous to fundamental works in [Mok88, CDP15, CH17, CH19] for manifolds with various non-negative properties (see also an approach in [HW15]). In the same spirit, Lei Ni and Fangyang Zheng introduced in [NZ18a, NZ18b] various notions of Ricci curvature and scalar curvature to obtain rational connectedness of compact Kähler manifolds. The following conjecture on quasi-positive holomorphic sectional curvature is widely open (e.g. [Yan20, Conjecture 1.9]), and in the special case when X is projective it was confirmed affirmatively by Matsumura ([Mat18b]):

Conjecture 0.17. *Let (X,ω) be a compact Kähler manifold. If it has quasi-positive holomorphic sectional curvature, then X is projective and rationally connected.*

As inspired by the Leray-Grothendieck spectral sequence for abstract vector bundles employed in [Yan18b, Yan20, Yan21], a new energy density was introduced in [Yan18a]. Let $f:(M,h)\to(N,g)$ be a holomorphic map between two Hermitian manifolds. Suppose that $\{z^\alpha\}_{\alpha=1}^m$ and $\{\eta^i\}_{i=1}^n$ are the local holomorphic coordinates around $p\in M$ and $q=f(p)\in N$ respectively. The generalized energy density

$$\mathscr{Y} = g_{i\overline{j}} f^i_\alpha \overline{f^j_\beta} \frac{W^\alpha \overline{W}^\beta}{\sum h_{\gamma\overline{\delta}} W^\gamma \overline{W}^\delta}$$

is defined over the projective bundle $\mathbb{P}(T_M)\to M$ where $\{W^1,\cdots,W^m\}$ are the holomorphic coordinates on the fiber T_pM with respect to the given trivialization. The complex Hessian of the new energy density has the following estimate.

Theorem 0.18. *Let $f:(M,h)\to(N,g)$ be a holomorphic map between two Hermitian manifolds. We have the following inequality on the projective bundle $\mathbb{P}(T_M)\to M$,*

$$\sqrt{-1}\partial\overline{\partial}\mathscr{Y} \geq \left(\sqrt{-1}\partial\overline{\partial}\log\mathscr{H}^{-1}\right)\cdot\mathscr{Y} - \frac{\sqrt{-1}R_{i\overline{j}k\overline{\ell}} f^i_\alpha \overline{f^j_\beta} f^k_\mu \overline{f^\ell_\nu} W^\mu \overline{W}^\nu dz^\alpha \wedge d\overline{z}^\beta}{\mathscr{H}}$$

where $\mathscr{H} = \sum h_{\gamma\overline{\delta}} W^\gamma \overline{W}^\delta$.

As applications of Theorem 0.18, several new rigidity theorems are established in [Yan18a], which have been widely open for geometers in the field since the classical works of Yau ([Yau78b]) and Royden ([Roy80]).

Theorem 0.19. *Let $f:(M,h)\to(N,g)$ be a holomorphic map between two Hermitian manifolds. Suppose M is compact. If (M,h) has positive (resp. non-negative) holomorphic sectional curvature and (N,g) has non-positive (resp. negative) holomorphic sectional curvature, then f is a constant map.*

Moreover, similar ideas can also work for harmonic maps and pluri-harmonic maps between Riemannian manifolds. As an application of rational connectedness, one has the following rigidity result.

Corollary 0.20. *Let $f : M \to N$ be a holomorphic map between two compact complex manifolds. If M is a Kähler manifold with uniformly RC-positive T_M and N is Kobayashi hyperbolic, or N contains no rational curve, then f is constant.*

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