Open Problems

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compiled by Kefeng Liu*

Note. This column is edited by Shing-Tung Yau (Harvard University). The readers are welcome to propose the solutions. The readers are welcome to propose the solutions. The authors may send their solutions to () and post the solutions in MathSciDoc (http://archive.ymsc.tsinghua.edu.cn/). The correct solutions will be announced and some souvenirs will be awarded.

Problem 2020006 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Given a Riemannian manifold, we can look at the graded ring of differential operators that commute with the Laplacian of the manifold. This ring can include all polynomials of the Laplacian. But it can be larger and is in general noncommutative. This is the case for the Euclidean version of the Kerr metric (as was demonstrated by Lars Andersson et al. on the existence of Killing sponsor tensor) However, it would be interested to find examples of compact manifolds with nontrivial graded ring and hopefully classify them.

Problem 2020007 (Differential Geometry). *Proposed by Aghil Alaee (Clark University and Harvard University), Martin Lesourd (Harvard University), and Shing-Tung Yau (Harvard University).*

A typical object of study of free boundary minimal surfaces are inequalities that combine area, Euler characteristic, and Morse index. These are readily obtainable in domains with positive scalar or Ricci curvature with boundaries satisfying various convexity assumptions. For instance, Chen, Fraser and Pang [1] proved an estimate for area of a compact orientable two-sided free boundary minimal surface with index 1, genus g, and $l \geq 1$ boundary components in a Riemannian manifold with non-negative scalar curvature. Moreover, Carlotto and Franz [2] obtain a diameter estimate for stable free boundary minimal surfaces, from which they then obtain a useful area bound.

In [1], we establish an index estimate and a diameter estimate for free boundary marginally outer trapped surface (MOTS), as generalizations of Chen-Fraser-Pang and Carlotto-Franz results for free boundary minimal surfaces, respectively.

These two instances of straightforward generalizations from minimal surfaces to MOTS suggest the following additional open problems:

(i) Obtain a general existence theorem for free boundary MOTS within domains satisfying appropriate convexity conditions.
(ii) Obtain curvature estimates for stable free boundary MOTS generalizing [4].
(iii) In either the closed or free boundary setting, study compactness phenomena for MOTS.

Reference

[1] J. Chen, A. Fraser and C. Pang, *Minimal immersions of compact bordered Riemann surfaces with free boundary*, Trans. Amer. Math. Soc. **367** (2015), 2487–2507.
[2] A. Carlotto and G. Franz, *Inequivalent complexity criteria for free boundary minimal surfaces*, Adv. Math. **373** (2020), 107322.
[3] A. Alaee, M. Lesourd and S.-T. Yau, *Stable surfaces and free boundary marginally outer trapped surfaces*, arXiv:2009.07933 (2020).

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[4] L. Andersson and J. Metzger, *Curvature estimates for stable marginally trapped surfaces*, J. Differential Geom. **84** (2010) 231–265.

Problem 2020008 (Differential Geometry). *Proposed by Martin Lesourd (Harvard University), Ryan Unger (Princeton University), and Shing-Tung Yau (Harvard University).*

Recall the following Liouville Conjecture for locally conformally flat (LCF) manifolds M^n with nonnegative scalar curvature $R \geq 0$.

Conjecture 1 (LCF Liouville Conjecture). *Let (M^n, g), $n \geq 3$, be a complete, LCF, $R \geq 0$ manifold, and $\Phi : M^n \to S^n$ a conformal map. Then Φ is injective and $\partial\Phi(M)$ has zero Newtonian capacity.*

Schoen-Yau [1] show this for $n \geq 7$, and, separately, for $4 \leq n \leq 6$ under various assumptions on the scalar and Ricci curvature. We note that in [1], it is shown that Conjecture 1 follows from the *positive mass conjecture with arbitrary ends*.

Conjecture 2 (Positive Mass Conjecture with Arbitrary Ends). *Let N^n be a complete manifold diffeomorphic to $\mathbb{R}^n$ and let X^n be a complete connected noncompact manifold. Let g be a smooth complete metric on $M^n = N^n \# X^n$ with nonnegative scalar curvature $R \geq 0$. If g is asymptotically flat on N^n with suitable fall-off, then the ADM mass of (M^n, g) measured in the asymptotically flat end N^n is nonnegative.*

In [2], we show that the Liouville theorem for locally conformally flat manifolds M_n with $R \geq 0$ follows from the impossibility of admitting a complete metric with $R > 0$ on its connected sum with n-torus $T^n \# M^n$.

Theorem 1. *The nonexistence of a complete smooth metric with $R > 0$ on $T^n \# M^n$ implies Conjecture 1.*

Theorem 1 and the recent work of Chodosh-Li [3], settles Conjecture 1 in the remaining cases.

By "suitable fall-off", one could seek to reproduce that of the standard positive mass theorem. Lokhamp-type compactification arguments do not seem to allow reducing Conjecture 2 to a statement about $T^n \# X^n$, but it turns out that we do not need the full strength of Conjecture 2 to prove Theorem 1.

Reference

[1] R. Schoen and S.-T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math. **92** (1988), 47–71.

[2] M. Lesourd, R. Unger and S.-T. Yau, *Positive scalar curvature on noncompact manifolds and the Liouville theorem*, arXiv:2009.12618 (2020).

[3] O. Chodosh and C. Li, *Generalized soap bubbles and the topology of manifolds with positive scalar curvature*, arXiv:2008.11888 (2020).