Dr. Tien-Yien Li's Three Seminal Papers

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Abstract. We present the most important mathematical contributions of Dr. Tien-Yien Li by discussing his three celebrated papers.

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Introduction

Dr. Tien-Yien Li, who coined the term "chaos" in his famous paper "Period three implies chaos" with Dr. James Yorke in 1975, passed away peacefully on June 25, 2020, at the age of 75.

Dr. Li was born on June 28 of 1945 in Sha County, Fujian Province of China. At age three, he followed his parents to Taiwan, where he received traditional Chinese education. He earned his B.S. in mathematics at the National Tsinghua University in Taiwan in 1968. He received his Ph.D. in mathematics from the University of Maryland in the United States in 1974 under the guidance of Dr. James Yorke.

Dr. Li joined the faculty of the Department of Mathematics at Michigan State University in 1976 and was promoted to the rank of full professor in 1983. He received the honorary title of University Distinguished Professor in 1998. He supervised 26 Ph.D. dissertations in the general areas of dynamical systems and numerical analysis. He retired as a University Distinguished Professor Emeritus in 2018 after spending 42 years at the university.

Dr. Li received the Guggenheim Fellowship in 1995, Michigan State University's Distinguished Faculty Award as well as Frame Teaching Award in 1996, College of Sciences Distinguished Alumni Award of the National Tsinghua University in 2002, Michigan State University College of Natural Science's Outstanding Academic Advisor Award in 2006, and National Tsinghua University's Outstanding Alumni Award in 2012.

Dr. Li was a trailblazer in several important fields of applied mathematics and computational mathematics. Some of his monumental accomplishments include: he and Yorke's paper, "Period three implies chaos," first formally encapsulated the concept of "chaos" in mathematics; his proof of Ulam's conjecture is the fundamental work in the computation of invariant measures of dynamical systems; his idea and numerical method with R. B. Kellogg and Yorke in computing Brouwer's fixed point opened a new era for the research in modern homotopy continuation methods.

Although he made numerous important contributions to other areas of mathematics during his academic career of five decades, such as the Cauchy problem of ordinary differential equations in Banach spaces, solving multivariate polynomial systems and algebraic eigenvalue problems, in this article, we only survey Dr. Tien-Yien Li's three most celebrated works, which were also done before his thirtieth birthday and which have had and will continue to have deep impacts on mathematics and its applications.

In the next section we describe his most well-known paper [6]. Section 3 will be on his pioneering work [4] in computational ergodic theory. Section 4 will present his idea and construction [2] of the first modern homotopy continuation method. The contents of the article are partly based on [1, 5].

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“Period Three Implies Chaos”

The concept of chaos began to evolve in the 1880s when the great French mathematician Henri Poincaré studied the three body problem, who found that there can be orbits that are non-periodic and yet not forever increasing nor approaching a fixed point. Another great French mathematician Jacques Hadamard of the same times also observed chaotic motion in the “Hadamard billiards.” In the 1940s, English mathematicians Mary Cartwright and John Littlewood found chaotic dynamics of some nonlinear differential equations. American meteorologist Edward Lorenz discovered chaos in his computer simulation of weather prediction in the early 1960s. All such observations and perspectives of chaos were mainly from physical sciences. However, the first formal formation of chaos in mathematics was given in Tien-Yien Li and James Yorke's paper “Period three implies chaos” in 1975.

One Friday afternoon in March 1973, when Tien-Yien Li entered his Ph.D. thesis advisor's office, Yorke immediately said to him, “I have a good idea for you.” This idea had evolved in his head after reading Lorenz's four papers concerning weather prediction, which had been passed to him by his meteorologist colleague Allen Feller in the Institute of Fluid Dynamics and Applied Mathematics, now called the Institute of Physical Sciences and Technology, at the University of Maryland. Yet he had not been able to prove it completely. Two weeks later, Tien-Yien Li, skillfully manipulating his calculus techniques, proved what is later known as the “Li-Yorke chaos.”

Li-Yorke Theorem. Let I be an interval and let $S: I \to I$ be continuous. Assume that there is a point $a \in I$ such that

$$S^3(a) \leq a < S(a) < S^2(a) \quad \text{or} \quad S^3(a) \geq a > S(a) > S^2(a),$$

then for every $n = 1, 2, \ldots$, there is a periodic point of period n in I. Furthermore, there is an uncountable set $A \subset I$, containing no periodic points, that satisfies the following conditions:

(1) for every $x \neq y$ in A,

$$\limsup_{n\to\infty} |S^n(x) - S^n(y)| > 0 \quad \text{and} \quad \liminf_{n\to\infty} |S^n(x) - S^n(y)| = 0; \tag{1}$$

(2) for every $x \in A$ and periodic point $p \in I$,

$$\limsup_{n\to\infty} |S^n(x) - S^n(p)| > 0. \tag{2}$$

In particular, when $S^3(a) = a$, namely S has a period-3 point, then (1) and (2) are satisfied, which explains the short title of the resulting paper. However, the more general assumption of $S^3(a) \leq a$ or $S^3(a) \geq a$ has far richer applicability. For example, in population dynamics, it is rare that the third year population $S^3(a)$ is exactly the same as a of the initial year.

The mathematical tool that Tien-Yien Li used to prove the theorem is the intermediate value theorem from calculus, and he applied it in a smart way. A well-known direct application of the intermediate value theorem is the Brouwer fixed point theorem in one dimension: if S is a continuous function on a closed and bounded interval J such that $S(J) \subset J$, then S has a fixed point in J. Tien-Yien Li discovered its “dual version” in which “$\subset$” is changed to “$\supset$”: if the continuous function S satisfies $S(J) \supset J$, then S has a fixed point in J. By using repeatedly another result of his discovery that if J_0 is a closed subinterval of $S(J)$, then there is a closed subinterval J_1 of J such that $S(J_1) = J_0$, together with the above fixed point theorem, Tien-Yien Li proved the Li-Yorke theorem.

After finishing it, according to Yorke's intention, they sent the paper to the *American Mathematical Monthly*, which is the most read mathematics journal in the world. However, it was rejected because its writing style lacked the appealing to the major pool of college students. The editor agreed that they may re-submit the paper if the authors could rewrite it to fit college students. Since Tien-Yien Li was busy with research on differential equations and the others, this paper sat untouched on his desk for nearly one year.

The year 1974 was a “special year” of biomathematics in the Department of Mathematics at the University of Maryland. In the first week of May, the department invited Robert May of Princeton University to lecture for a week. On the last day, he lectured about the logistic model $S_\alpha(x) = \alpha x(1-x)$ and reported on its iteration sequences' complicated dynamical behavior as the parameter α is near 4, yet he did not offer an explanation, thinking that the phenomenon is perhaps caused by computation errors. After Yorke heard this lecture, he gave May the paper of the Li-Yorke theorem on their way to the airport. May was stunned upon reading the conclusion of the paper, and he recognized that this theorem had fully explained his uncertainties. At once, Yorke returned from the airport and contacted Tien-Yien Li, “We should rewrite this paper immediately.” The task was completed within two weeks, and it was accepted by the *American Mathematical Monthly*. The paper appeared in the December issue of 1975.

In the Li-Yorke theorem, expressions of (1) mean that for any two points $x \neq y$ in A, the distance sequence $\{|S^n(x) - S^n(y)|\}$ has a subsequence that converges to 0 and a subsequence that converges to a positive number, and inequality (2) indicates that for any $x \in A$ and periodic point p of S, the sequence $\{|S^n(x) - S^n(p)|\}$ has a subsequence that converges to a positive number. These fully exhibit the sensitive dependence on initial conditions and the resulting unpredictable nature for the eventual behavior of the dynamics of S, thus giving strictly a mathematical definition of chaos. The Li-Yorke theorem thoroughly unveiled the nature and characteristics of chaos the first time in mathematics, and was credited by Freeman Dyson of the Institute of Advanced Studies as "one of the immortal gems in the literature of mathematics" in the 2008 Einstein Lecture article "Birds and Frogs." The paper "Period three implies chaos" has been cited 4882 times as of September 19, 2020 according to Google Scholar.

Ulam's Conjecture

Chaos theory began in ergodic theory, which is a branch of mathematics that studies statistical properties of deterministic systems. An important topic in ergodic theory concerns the existence and computation of an absolutely continuous invariant measure associated with a mapping S, which is reduced to the existence and computation of a fixed density function of the corresponding Frobenius-Perron operator $P_S : L^1(0,1) \to L^1(0,1)$ defined by

$$P_S f(x) = \frac{d}{dx}\int_{S^{-1}([0,x])} f(t)dt, \ \ \forall\, f \in L^1(0,1),\ x \in [0,1]$$

when $S : [0,1] \to [0,1]$. To a chaotic dynamical system, such an invariant measure gives the probability distribution of chaotic orbits in the phase space, and it is intimately related to crucial mathematical concepts such as entropy and the Lyapunov exponent.

In 1960, Polish-born American mathematician Stan Ulam, father of the American hydrogen bomb, proposed a numerical scheme in his famous book [7], entitled *A Collection of Mathematical Problems*, to calculate a fixed density function of the Frobenius-Perron operator associated with a nonlinear mapping $S : [0,1] \to [0,1]$. He partitioned $[0,1]$ into n subintervals $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$. Next, he defined an $n\times n$ row stochastic matrix $P_n = [p_{ij}]$ whose (i, j) entry is

$$p_{ij} = \frac{m\{[x_{i-1},x_i]\cap S^{-1}([x_{j-1},x_j])\}}{x_i - x_{i-1}},$$

where m is the Lebesgue measure. The number p_{ij} quantifies the fraction of those points in the i-th subinterval $[x_{i-1},x_i]$ that are mapped into the j-th subinterval $[x_{j-1},x_j]$ under S. In Ulam's method, one computes a normalized nonnegative left eigenvector v_n of P_n with respect to eigenvalue 1, so that the corresponding piecewise constant function f_n with function values on the subintervals given by the components of v_n is a density function. This density function f_n can be considered as an approximate fixed density function of the Frobenius-Perron operator P_S. For the convergence of this numerical scheme based on a probability argument, Ulam presented his famous conjecture: if P_S has a fixed density function, then the sequence $\{f_n\}$ converges to a fixed density function f^* of P_S as n approaches infinity.

In 1973, Polish Academician Andrzej Lasota and Yorke solved in their paper [3] another problem that Ulam proposed in *A Collection of Mathematical Problems* by proving that, if $S : [0,1] \to [0,1]$ is a piecewise C^2 mapping such that $\inf_{x\in[0,1]} |S'(x)| > 1$, then the corresponding Frobenius-Perron operator P_S has a fixed density function. The key to proving this theorem is using the Yorke inequality relating the variations of a function and its product with the characteristic function of a subinterval. For the given mapping S, the Yorke inequality implies that a positive constant C exists such that for all functions f of bounded variation, there holds the following Lasota-Yorke inequality

$$\bigvee_0^1 P_S f \le \frac{2}{\inf_{x\in[0,1]}|S'(x)|}\bigvee_0^1 f + C\int_0^1 |f(x)|dx.$$

When Tien-Yien Li read the aforementioned Lasota-Yorke theorem, he began to think about how to numerically compute a fixed density function, the existence of which is guaranteed. He keenly sensed that the concept of functions of bounded variation and Helly's lemma in real analysis on a sequence of functions of uniform bounded variation must play a key role in proving the convergence of the numerical method. First, he defined a finite dimensional operator Q_n associated with a partition $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ of the interval $[0,1]$. The operator Q_n maps each $f \in L^1(0,1)$ into a piecewise constant function that takes the average value of f on each $[x_{i-1},x_i]$ as its value on it. Moreover, Q_n is not only a Galerkin projection that projects the L^1-space onto the subspace of piecewise constant functions, but also a Markov operator that preserves the positivity and integral of nonnegative functions. If we compose Q_n with the Frobenius-Perron operator P_S to form $P_n = Q_nP_S$, then the matrix representation of P_n restricted to the subspace of all piecewise constant functions under the canonical density functions basis is a row stochastic matrix. Utilizing Brouwer's fixed point theorem, Tien-Yien Li directly proved that P_n has a piecewise constant fixed density function for every natural number n, and with the help of the Lasota-Yorke inequality and Helly's lemma, he proved the convergence of the numerical method that he constructed for the Lasota-Yorke class of interval mappings. Specifically, he proved that the sequence $\{f_n\}$ of approximate fixed density functions contains a subsequence that converges in L^1-norm to a fixed density function f^* of the Frobenius-Perron operator.

Actually Tien-Yien Li independently invented Ulam's method. He had not known that the matrix representation of the piecewise constant approximation method that he constructed is the same as what Ulam proposed in his book, and he in fact proved Ulam's conjecture for the Lasota-Yorke class of one dimensional mappings. After he finished his paper, he was told about Ulam's work around 15 years before. For this reason, he added "a solution to Ulam's conjecture" to his paper's original title "Finite approximation for the Frobenius-Perron operator."

Ulam's method and Tien-Yien Li's solution of Ulam's conjecture initiated a new area of computational ergodic theory. In the following four decades, the computation of invariant measures has become an active branch of ergodic theory and nonlinear analysis. In the literature related to Ulam's method and its variants for the computation of invariant measures, the paper by Tien-Yien Li, published by the Journal of Approximation Theory in 1976, became one of the most essential and most widely cited papers. In addition, his thought process inspired his student Jiu Ding and collaborator Aihui Zhou to prove the convergence of Ulam's method for the Góra-Boyarsky class of multi-dimensional piecewise expanding transformations 20 years later.

Modern Homotopy Continuation Method

Solving nonlinear equations numerically is of great importance in science and engineering. Newton's method is a classic numerical scheme and its modern theory was mainly developed by the Soviet mathematician Leolid Kantorovich and his school. But this method and its variants have a shortage of mere local convergence, that is, only when the initial point is near the unknown solution can the convergence of the method be guaranteed. In other words, such nonlinear solvers lack global convergence in general. The idea of homotopy continuation helps eliminate the drawback. Suppose we want to solve the equation $f(x)=0$, where $f:\mathbb{R}^n\to\mathbb{R}^n$ is a nonlinear mapping. We first choose a "trivial equation" $f_0(x)=0$ whose solution x_0 is available, say $f_0(x)=x-x_0$. Then we define a homotopy mapping $H:\mathbb{R}^n\times[0,1]\to\mathbb{R}^n$ by

$$H(x,t)=(1-t)f_0(x)+tf(x). \tag{3}$$

The traditional homotopy continuation method, which had emerged as early as in the 1950s, is based on the assumption that the inverse image $H^{-1}(0)$ of 0 under H can be represented as a curve $(x(t),t)\in\mathbb{R}^n\times[0,1]$ with $0\le t\le 1$, which connects the known zero point x_0 of f_0 and a zero point x^* of f.

In 1953, the Soviet mathematician D. Davidenko introduced the initial value problem $x'(t)=-H_x(x,t)^{-1}H_t(x,t)$ and $x(0)=x_0$ to numerically solve the homotopy equation, which was obtained by differentiating the identity $H(x(t),t)\equiv 0$. By numerically integrating the above initial value problem from $t=0$ to $t=1$, a zero point x^* of f can be found. However, this method has the following fatal weakness: in general, the homotopy curve $x(t)$ may not always be monotonic in t. In other words, it may turn around with respect to t and $H_x(x,t)^{-1}$ may not exist at the turning point.

But, using ideas from differential topology, the field of modern homotopy continuation methods was born from a graduate course that Tien-Yien Li took. As mentioned in Section 2, the simplest case of Brouwer's fixed point theorem is a consequence of the intermediate value theorem, but for dimension more than one, the proof of Brouwer's fixed point theorem, which was mainly due to the Dutch mathematician L. E. J. Brouwer in his 1912 proof for the case of dimension 2, is not trivial. Now, anyone who has studied algebraic topology or nonlinear functional analysis would know the famous Brouwer's fixed point theorem: a continuous mapping g from an n-dimensional closed ball $\mathbb{D}^n$ of the Euclidean space $\mathbb{R}^n$ into itself must have a fixed point. A short and beautiful proof of this theorem was given by Morris W. Hirsh in 1963 with an argument by contradiction. Suppose g is smooth and has no fixed point. Then for each $x\in\mathbb{D}^n$ let $f(x)$ be the intersection of the line segment from $g(x)$ to x extended to the sphere $\mathbb{S}^n$. It is easy to see that $f(x)=x$ if x is on the sphere. Thus, we obtain a smooth mapping from the closed ball $\mathbb{D}^n$ onto its boundary $\mathbb{S}^n$ such that its restriction to $\mathbb{S}^n$ is an identity mapping. However, differential topology tells us this is impossible.

In 1973, while Tien-Yien Li attended Professor Bruce Kellogg's course "Numerical Solutions of Nonlinear Equations" and heard the above proof of Brouwer's fixed point theorem, a marvelous idea emerged: in Hirsh's proof by contradiction, if g were to have no fixed point at all, then for the mapping f defined above, for almost all $y\in\mathbb{S}^n$ the smooth curve $f^{-1}(y)$, which is the inverse image of y under f, would have no place to reach. Thus, g must have a fixed point. However, if we admit g has fixed points in the first place, f can still be defined except on those fixed points of g. Apparently for $y\in\mathbb{S}^n$, the curve $f^{-1}(y)$ must go toward the set of fixed points of g. More precisely, let F be the nonempty set of all fixed points of a smooth mapping $g:\mathbb{D}^n\to\mathbb{D}^n$, we can define a smooth mapping $f:\mathbb{D}^n\setminus F\to\mathbb{S}^n$ from the n-dimensional manifold $\mathbb{D}^n\setminus F$ to the $(n-1)$-dimensional sphere $\mathbb{S}^n$. From Sard's theorem of differential topology, y is a regular value of f for almost all $y\in\mathbb{S}^n$. It follows that $f^{-1}(y)$ is a one dimensional manifold starting from y, that is, $f^{-1}(y)$ is a smooth curve. The other end of this curve can neither come back to the sphere nor stop inside $\mathbb{D}^n \setminus F$. Therefore it must approach the fixed point set F of g. If this curve can be numerically followed, a fixed point of g can be calculated. Under the encouragement of Professors Kellogg and Yorke, Tien-Yien Li began to implement this idea on computer.

In the next two months, he spent nearly everyday with a computer for which the data could only be inputted with cards, each time without success. The stacks of paper that the computer spit out foreshadowed the program's failure. Tien-Yien Li was not defeated; he persevered in modifying the program. He modified and fixed, taking small steps from a computing novice down the path to expertise. At last, he beheld a single sheet of output from the computer, and on that sheet was a successful computation of a Brouwer's fixed point! He finally made it! Thus, a new numerical method for computing Brouwer's fixed points was born. It also paved the way for the modern homotopy continuation method.

The revolutionary idea of the resulting Kellogg-Li-Yorke paper [2] is: as long as 0 is a regular value of the homotopy mapping H given by (3), the implicit function theorem ensures that the smooth homotopy curve must exist, and in this case the coordinates vector variable x and the parameter variable t possess the same role. They may both be viewed as functions of the curve's arc length s for instance. Therefore, regardless of whether the curve "turns back" with respect to t or not, one can numerically follow the homotopy curve and find a solution by using the predictor-corrector technique. This is an important application of modern theoretical mathematics, especially differential topology, to the field of computational mathematics.

Interestingly, Kellogg-Li-Yorke's calculation of Brouwer's fixed point was not the first time it was done. They did not know that in 1967, Yale University's economics professor Herbert Scarf reduced the equilibrium point for a model in econometrics to a fixed point problem of a continuous mapping f from an n-dimensional standard simplex into itself. According to Brouwer's fixed point theorem, such a fixed point does exist. Scarf used the so-called simplicial triangulation of the simplex and then utilized Lemke's complementarity pivoting procedure to find an approximate fixed point, resulting in a simplicial fixed point algorithm. In the 1970s, this algorithm was extended to a class of simplicial algorithms to solve systems of nonlinear equations, which became a hot research topic during that period. In 1974, when the organizing committee of the First International Conference on Computing Fixed Points with Applications held at Clemson University found out Kellogg-Li-Yorke's new method, the committee immediately provided them with two airline tickets so that they may report their findings at the conference. As Scarf wrote in the Introduction of the conference proceedings *Fixed Point Algorithms and Applications*:

> For many of us one of the great surprises of the conference at Clemson was the paper by Kellogg, Li and Yorke which presented the first computational method for finding a fixed point of a continuous mapping making use of the considerations of differential topology instead of our customary combinatorial techniques. ...

Today, Kellogg, Li, and Yorke together are widely regarded as the originators of the modern homotopy continuation method for solving nonlinear problems, and particularly, with his collaborators and students, Dr. Tien-Yien Li had contributed tremendously to this important field via his extensive and deep research on polynomial systems and algebraic eigenvalue problems.

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References

[1] J. Ding, "Legendary mathematician Tien-Yien Li," *Math. Culture* 2(3), 15–29, 2011.

[2] R. Kellogg, T.-Y. Li and J. Yorke, "A constructive proof of the Brouwer fixed-point theorem and computational results," *SIAM J. Numer. Anal.* 13(4), 473–483, 1976.

[3] A. Lasota and J. Yorke, "On the existence of invariant measures for piecewise monotonic transformations," *Trans. Amer. Math. Soc.* 186, 481–488, 1973.

[4] T.-Y. Li, "Finite approximation for the Frobenius-Perron operator, a solution to Ulam's conjecture," *J. Approx. Theory* 17, 177–186, 1976.

[5] T.-Y. Li, "The story of the Li-Yorke chaos," 13–16, 1988.

[6] T.-Y. Li and J. Yorke, "Period three implies chaos," *Amer. Math. Monthly* 82(1), 985–992, 1975.

[7] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Math. 8, Interscience, 1960.