Open Problems

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—*The Editors*

Problem 2020001 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

The classification of Einstein metrics which are not Kähler has been difficult because the number of variables is large. Kähler metric is determined up to the Kähler class and the potential and therefore by one single function.

One can enlarge the class of Kähler metrics by conformal change. It would be interested to figure out what class of Einstein metrics are actually conformally Kähler. Four manifolds are the first class of manifolds that show some rigidity. The first example was due to N. Hitchin [1] who showed that the only metric over K3 surfaces are the Ricci-flat Kähler-Einstein metric constructed by me. There are much more examples, and a recent example is given by Peng Wu [2] where he proved that a simply connected four dimensional manifold with positive scalar curvature with $\det(W^+) > 0$ is conformally Kähler, where W^+ is the self-dual Weyl curvature. Claude LeBrun [3] gives a different proof recently. He also proved that four-dimensional simply connected Ricci-flat manifold must be flat or the Calabi-Yau manifold if the Weyl tensor is not zero everywhere. It would be interesting to find similar theorem for Einstein metric with negative scalar curvature. The question is to find a systematic way to classify Einstein metrics which are conformally Kähler.

Reference

[1] N. Hitchin, *Compact four-dimensional Einstein manifolds*, J. Differ. Geom. **9** (1974), 435–441.

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[2] P. Wu, *Einstein four-manifolds with self-dual Weyl curvature of nonnegative determinant*, IMRN (201910).

[3] C. LeBrun, *Einstein manifolds, self-dual Weyl curvature*, and conformally Kähler geometry, arXiv:1908.01881.

Problem 2020002 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

There is an Atiyah-Singer mod 2 index theorem [1] for five-dimensional manifolds which are used extensively by works in condensed matter physics community. It is related to the Stiefel-Whitney classes w_2 and w_3 but the argument is global. Is there a local version of such a mod 2 index theorems?

Reference

[1] M.F. Atiyah and I.M. Singer, *The index of elliptic operators. V*, Ann. Math. Second Series **93** (1971), 139–149.

Problem 2020003 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Let X be a Riemann surface of genus $g \geq 2$. We denote by $\mathcal{C}$ the space of all complex structures on X. Let $\mathrm{Diff}(X)$ be the group of diffeomorphisms on X and $\mathrm{Diff}^0(X)$ be the path-connected component of the identity. The Teichmüller space of X is defined to be

$$\mathrm{Teich}(X) := \mathcal{C}/\mathrm{Diff}^0(X)$$

The quotient $\mathrm{MC}(X) := \mathrm{Diff}(X)/\mathrm{Diff}^0(X)$ is called the mapping class group of X. By choosing a base for the quadratic holomorphic differentials, one can prove, using the Teichmüller theorem, that the $\mathrm{Teich}(X)$ has a complex structure and homeomorphic to the ball with real $6g-6$ dimension with the sphere as a boundary.

Would it be possible to holomorphically embed the Teichmüller space of a Riemann surface of genus g into a bounded domain in $\mathbb{C}^{3g-3}$ so that the mapping class group is represented as a discrete subgroup of complex linear fractional transformations acting on the bounded domain?

We can also consider meromorphic maps rather than holomorphic ones. More concretely, does there exist a meromorphic mapping from $\mathrm{Teich}(X)$ onto a subdomain in $\mathbb{C}P^{3g-3}$ so that $\mathrm{MC}(X)$ is realized by a discrete group of the group of *birational transformations* of $\mathbb{C}P^{3g-3}$?

Problem 2020004 (Differential Geometry). *Proposed by Michael McBreen (Harvard University), Artan Sheshmani (Harvard University) and Shing-Tung Yau (Harvard University).*

There have been conjectures relating quantizations of a symplectic resolution X to the enumerative geometry of a symplectic dual resolution $X^!$. In [1], we investigated a specific enumerative problem on X, namely the Betti numbers of the moduli of twisted hypertoric quiver sheaves on a rational curve, which one may think of as a kind of refined Donaldson-Thomas invariant.

We define a symplectic ind-scheme $\widetilde{\mathscr{L}}X$, which we view as a model of the universal cover of the loop space of X. We identify its dual $\widetilde{\mathscr{L}}X^!$ with a periodic analogue $\mathscr{P}X^!$ of $X^!$. Our main result expresses the generating function for twisted DT invariants of the hypertoric space X as a certain graded trace of an indecomposable tilting module over the quantization of $\mathscr{P}X^!$.

We should note that to avoid dealing with the potential pathologies of infinite dimensional spaces, we work extensively with finite dimensional and finite type approximations to $\widetilde{\mathscr{L}}X$ and $\mathscr{P}X^!$, and limits of these. It would be interesting to work directly on the limit spaces, and develop in this context the full analogues of the finite dimensional theory – module categories and Koszul dualities. A second interesting direction is to replace the hypertoric space X by a Nakajima quiver variety, or more generally the Higgs branch of a non-abelian reductive group G.

Reference

[1] M. McBreen, A. Sheshmani and S.-T. Yau, *Twisted Quasimaps and Symplectic Duality for Hypertoric Spaces*, arXiv:2004.04508.

Problem 2020005 (Differential Geometry). *Proposed by Shinobu Hosono (Gakushuin University), Tsung-Ju Lee (Harvard University), Bong Lian (Brandeis University) and Shing-Tung Yau (Harvard University).*

In [1, Conjecture 6.3], we conjectured that the mirror of a $K3$ family is given by certain double covers over a del Pezzo surface of degree 6, which is a blow-up of three torus invariant points on $\mathbb{P}^2$. In [2], we studied its further generalizations.

Consider a nef-partition $(\Delta,\{\Delta_i\}_{i=1}^r)$ and its dual nef-partition $(\nabla,\{\nabla_i\}_{i=1}^r)$ in the sense of Batyrev and Borisov (see [2, §1.2]). Let $\mathbf{P}_\Delta$ and $\mathbf{P}_\nabla$ be the toric varieties defined by Δ and ∇. Let $X\to\mathbf{P}_\Delta$ and $X^\vee\to\mathbf{P}_\nabla$ be maximal projective crepant partial resolutions (MPCP resolutions for short hereafter) of $\mathbf{P}_\Delta$ and $\mathbf{P}_\nabla$. The nef-partitions on $\mathbf{P}_\Delta$ and $\mathbf{P}_\nabla$ determine nef-partitions on X and $X^\vee$. Let $E_1,\ldots,E_r$ and $F_1,\ldots,F_r$ be the sum of toric divisors representing nef-partitions on X and $X^\vee$, respectively. We will assume that X and $X^\vee$ are both *smooth*. Said differently, both Δ and ∇ admit a *regular triangulation*.

Let $s_j\in\mathrm{H}^0(X,2E_j)$ be a smooth section and Y be the double cover over X branched along $s_1\cdots s_r$. Deforming the sections s_j yields a family of Calabi-Yau double covers over X, which is parameterized by a suitable open set in the product of $\mathrm{H}^0(X,E_j)$. We now elaborate how to define *a partial gauge fixing* for such a family (see [2, §2.1]).

A *partial gauge fixing* is a decomposition of the section s_j into a product of a canonical section of E_j and a smooth section of E_j. In other words, $s_j=s_{j,1}s_{j,2}$ with $s_{j,k}\in\mathrm{H}^0(X,E_j)$ such that $\mathrm{div}(s_{j,1})\equiv E_j$ and $\mathrm{div}(s_{j,2})$ is smooth. The original double cover family will restrict to a *subfamily* parametrized by

$$V\subset\mathrm{H}^0(X,E_1)\times\cdots\times\mathrm{H}^0(X,E_r).$$

A parallel construction can be applied on the dual side. Let $\mathcal{Y}\to V$ and $\mathcal{Y}^\vee\to U$ be partial gauge fixings for those families. Let Y and $Y^\vee$ be the fiber of these families.

We observe that Y and $Y^\vee$ form a topological mirror pair.

Theorem. *We have* $\chi_{\mathrm{top}}(Y)=(-1)^n\chi_{\mathrm{top}}(Y^\vee)$, *where* $n=\dim Y$ *and* $\chi_{\mathrm{top}}(-)$ *denotes the topological Euler characteristic.*

Since Y and $Y^\vee$ are *orbifolds*, the Hodge numbers $h^{p,q}(Y)$ are well-defined. Moreover, by construction, $X\setminus B$ is affine, where B is the branched locus of the cover $Y\to X$. It follows that $h^{p,q}(Y)=h^{p,q}(X)$ for all p,q with $p+q\neq n$. In particular, when $n=3$, we can prove

Theorem. *We have* $h^{p,q}(Y)=h^{3-p,q}(Y^\vee)$ *for all* p,q.

Based on these results, we propose the following conjecture, which can be served as a generation of [1, Conjecture 6.3].

Conjecture. Y *is mirror to* $Y^\vee$.

We shall emphasize that none of Y and $Y^\vee$ is smooth. The conjecture is served as an extension of the classical mirror correspondence to *singular Calabi-Yau varieties*.

Reference

[1] S. Hosono, B. Lian and S.-T. Yau, *K3 surfaces from configurations of six lines in $\mathbb{P}^2$ and mirror symmetry II—λ_{K3}-functions*, IMRN (201903).

[2] S. Hosono, T.-J. Lee, B. Lian and S.-T. Yau, *Mirror symmetry for double cover Calabi-Yau varieties*, arXiv:2003.07148.

Problem 2020006 (Differential Geometry). *Proposed by Shing-Tung Yau (Harvard University), Quanting Zhao (Central China Normal University) and Fangyang Zheng (Chongqing Normal University).*

We call a Hermitian manifold (M^n, g) whose Strominger connection is Kähler-like, in the sense that its curvature tensor obeys all the symmetries of the curvature of a Kähler manifold, a *Strominger Kähler-like* manifold, or a *SKL* manifold in short.

It has been proved in [1, Theorem 6 and 7] that, if (M^n, g) is a compact SKL manifold with g not Kähler, then M^n cannot admit any balanced metric, or more generally, it can not admit any strongly Gauduchon metric. Furthermore, it has been shown in [1, Theorem 5] that, when $n = 2$, the SKL condition is equivalent to the Vaisman condition, which means that the Lee form is parallel under the Riemannian (Levi-Civita) connection.

In [2], we proved the uniqueness of SKL metrics within a conformal class. Note that since SKL metrics are Gauduchon by [1, Theorem 8], so when M^n is compact, any SKL metric on M^n will be unique (up to constant multiple) within its conformal class. The same is true for Riemannian Kähler-like or Chern Kähler-like metrics as proved in [3, Theorem 4]. When M^n is not compact, however, Riemannian Kähler-like or Chern Kähler-like metrics are no longer unique within a conformal class, but SKL metrics are, provided that the dimension is at least 3:

Theorem. *Let (M^n, g) be any Hermitian manifold with $n \geq 3$. Then within the conformal class of g, there is at most one SKL metric, up to constant multiples.*

As mentioned above, in the case of $n = 2$, a SKL metric is actually Vaisman, namely a Hermitian metric which is locally conformal Kähler with its (real) Lee form parallel under the Levi-Civita connection. Hence, on the universal cover, any SKL metric g on M^2 is conformal to a Kähler metric, and thus is not unique within its conformal class when g is not Kähler. When $n \geq 3$, however, the above theorem implies that any non-Kähler SKL metric is never locally conformal Kähler. We speculate that there cannot exist any other locally conformal Kähler metrics as well:

Conjecture. *If (M^n, g) is a compact SKL manifold with g not Kähler and $n \geq 3$, then M^n does not admit any locally conformal Kähler metric.*

As a partial evidence, we prove the following:

Theorem. *Let (M^n, g) be a compact SKL manifold with g not Kähler. If $n \geq 3$, then M^n cannot admit any Vaisman metric.*

Reference

[1] Q. Zhao and F. Zheng, *Strominger connection and pluriclosed metrics*, arXiv:1904.06604.

[2] Q. Zhao, F. Zheng and S.-T. Yau, *On Strominger Kähler-like manifolds with degenerate torsion*, arXiv:1908.05322.

[3] B. Yang and F. Zheng, *On curvature tensors of Hermitian manifolds*, arXiv:1602.01189, Comm. Anal. Geom. **26** (2018), 1193–1220.