Lie Theory, from Lie to Borel and Weil

by Wilfried Schmid*†

In 1913, Hermann Weyl published a monograph, “Die Idee der Riemannschen Fläche” [28], which for the first time introduced the notion of a manifold. Basic notions of topology had been established by Poincaré (“analysis situs”) around the turn of the century, and Weyl used those freely. In the monograph, Weyl concentrated on Riemann surfaces, of course – i.e., one dimensional complex manifolds – but he also gave various other examples of manifolds, such as the Möbius strip.

Lie developed the notion of a Lie group – or what he called “Transformationsgruppen” – in the years 1874–1893 [15], long before Weyl’s definition of a manifold. How could he do that?

Before answering this question, I shall briefly describe Lie’s career. After studying in Oslo (then called Kristiana) without a strong focus, he published his first mathematical paper in 1869, in Crelle’s Journal. Awarded a generous scholarship for travel, he went to the University of Berlin, where he met, and became friends with, Felix Klein. One can almost say that Klein and Lie then consciously mapped out their research programs – both viewing groups as a powerful tool in differential equations and geometry. Lie returned as professor to Oslo, Klein became professor in Erlangen, then moved to Leipzig, and later Göttingen. Lie became Klein’s successor in Leipzig in 1886. Three years earlier, Klein’s student Friedrich Engel (1861–1941) had become Lie’s “Assistent”. That was a beginning academic position in Germany, roughly comparable to that of an assistant professor in the US today, but attached to one full professor, to whom the Assistent owed certain responsibilities. Engel had significant influence on Lie – Lie’s papers became far more rigorous after Engel joined him.

In the following I shall concentrate on the definition of (what are now called) Lie groups. Lie, of course, viewed them as an important tool and pursued many applications to geometry and differential equations. For Lie, a transformation group is a pair of real or complex analytic maps, only locally defined on open sets in $\mathbb{R}^m \times \mathbb{R}^n$ or $\mathbb{C}^m \times \mathbb{C}^n$,

$$y = F(x,a) \quad (\text{taking values in } \mathbb{R}^m,\ \text{resp. } \mathbb{C}^m)\,,$$

with a law of composition $c = \phi(a,b)$ satisfying

$$\begin{aligned} &F(F(x,a),b) = F(x,c) \ \text{ when } \ c = \phi(a,b)\,, \quad \text{and} \\ &\qquad \phi(a,\phi(b,c)) = \phi(\phi(a,b),c) \ \text{ whenever defined} \end{aligned} \tag{1}$$

Usually, but not always, a “unit” and “inverses” are required to exist:

$$\begin{aligned} &\phi(e,a) = \phi(a,e) = a \ \text{ for each } a\,, \ \text{ and } \ F(x,e) \equiv x; \\ &\text{also, for each } a\text{, there exists } b \text{ such that} \\ &\phi(a,b) = \phi(b,a) = e. \end{aligned} \tag{2}$$

In current terminology, one would call this the germ of a Lie group acting on the germ of a manifold.

As a special case, one gets the group acting on itself – in which case $F = \phi$ – either by left or by right translation. With $F(x,a)$ and $\phi(a,b)$ as above, the “infinitesimal transformations” are

$$X_i \;=\; \sum\nolimits_j \frac{\partial \phi_j(a,b)}{\partial b_i}\Big|_{b=e} \frac{\partial F}{\partial a_j}.$$

Lie formulated “three fundamental theorems”. The first is the infinitesimal version of the group law (1).

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The second asserts that the linear span of the X_i are closed under taking commutators

$$[X_i, X_j] = X_i X_j - X_j X_i = \sum_k c_{i,j,k} X_k,$$

in other words, they span a Lie algebra (Weyl's terminology). Lie's third theorem asserts that every system of structural constants $c_{i,j,k}$ satisfying the obvious condition (skew symmetry, Lie triple identity) arises from a transformation group.

Lie's first, early proof of his third theorem associates to each system of structural constants $c_{i,j,k}$ vector fields

$$X_i = \sum_{j,k} c_{j,i,k}\, x_j \frac{\partial}{\partial x_k}$$

When the $c_{i,j,k}$ satisfy the triple identity, the X_i span a Lie algebra of vector fields on $\mathbb{R}^m$, in modern terminology, the Lie algebra of the adjoint group. At the time, Lie overlooked the possibility of a nontrivial center, in which case the group to be constructed does not coincide with the adjoint group. Years later, he realized the Lie algebra to be constructed, and thereby indirectly the corresponding transformation group, as a Lie algebra of functions under Poisson bracket. To this day, constructing the Lie group corresponding to a given Lie algebra is not entirely trivial.

Many aspects of Lie groups and actions of Lie groups on manifolds that are obvious today were not in Lie's time. As just one example, the fact that the left translation action of a Lie group on itself commutes with the right translation action could not be more obvious today. It was not obvious to Lie, who thought in terms of his own definition. Lie credits Engel with observing that the two actions commute.

As another example, Lie tried to do as much as he could without requiring the existence of an identity. Today that seems pointless. The role of the "infinitesimal translations" X_i, though clear enough to Lie, became far more transparent after Hermann Weyl introduced the notion of a Lie algebra.

Friedrich Schur (1856–1932) – not to be confused with Issai Schur, to whom he was unrelated – had been a student of Karl Weierstrass in Berlin. It showed: Weierstrass was perhaps the first well known mathematician to insist on complete rigor in analytic arguments, and transmitted that insistence to his students. Schur started out as "Assistent" to Klein in Leipzig. In 1889–1893, not long after Lie had defined the notion of a transformation group, Schur wrote three papers [24–26], all published in Mathematische Annalen, which laid out an alternate, rigorous, approach to the subject. His starting point is an observation, first made by Lie, that there is a "canonical" choice of parameters a for Lie's map $\phi(a,b)$, namely those for which straight lines $t \mapsto ta$ correspond to one parameter subgroups. In terms of those coordinates, $\phi(a,b)$ can be expressed as a power series, absolutely and uniformly convergent in a neighborhood of the origin.

The coefficients of this power series depend polynomially on Lie's structural constants $c_{i,j,k}$, but are otherwise universal. In effect, this is the Baker-Campbell-Hausdorff formula in disguise! He also treats the case of the map $F(x,a)$ in Lie's definition, and thus rigorously proves Lie's third theorem, almost simultaneously with Lie. Schur also addresses the differentiability requirements for transformation groups. Lie assumes real (or complex) analyticity, but Schur observes that C^2 differentiability is enough. This is the origin of Hilbert's fifth problem, of course: even being locally Euclidean is enough. The difference in styles between Lie and Schur is striking. In Engel's obituary for Schur [13], he wrote that Lie and Schur had very different ideas of what was easy and what was not. Schur was recognized by his contemporaries – he received an honorary Doctorate from the Technical University of Karlsruhe, for example – but in later times became underappreciated.

Killing (1847–1923) – like Schur a student of Weierstrass – wrote a series of papers around 1890, contemporaneously with Schur, in the very early days of Lie theory [14]. In these he established, or came close to establishing, many structural results about (what we now call) Lie algebras: criteria for semsimplicity (Killing introduced the term "halbeinfach", i.e., semisimple), the decomposition of reductive Lie algebras into simple factors and the center, but most impressively by far, a classification of simple Lie algebras over $\mathbb{C}$ barely a few years after the notion (but not the name) of Lie algebra had been introduced. His classification contained a minor error and the exposition is often obscure. But the classification of simple Lie algebras was so totally unexpected that it could not possibly have been "guessed". After his PhD in Berlin, Killing taught at a "Gymnasium" (i.e., High School) before becoming Professor in Münster in 1892.

If Friedrich Schur was underappreciated, Killing has been almost ignored. Lie made a habit of reviewing the work of others on "my theory of groups". He was especially savage towards Killing: "The correct theorems (in a particular paper of Killing) are due to Lie, the false ones due to Killing". And "... (several papers of Killing) contain not so many results that are correct and new. Proved, correct, and new are even fewer".

Élie Cartan is generally credited with the classification of simple Lie algebras. He did put Killings classification over $\mathbb{C}$ on a solid footing in his thesis, and classified simple Lie algebras over $\mathbb{R}$, which Killing did not touch at all. But Killing deserves far more recognition than he got. Killing introduced the notion of a root, i.e., a root of the characteristic equation

$$\det(\operatorname{ad}X - \alpha) \;=\; 0\,, \quad \text{where} \quad \operatorname{ad}X(Y) = [X,Y]\,,$$

but in term's of Lie's notation. Then, as now, the study of roots was key to understanding of the structure of a simple Lie algebra.

Campbell (1862–1924) studied at Oxford, and became a fellow and then tutor there. In 1887, he published two papers on what is now known as the Campbell-Baker-Hausdorff formula [5]. The opening paragraph of the second neatly describes their point of view: "If x and y are operators which obey the ordinary laws of algebra, we know that $e^y e^x = e^{y+x}$. I propose to investigate the corresponding theorem when the operators obey the distributive and associative laws, but not the commutative". In doing so, he treated the equation

$$e^Y e^X \;=\; e^Z\,, \quad \text{with } Z \;=\; Z(X,Y)\,, \tag{3}$$

as a formal identity. By long calculations, he expressed Z in terms of X, Y, and repeated brackets, with universal coefficients. He cites Schur's paper, because his calculations lead to the same coefficients as occur in Schur's proof of Lie's third theorem. But remarkably, he says nothing about the logical connection. Unlike Schur, he does not address the question of convergence.

Lie had already used the exponential series in the context of Lie groups. But Campbell was the first to use the exponential *notation* in the context of Lie groups. In 1903, Campbell published an "Introductory Treatise on Lie's Theory of Finite Continuous Transformation Groups" [6], which was instrumental in introducing Lie's ideas to the English speaking mathematical world.

Poincaré (1854–1912) had a "modern" interest in group actions in geometry, just one of his very many mathematical interests, of course [17]. The problem of characterizing physical space by suitable axioms was one of the grand themes of mathematics in his time. In "Sur les hypothèses fondamentales de la Géométrie" (1887) [18], he approached the problem by observing that (in the plane) Euclidean, hyperbolic, and elliptic geometry have one feature in common: their groups of motion act transitively, with one-dimensional isotropy groups. He used Lie's infinitesimal methods, but was reproached by Lie – in uncharacteristically gentle terms – for being unaware of Lie's own classification of (local, of course) group actions of three dimensional transformation groups on the plane. It may not be an accident that he waited until Lie's death in 1899 to write about "transformation groups" in Lie's sense.

Poincaré published an announcement, outlining the proof of Lie's third theorem (existence of a "transformation group" when given its "infinitesimal group") in 1899 [19], around the time of Lie's death. The details followed a few months later, in a paper dedicated to Stokes [20]. By then, he knew about earlier work by Friedrich Schur and Campbell: he cites both, and comments on the overlap. In effect, he introduced what is now called the "universal enveloping algebra", as the algebra spanned by all formal noncommuting products of generators of the "infinitesimal transformation group" – i.e., Lie algebra – of the group in question, modulo the relations forced by the relation

$$XY \;-\; YX \;-\; [X,Y] \;=\; 0 \tag{4}$$

for all generators X, Y. Repeated application of this identity makes any non-commutative polynomial equivalent to one whose homogeneous components are symmetric in the variables.

The crux of the Poincaré-Birkhoff-Witt theorem is the assertion that there are no "hidden relations", or in current terminology, that the associated graded algebra is isomorphic to the symmetric algebra of the Lie algebra in question. His proof is complicated and leaves much unsaid. However, the idea becomes clear if one works it out in the case of a Lie polynomial of degree three. Poincaré's proof seems to have been forgotten for almost 40 years. Garrett Birkhoff [2] and Ernst Witt [30] published proofs independently in 1937, in the most general setting – for possibly infinite dimensional Lie algebras, and in any characteristic of the ground field. Only Cartan-Eilenberg's book on homological algebra affixed Poincaré's name to the theorem. For Poincaré, the result was not an exercise in algebra. Rather, he used it to make sense of Campbell's formal identity

$$e^Y e^X \;=\; e^Z\,; \tag{5}$$

after all, at the time a global Lie group in which one could make sense of (5) had not yet been constructed.

What is the connection between the Poincaré-Birkhoff-Witt theorem and the relation (5)? The differentiated version of (5) can be written, in symbolic notation, as

$$e^Y e^{\delta X} \;=\; e^{Y+\delta Y}\,, \quad \text{with} \quad \delta X \;=\; \frac{1 - e^{-\operatorname{ad}Y}}{\operatorname{ad}Y}\,\delta Y\,; \tag{6}$$

here $\operatorname{ad}Y(Z) = [Y,Z]$ is (now) standard notation, and

$$\frac{1 - e^{-\operatorname{ad}Y}}{\operatorname{ad}Y} \;=\; \sum\nolimits_{n\geq 0}\, (-1)^n\, \frac{(\operatorname{ad}Y)^n}{(n+1)!}\,.$$

This series evidently has a positive radius of convergence. Poincaré proves (6) by means of a residue argument, then uses (6) to show that the quantity on the right in (5) is the exponential of a quantity $Z = Z(X,Y)$ whose power series converges for all small X and Y.

Although his argument implies Campbell's universal formula for the coefficients, Poincaré does not pursue this.

To follow the chronological order, Baker (1866–1956) won a scholarship to St. Johns College, Cambridge, became a fellow there, and in 1914 became Lowndean Professor of Astronomy at Cambridge. One of his earliest publications was an 1905 article on "Alternants and continuous groups" [1] in the Proceedings of the London Mathematical Society; "alternants" being then used as terminology for the Lie bracket of X and Y, which Baker denoted by (X,Y). After referring to Friedrich Schur and Campbell, he derives explicit formulas for Z in the relation

$$e^X e^Y = e^Z, \tag{7}$$

in terms of X and Y – as had Campbell (and implicitly, Schur) – but much more efficiently and elegantly than Campbell. That was his last publication on Lie groups; at his time, he was better known as author of textbooks on algebraic geometry.

After the 1899 announcement and the detailed paper on Lie's third theorem, on (what is now known as) the Baker-Campbell-Hausdorff formula, and on the Poincaré-Birkhoff-Witt theorem, Poincaré returned to the subject of transformation groups in two papers in 1901 [21] and 1908 [22]. His main analytic tool was the residue calculus, which he used to re-prove some of Killing's results on the root space decomposition of a semisimple Lie algebra. More significantly, he attempted to understand global questions about semisimple Lie groups, before Hermann Weyl's notion of manifold had been formalized. That makes these papers frustrating for a present-day reader. In the semisimple case, he saw that the multiplication rule given by the Campbell-Baker-Hausdorff formula could be analytically continued. In effect, he treated the Lie algebra as (what we now regard as) the universal covering group, with a multi-valued multiplication rule, which ramifies along certain hypersurfaces.

One of his examples is instructive. He considers two "infinitesimal rotations" X and Y for the rotation group $SO(3)$, about axes ℓ_X, ℓ_Y in general position. He normalizes X so that e^X is a full rotation through an angle 2π. As element of $SO(3)$, e^X is the identity, but not in (what he regarded as) the parameter group in Lie's sense, i.e., the faux universal covering group. Since e^Y represents a rotation about the axis $e^X\,\ell_Y = \ell_Y$, through the same angle as

$$e^X e^Y e^{-X} = e^Y,$$

regardless how Y is scaled. On the other hand, in general $e^Y\,\ell_X \neq \ell_X$, hence

$$e^{-Y} e^X e^Y \neq e^X, \quad \text{so} \quad e^X e^Y e^{-X} \neq e^Y$$

in the parameter group. That's a paradox, but not a contradiction: the group laws are only required to hold locally, near the identity. A plausible explanation had to await the understanding of the universal covering group, which became possible only after Weyl's definition of manifold in 1913.

Élie Cartan (1869–1951), the son of a village blacksmith, entered the elite École Normale Supérieure in 1888, graduating in 1891. Among his teachers were Hermite, Darboux, Picard, Goursat, and Poincaré. After one year of compulsory military service, he returned to the ENS for two more years of graduate study. He met Lie in 1892, when Lie was visiting Paris at the invitation of Darboux. Cartan received his doctorat d'État in 1894. The subject of his thesis was a rigorous reworking of Killing's classification of the simple Lie algebras over $\mathbb{C}$ [7]. After junior positions in Montpellier and Lyon, Cartan became professor in Nancy and then Paris. Chern was his student (though Cartan was not Chern's PhD advisor). Following his retirement in 1940, Cartan taught classes at the École Normale Supérieure des Jeunes Filles. Perhaps of interest: Josiane Serre, wife of Jean Pierre, was that institution's last Director, before the ENS became coeducational in 1985; four years later she served for one year of interim director of the ENS.

As was already noted, Cartan's thesis rigorously classified the simple complex Lie algebras. The key to understanding the structure of a simple Lie algebra $\mathfrak{g}$ is the notion of *Cartan subalgebra* – the centralizer of a generic $X \in \mathfrak{g}$. Cartan subalgebras are abelian, and any two of them are conjugate under the automorphism group $\mathrm{Aut}(\mathfrak{g})$. Since $\mathfrak{g}$ is simple, Lie's original proof of his "third fundamental theorem" applies, and $\mathfrak{g}$ is the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$. In particular, any Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ acts on $\mathfrak{g}$ faithfully, and

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\oplus_{\alpha\in\Phi}\, \mathfrak{g}^\alpha\right), \quad \text{with} \quad \Phi \subset \mathfrak{h}^* \quad \text{the set of roots},$$

and for $\alpha \in \Phi$,

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g} \mid [H,X] = <\alpha, H> X \ \text{for all}\ H \in \mathfrak{h}\}$$

is the α-root space. The root spaces are one dimensional, and

$$[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \mathfrak{g}^{\alpha+\beta} \quad \text{if} \quad \alpha+\beta \quad \text{is also a root.}$$

Φ contains $\mathbb{Z}$-bases for the $\mathbb{Z}$-linear span of Φ, so-called systems of simple roots Ψ, all of them conjugate under the action of the normalizer of $\mathfrak{h}$ in $\mathrm{Aut}(\mathfrak{g})$.

Up to conjugacy, the choices of Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and of simple root system $\Psi \subset \Phi$ are unique. The $\mathbb{R}$-linear span of Ψ carries a canonical inner product, the Killing form; the possible angles between two simple roots are $(n-1)\pi/n$, $n = 2, 3, 4$ or 6, with possible ratios of squared lengths, related to the angles, equal to 1, 2, or 3. This then leads to a classification of the possible simple Lie algebras over $\mathbb{C}$. The substance of these argument can be already be found in Killing's writings, though with gaps, obscurely presented almost to the point of unintelligibly; to give just a single example of many, for Killing roots were multiple valued functions on the Lie algebra $\mathfrak{g}$, with branches along the set of non-semisimple elements of $\mathfrak{g}$. Also, unlike Killing, Cartan actually *constructed* the exceptional Lie algebras, though not in his thesis, but 15 years later, in 1909 [8].

Until Cartan, all those proving results about Lie groups followed Lie's terminology: they distinguished between the *parameter group* and the associated *transformation group*, even when the groups in question were obviously globally defined, linear groups – recall that in general, Lie groups had been defined as germs of groups, acting on germs of spaces. That changed after Weyl's definition of manifold in 1913: in his later writings, Cartan viewed Lie groups no longer as germs of groups, but as Lie groups in the modern sense. $\mathrm{Aut}(\mathfrak{g})^0$, the connected component of the identity in the automorphism group of a simple complex Lie algebra $\mathfrak{g}$, has trivial center. A rather delicate aspect of the classification of exceptional simple Lie groups is to determine the center of the universal cover of $\mathrm{Aut}(\mathfrak{g})^0$. Cartan solved that problem in a paper in 1927 [11].

The classification of simple Lie algebras over $\mathbb{R}$ is much more involved than the classification over $\mathbb{C}$. Cartan dealt with that issue in 1914, and more importantly, described their structure, as well as the structure of the corresponding Lie groups [10]. Every connected simple, real Lie group G has a unique conjugacy class of maximal compact subgroups, as proved by Cartan. Maximal compact subgroups $K \subset G$ are connected and are either semisimple – i.e., a direct product simple factors – or have a one dimensional center. Moreover, the center of G is contained in the center of K. In particular, every connected, complex, simple Lie group $G_{\mathbb{C}}$ contains a connected, simple, compact Lie group $U_{\mathbb{R}}$, because $G_{\mathbb{C}}$ can be viewed as a real group. Thus, up to conjugacy,

$$G_{\mathbb{C}} \longleftrightarrow U_{\mathbb{R}}$$

establishes a bijection between connected, complex, simple Lie groups and connected, compact, simple Lie groups.

One minor complication of the study of connected, simple, real Lie groups is the fact that they need not be linear groups, i.e., subgroups of $GL(n,\mathbb{R})$ for some n; the covering groups of $SL(2,\mathbb{R})$ provide the simplest example of this phenomenon. The problem is minor because coverings of the adjoint group of a connected, simple, real group correspond bijectively to coverings of its maximal compact subgroup $K_{\mathbb{R}}$, and those are well understood, of course.

For each linear, connected, real, simple Lie group, Cartan constructs an involution, now called the Cartan involution,

$$\theta : G \longrightarrow G, \quad \text{such that} \quad K = \{ g \in G \mid \theta g = g \}$$

is a maximal compact subgroup. The pair (G,θ) determines (G,K) and vice versa, up to isomorphism, of course. Thus, to problem of classifying the connected, real, simple Lie groups, turns into the problem of classifying involutions of connected, simple, complex Lie groups, which Cartan was able to carry out with considerable effort.

The preceding remarks, properly stated, apply also to groups G that are semisimple, rather than simple. They imply important structural information about real, semisimple Lie groups G.

The Cartan involution θ induces an involution on the Lie algebra $\mathfrak{g}$ of G, which is denoted by the same letter, and

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \text{with } \mathfrak{p} = (-1) - \text{eigenspace of } \theta,$$

and $\mathfrak{k} = (+1)$-eigenspace, which is then also the Lie algebra of a maximal compact subgroup. In this situation,

$$\exp : \mathfrak{p} \longrightarrow G$$

is a diffeomorphism onto its image, and

$$K \times \mathfrak{p} \ni (k,X) \longmapsto k \cdot \exp(X)$$

establishes a homeomorphism onto G. That is called the Cartan decomposition of G. In particular, $G/K \cong \mathfrak{p}$ as topological space.

This is a brief summary of Cartan's work on Lie groups – an important part of, but far from all, of his work.

Hermann Weyl (1885–1955) was born near Hamburg. He studied mathematics in Munich and Göttingen, where he received his PhD in 1908 as student of David Hilbert. After holding junior positions in Göttingen, he became professor at the Eidgenössische Technische Hochschule (ETH) in Zürich, in 1913, the same year in which his book “Die Idee der Riemannschen Fläche” appeared [28]. He became Hilbert's successor in Göttingen in 1930. The Institute for Advanced Study in Princeton offered him a professorship in 1932, as Hitler's rise to power in Germany was imminent. He accepted, then revoked the acceptance. When Hitler became German Chancellor in 1933, exposing his Jewish wife to imminent danger, the Institute renewed the offer, which Weyl now gratefully accepted.

While Lie theory and representation theory was an important part of Weyl's work, he made many other contributions to other areas of mathematics and mathematical physics. After his retirement in 1951, he divided his time between Princeton and Zürich.

In 1927, Weyl and his student Fritz Peter published their proof of (what is now called) the Peter-Weyl theorem, patterned after, and a generalization of, the Schur orthogonality relations for finite groups [16]. As an aside, I should mention that Fritz Peter (1899–1949) was never again heard from in mathematics; he became a teacher at the "Gymnasium Schloss Salem", then an elite boarding school, near the German-Swiss border, but beset by scandals in recent years.

If G is a compact Hausdorff group, with Haar measure dg normalized to have total measure 1, the Peter-Weyl theorem asserts that

$$L^2(G) \;\simeq\; \textstyle\sum_{\iota\in\widehat{G}} V_\iota \otimes V_\iota^* \quad \text{(Hilbert space direct sum)}.$$

Here $\widehat{G}$ is the set of irreducible unitary representations of G modulo isomorphism, (π_ι, V_ι) the representation indexed by ι, and (π_ι^*, V_ι^*) the dual representation. Under the isomorphism the left, respectively right action of G on $L^2(G)$ corresponds to the action π_ι on the V_ι, respectively π_ι^* on the V_ι^*. It is an isometry when the inner products on the $V_\iota \otimes V_\iota^*$ are suitably renormalized.

Under the Peter Weyl isomorphism, the tensor product $u_\iota \otimes v_\iota^* \in V_\iota \otimes V_\iota^*$ corresponds to the function

$$f_{u_\iota \otimes v_\iota^*}(g) \;=\; <\pi_\iota(g^{-1})u_\iota\,,\, v_\iota^*> .$$

As a formal consequence of the Peter-Weyl theorem, the *character* of a finite dimensional irreducible unitary representation π of the compact Hausdorff group G,

$$\chi_\pi(g) \;=\; \operatorname{tr} \pi_\pi(g)\,,$$

determines the representation π up to isomorphism. Any finite dimensional representation π of G can be made unitary (Weyl's *unitary trick*). Thus, to describe the irreducible finite dimensional representations of G, it suffices in principle, at least, to describe the irreducible characters.

Chronologically, though not logically, the Weyl character formula [29] precedes the Peter-Weyl theorem. If G is a connected, compact Lie group, any two maximal compact tori $T \subset G$ are conjugate. Choose one. Any $g \in G$ is conjugate to some $t \in T$. It therefore suffices to know the restriction of the irreducible characters to T.

Weyl shows that the identity component $N_G^0(T)$ of the normalizer of T coincides with the centralizer $Z_G(T)$, and that the centralizer in turn coincides with T. Hence the *Weyl group* of (G,T),

$$W = N_G(T)/N_G^0(T) = N_G(T)/Z_G(T) = N_G(T)/T\,,$$

is a finite group which acts faithfully on T, hence also on the Lie algebra $\mathfrak{t}$, as well as on the *weight lattice*

$$\Lambda \;=\; \{\,\lambda \in i\mathfrak{t}^* \mid \lambda \text{ lifts to a character } e^\lambda : T \to \mathbb{C}^*\,\}.$$

As mentioned earlier, the set of roots $\Phi \subset \Lambda$ consists of the non-zero weights by which T acts on the complexified Lie algebra,

$$(\mathfrak{g}/\mathfrak{t}) \otimes_{\mathbb{R}} \mathbb{C} = \oplus_{\alpha\in\Phi}\, \mathfrak{g}^\alpha,$$
$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g} \mid \mathrm{Ad}\, t(X) = e^\alpha(t)X \text{ for } t \in T\}$$

The *root spaces* $\mathfrak{g}^\alpha \subset \mathfrak{g}$ are one-dimensional. The *Weyl integration formula* asserts that for any $f \in C(G)$,

$$\int_G f(g)\,dg$$
$$= \frac{1}{\#W}\int_T\int_G f(gtg^{-1}) \textstyle\prod_{\alpha\in(\Phi/\pm)}(e^{\alpha/2}(t) - e^{-\alpha/2}(t))^{-2}\,dg\,dt\,.$$

To simplify the following statements very slightly, I shall assume that the compact Lie group G is connected, semisimple, and simply connected. One calls $\lambda \in \Lambda$ *regular* if λ is not perpendicular to any $\alpha \in \Phi$. Then for each regular $\lambda \in \Lambda$, there exists a unique (up to isomorphism) irreducible, finite dimensional representation π_λ whose character χ_λ, when restricted to T, is given by the formula

$$\begin{aligned} \chi_\lambda(t) \;&=\; (\Delta(t))^{-1} \textstyle\sum_{w\in W} \epsilon(w)\, e^{w\lambda}(t) \quad \text{with} \\ \epsilon(w) \;&=\; \operatorname{sgn}(\det\{w : \mathfrak{t} \to \mathfrak{t}\}) \quad \text{and} \\ \Delta(t) \;&=\; \textstyle\prod_{\alpha\in\Phi,\,(\alpha,\lambda)>o} (e^{\alpha/2}(t) - e^{-\alpha/2}(t))\,. \end{aligned}$$

Every irreducible representation of G is isomorphic to one of these, and $\pi_{\lambda_1} \cong \pi_{\lambda_2}$ if and only if $\lambda_2 = w\lambda_1$ for some $w \in W$.

In particular then, the set of irreducible representations of the compact, connected, simply-connected group G can be naturally identified with

$$W \backslash \{\,\lambda \in \Lambda \mid \lambda \text{ is regular}\,\}.$$

The proof of the Weyl character formula depends on Weyl's structural statements about compact Lie groups mentioned earlier, the Peter-Weyl theorem, the Weyl integration formula, and the observation that the restriction of an irreducible character χ_λ to T must be a linear combination of characters e^μ of T with positive integral coefficients; the theorem follows readily from these ingredients.

Cartan had already classified the irreducible finite dimensional representations of a complex semisimple Lie algebra $\mathfrak{g}$ in 1913 [9]. In terms of the Weyl character formula, the classification can be stated as follows. Let V_λ be the irreducible representation with character χ_λ, as described above. Then $\mu \in \Lambda$ is called a *weight* of V_λ if the μ*-weight space* V_λ^μ,

$$V_\lambda^\mu =_{\text{def}} \{\, v \in V_\lambda \mid t\,v = e^\mu(t)v \text{ for all } t \in T\,\} \quad \text{is nonzero}.$$

Recall that the parameter $\lambda \in \Lambda$ of V_λ must be regular. Hence the root system Φ can be expressed as the disjoint union

$$\Phi = \Phi_\lambda^+ \cup (-\Phi_\lambda^+), \quad \text{where } \Phi_\lambda^+ = \{\alpha \in \Phi \mid (\lambda, \alpha) > 0\}.$$

Define

$$\rho_\lambda = \tfrac{1}{2} \textstyle\sum_{\alpha \in \Phi_\lambda^+} \alpha.$$

It is not difficult to show that

$$(\lambda - \rho_\lambda, \alpha) \geq 0 \quad \text{for all } \alpha \in \Phi^+.$$

Theorem (Theorem of the highest weight). *Suppose G is a connected, compact, semisimple, and simply connected Lie group. The irreducible representation (π_λ, V_λ) with irreducible character χ_λ – so according to our convention $\lambda \in \Lambda$ is regular – has $\lambda - \rho_\lambda$ as weight of multiplicity one. Moreover,*

a) if A is a nonempty sum of roots in Φ_λ^+, $\lambda - \rho_\lambda + A$ is not a weight;
b) any weight can be expressed as $\lambda - \rho_\lambda - A$, with A denoting a sum of roots in Φ_λ^+.

If a representation with character χ_λ is known to exist – as was the case for Weyl – the statement is a straightforward consequence of the Poincaré-Birkhoff-Witt theorem. But for Cartan, who needed to establish the existence of π_λ first, it was a significant accomplishment.

On the surface, the Weyl character formula is merely an existence theorem: it sets up a parametrization of the irreducible representations of G. The theorem of the highest weight, on the other hand, implies much, if not all, the known structural information about irreducible representations of compact Lie groups.

After enumerating the irreducible representations of G – either by means of the Weyl character formula, or via Cartan's highest weight theorem – and establishing important structural information as consequence of the theorem of the highest weight, there remains the issue of providing "models" of these representations. This isn't simply an aesthetic requirement: not just in the case of compact (or complex semisimple) Lie groups, a good model of a mathematical structure can, and often does, lead to new insights. The Borel-Weil theorem [3, 27], and later the Borel-Weil-Bott theorem [4], provides such a model for irreducible representations of compact Lie groups.

Both Borel and Weil are "almost" contemporaries, in the sense that some of us – myself included – did meet them. André Weil (1906–1998) was born to Alsatian Jewish parents, who moved to Paris after the Franco-Prussian war of 1970. He studied in Göttingen and Paris, where he received his doctorate in 1928. He then spent two years at the Aligarh Muslim University in Uttar Pradesh, which is explained by his interest in Sanskrit and Hinduism. At the beginning of WW2, he happened to be in Finland, returned to France before it was occupied, then continued to the US via Marseille. After teaching at Lehigh University for two years, which he disliked intensely, he ended up at the Institute for Advanced Study via São Paulo and the University of Chicago. Lie theory is a very minor aspect of his work, of course.

Armand Borel (1923–2003) was born in Chaux-de-Fonds near Bern, in the French speaking part of Switzerland. His Swiss nationality shielded him from the turmoil of WW2 during his youth. Borel studied at the ETH Zürich and in Paris. He received his Doctorat d'Etat in Paris, with Jean Leray as advisor. Soon after he was appointed permanent member of the Institute for Advanced Study, where he stayed until his retirement. After retiring, he became very interested in making new developments in mathematics accessible to working mathematicians, for example by running seminars in Bern (because of its central location on Switzerland) that resulted in monographs on D-modules and on intersection cohomology; both, it might be noted, have become important tools in representation theory.

Let $G_{\mathbb{C}}$ be a connected, complex, semisimple Lie group, and $U \subset G_{\mathbb{C}}$ a compact real form – i.e., a connected, compact, semisimple Lie group whose complexification is $G_{\mathbb{C}}$. Pick a maximal torus $T \subset U$; not a significant choice, since any two are conjugate in U. Its complexification is then a Cartan subgroup of $G_{\mathbb{C}}$. The choice of a system of positive roots Φ^+ – again not a significant choice since any two are conjugate under $N_U(T)$ – determines a "Borel subgroup" $B \subset G_{\mathbb{C}}$, i.e., a maximal solvable subgroup, one that contains T and whose Lie algebra contains the root spaces $\mathfrak{g}^{-\alpha}$ indexed by the negative roots $-\alpha \in -\Phi^+$. In this situation,

$$X =_{\text{def}} U/T = G_{\mathbb{C}}/B \quad \text{(the "flag variety" of } G)$$

is a compact complex manifold, acted on transitively by U and $G_{\mathbb{C}}$. Any $\lambda \in \Lambda$ determines a holomorphic character $e^\lambda : B/[B,B] \longrightarrow \mathbb{C}^*$ of B, and hence a "homogeneous holomorphic line bundle"

$$\mathcal{L}_\lambda \longrightarrow X = G_{\mathbb{C}}/B,$$

a holomorphic line bundle to which the action of G on X lifts, and on whose fiber at the identity coset B acts via e^λ. Since the group $G_{\mathbb{C}}$ acts on X and $\mathcal{L}_\lambda$, it acts on the cohomology groups of $\mathcal{O}(\mathcal{L}_\lambda)$.

Theorem (Borel-Weil theorem). *For $\lambda \in \Lambda$, if $(\alpha, \lambda) \geq 0$ for all $\alpha \in \Phi^+$,*

$$H^p(X, \mathcal{O}(\mathcal{L}_\lambda)) \begin{cases} \textit{is irreducible of highest weight } \lambda \textit{ if } p = 0 \\ \textit{vanishes for all } p \neq 0 \end{cases}$$

This, in particular, gives a concrete geometric description of all irreducible representations of a connected compact Lie group U – or equivalently, of a connected complex semisimple Lie group $G_{\mathbb{C}}$. The proof is a fairly direct consequence of the Kodaira vanishing theorem and the theorem of the highest weight.

With $X = G_{\mathbb{C}}/B = U/T$ and Φ^+ as before, let ρ denote the half-sum of all the positive roots.

Theorem (Borel-Weil-Bott theorem). *$H^p(X, \mathcal{O}(\mathcal{L}_\lambda))$ vanishes for all p if $\lambda + \rho$ fails to be regular. If, on the other hand, $\lambda + \rho$ is regular, choose $w \in W$ so that $(\alpha, w(\lambda + \rho)) > 0$ for all $\alpha \in \Phi^+$, and define*

$$p_\lambda = \#\{\, \alpha \in \Phi^+ \mid (\alpha, \lambda + \rho) < 0 \,\}.$$

$$H^p(X, \mathcal{O}(\mathcal{L}_\lambda)) \;=\; \begin{cases} \textit{is irreducible of highest weight} \\ \quad w(\lambda+\rho)-\rho \ \textit{if}\ p = p_\lambda \\ \textit{vanishes for all}\ p \neq p_\lambda \end{cases}$$

Bott reduces this statement to the Borel-Weil theorem by means of spectral sequences associated to fibrations of X over so-called generalized flag varieties, with $\mathbb{P}^1$ fibres. The theorem has many applications in representation theory – e.g., in the proof of the Beilinson-Bernstein vanishing theorem – and in calculations in complex algebraic geometry.

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