Stephen Yau's Work in Several Complex Variables

by H. Blaine Lawson

Editor's Note: H. Blaine Lawson was a 1973 recipient of the American Mathematical Society's Leroy P. Steele Prize, and was elected to the National Academy of Sciences in 1995. He is a former recipient of both the Sloan Fellowship and the Guggenheim Fellowship, and has delivered two invited addresses at International Congresses of Mathematicians, one on geometry, and one on topology. He has served as Vice President of the American Mathematical Society, and is a foreign member of the Brazilian Academy of Sciences. In 2012 he became a fellow of the American Mathematical Society. He was elected to the American Academy of Arts and Sciences in 2013.

H. Blaine Lawson is a mathematician best known for his work in minimal surfaces, calibrated geometry, and algebraic cycles. He is currently a Distinguished Professor of Mathematics at Stony Brook University. He received his PhD from Stanford University in 1969 for work carried out under the supervision of Robert Osserman.

The idea of discussing Stephen Yau's contributions to mathematics, and to complex analysis in particular, for this ceremony in his honor, was very attractive to me. However, he has worked in many areas and written over 230 research articles, so any brief synopsis would be impossible. However, there is part of Stephen's work that I have always admired, and in my mind it epitomizes his research. It calls on a wide range of subjects, and it leads to unexpected consequences.

To begin let's consider a compact complex manifold with smooth boundary. For a simple example, let $X_t = \{z \in \mathbb{C}^n : \|z\| \leq 1 \text{ and } \sum_j z_j^d = t\}$ for $|t|$ small. The boundaries are all diffeomorphic, however when $t = 0$ the cobounding manifold X_0 acquires a singularity at the origin. The natural question is: Are there invariants that could detect this change at $t = 0$?

Now the boundary of a complex manifold X carries more than its differentiable structure. In fact it is almost a complex manifold. At each point $x \in \partial X$ the tangent space $T_x = T_x(\partial X)$ carries a complex subspace $H_x \equiv T_x \cap J(T_x)$ of real codimension one (where J denotes the complex structure of X). One might ask whether there exists something like the Dolbeault $\bar{\partial}$-cohomology groups for such spaces. The answer is yes; such groups were defined and analysed by J. J. Kohn and H. Rossi (Ann. of Math. *78* (1963), p. 112, *79* (1964), p. 450 and *81* (1965), p. 451). They are invariant under diffeomorphisms which preserve $(H, J|_H)$. However, the analysis is hard, and these groups can be infinite dimensional. A case where all these groups are finite dimensional is when ∂X is strictly pseudoconvex. This is where an intrinsically defined Levi form on H_x is positive definite for all $x \in \partial X$. However, even in this case, computations were difficult.

Now it was known at that time that a strictly pseudo-convex manifold was the boundary of a Stein analytic space with only a finite number of singularities, all of which were interior, i.e., not at the boundary. One could resolve the singularities to obtain a smooth complex manifold, but one would lose the Stein property of being embeddable into $\mathbb{C}^N$.

Stephen Yau's great result (Ann. of Math. *113* (1981), p. 67) was a computation of the Kohn-Rossi cohomology groups explicitly in terms of the local invariants of these singularities. These invariants (certain Brieskorn numbers) are computable, and particularly so when the singularity is of hypersurface type. To be more explicit, suppose that $V = \{z : f(x) = 0\}$ has an isolated singularity at $z = 0$, where $f \in \mathbb{C}[z_0, \cdots, z_n]$ and $df \neq 0$ on $\{0 < \|z\| < \epsilon\}$ for some $\epsilon > 0$. Then we define

$$\tau \equiv \dim\Big\{\mathbb{C}[z_0, \cdots, z_n]/(f, \frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n})\Big\}.$$

Now suppose that ∂X bounds a Stein space V of complex dimension n with isolated singularities

$z_1, \cdots, z_m$ all of hypersurface type. Then, in this case, Stephen's result says that the Kohn-Rossi cohomology $H^{p,q}_{KR}(\partial X) = 0$ if $p+q \leq n-2$ or $p+q \geq n+1$, and in remaining cases it is equal to $\tau_1 + \cdots + \tau_m$ where τ_k is the above invariant for z_k.

This result had a nice application. In 1975 Reese Harvey and I had established a general result characterizing the boundaries of complex varieties in a Stein manifold Y. A special case is when $M^{2n-1} \subset Y$ is a smooth submanifold of real dimension $2n-1$ and Y has complex dimension $n+1$ (so we are looking for boundaries of hypersurfaces). If $n > 1$ then M^{2n-1} bounds if and only if M^{2n-1} satisfies the *maximal complexity* condition above. If, in addition, M^{2n-1} is strictly pseudo-convex, then M^{2n-1} bounds a Stein space X, which is immersed in Y (see Ex. 9.1 in Ann. of Math. *102* (1975), p. 233), with a finite number of isolated singularities which are interior and of hypersurface type. Stephen's result says that X is non-singular exactly when the Kohn-Rossi cohomology of M^{2n-1} is trivial.

Over the years Stephen Yau has written many very good papers in the area of complex analysis and geometry. Some of these concern strictly pseudo-convex manifolds and morphisms between them. Some concern various aspects of isolated singularities on complex varieties. There are results about four dimensional $\mathcal{N}=2$ superconformal field theories, embedding problems for CR-manifolds, and much more. He has had a wonderful career.