Some Contributions of Stephen S.-T. Yau to Several Complex Variables

by Joseph J. Kohn

Editor's Note: Joseph J. Kohn is a Professor Emeritus of mathematics at Princeton University, where he does research on partial differential operators and complex analysis. Since 1966 he is a member of the American Academy of Arts and Sciences and since 1988 a member of the National Academy of Sciences. In 2012 he became a fellow of the American Mathematical Society. He won the AMS Steele Prize in 1979 for his paper Harmonic integrals on strongly convex domains. In 1990 he received an Honorary Doctorate from the University of Bologna. In 2004, he was awarded the Stefan Bergman Prize for Influential Research.

It gives me great pleasure to write about Stephen S.-T. Yau in connection with the Chern Award. Yau is an extraordinary mathematician not only because of his talent but also because of his breadth of interest and his prolific productivity. His originality and hard work fully deserve the honorable recognition provided by the award. Yau has numerous publications in the theory of several complex variables and its applications to geometry, algebraic geometry and mathematical physics. He also has numerous impressive publications in other branches of both pure and applied mathematics. Here I will describe, only briefly some, of his remarkable connected with CR manifolds and the study of singularities. This work illustrates his originality, powerful technique, and mastery of the subject. It is particularly striking since it brings out deep connections between algebraic geometry, CR geometry, and the theory of several complex variables.

Yau's joint work with John Mather (see [1]) proves a fundamental result. They show that two germs of complex analytic hypersurfaces with isolated singularities are biholomorphically equivalent if and only if they have the same dimension and their moduli algebras are isomorphic.

The Plateau problem is one of the basic problems in geometry, it deals with finding the conditions under which compact manifolds are boundaries of varieties and manifolds with minimal volumes. Yau has made impressive contributions to this problem for the case of real submanifolds of Stein manifolds. The starting point of Yau's research on this problem is a fundamental result of Harvey and Lawson [2]. Let X be a $2n-1$ dimensional real submanifold an n-dimensional Stein manifold W such that at each point $P \in X$ we have $dim_{\mathbb{C}}(T_P^{(1,0)}(X)) = n-1$, where $T_P^{(1,0)}(X) = T_P^{1,0}(W) \cap \mathbb{C}T_P(X)$. Then if X a CR manifold and the result states (approximately) that if X is connected then it is the boundary (in the sense of currents) of an irreducible complex n-dimensional subvariety V of W-X. In [3] Yau studies the case when the Levi form on X is not identically zero at any point and obtains a simpler proof with the stronger result that X is then the topological boundary of V. If V is a complex manifold (i.e. has no singularities) then it solves the Plateau problem. One of the main results of [3] is the following.

Theorem: *Let X be a compact, orientable, real manifold of dimension $2n-1, n \geq 3$ and $X \subset W$ where W is a Stein manifold and X is strictly pseudoconvex. then X is the boundary of the complex submanifold $V \subset W - X$ if and only if the $\bar{\partial}_b$-cohomology groups $H_b^{p,q}(X)$ are zero for $1 \leq q \leq n-2$.*

$\bar{\partial}_b$-cohomology on CR manifolds was introduced in [4]. This cohomology is defined by the operator $\bar{\partial}_b$:

$\mathcal{B}^{p,q}(X) \to \mathcal{B}^{p,q+1}(X)$ where for $P \in X$ we define $\mathcal{B}_P^{p,q}(X)$ to be the restriction of the (p,q)-forms in W at P to

$$\underbrace{T_P^{1,0}(X) \times \cdots \times T_P^{1,0}(X)}_{p} \times \underbrace{T_P^{0,1}(X) \times \cdots \times T_P^{0,1}(X)}_{q}$$

then $\bar{\partial}_b$ is the corresponding restriction of the domain of $\bar{\partial}$ followed by a projection to $\mathcal{B}^{p,q+1}(X)$, so that $\bar{\partial}_b^2 = 0$ and we define $H_b^{p,q}(X)$ to be the corresponding cohomology. The proof of the theorem follows from another important result in [3] which relates the dimension of the $\bar{\partial}_b$ cohomology to the Brieskorn invariants of singularities.

In [5] Luk and Yau take up the case of $n = 2$, that is $dimX = 3$. In this case they study the singularities of V by an ingenius use of a spectral sequence introduced by Tanaka which connects $H_b^{p,q}(X)$ with the holomorphic DeRham cohomology. This technique enables them to analyze the singularities of V. Further work on the case of $n = 2$ was done by Du and Yau in [6]. They consider the case when $X \subset \mathbb{C}^3$ with $dimX = 2$, they introduce a CR invariant $g^{(1,1)}(X)$ and prove that vanishing of this invariant suffices to prove the interior regularity of the Harvey-Lawson solution of the complex Plateau problem.

References

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