# Hodge Bundles on Smooth Compactifications of Siegel Varieties and Applications

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## Contents



Siegel varieties are locally symmetric varieties. They are important and interesting in algebraic geometry and number theory. We construct a canonical Hodge bundle on a Siegel variety so that the holomorphic tangent bundle can be embedded into the Hodge bundle; we obtain that the canonical Bergman metric on a Siegel variety is same as the induced Hodge metric and we describe asymptotic behavior of this unique Kähler-Einstein metric explicitly; depending on these properties and the uniformitarian of Kähler-Einstein manifold, we study extensions of the tangent bundle over any smooth toroidal compactification (Theorem 1.4, Theorem 1.6 and Theorem 1.10 in Section 1). We apply these results of Hodge bundles, to study dimension of Siegel cusp modular forms and general type for Siegel varieties (Theorem 2.4 and Theorem 2.7 in Section 2).

Throughout this paper, $g$ is an integer more than two.

In this paper, we fix a real vector space $V_{\mathbb{R}}$ of dimensional $2g$ and fix a standard symplectic form $\psi = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}$ on $V_{\mathbb{R}}$. For any non-degenerate skew-symmetric bilinear form $\widetilde{\psi}$ on $V_{\mathbb{R}}$, it is known that there is an element $T \in \mathrm{GL}(V_{\mathbb{R}})$ such that ${}^tT\widetilde{\psi}T = \psi$. We also fix a symplectic basis $\{e_i\}_{1\le i\le 2g}$ of the standard symplectic space $(V_{\mathbb{R}}, \psi)$ such that

(0.0.1)
$$\psi(e_i, e_{g+i}) = -1 \text{ for } 1 \le i \le g, \text{ and } \psi(e_i, e_j) = 0 \text{ for } |j-i| \neq g.$$

- Denote by $V_{\mathbb{Z}} := \oplus_{1\le i\le 2g}\mathbb{Z}e_i$, then $V_{\mathbb{R}} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ and $V_{\mathbb{Z}}$ is a standard lattice in $V_{\mathbb{R}}$. In this paper, we fix the lattice $V_{\mathbb{Z}}$ and fix the rational space $V_{\mathbb{Q}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$.
- For any $\mathbb{Z}$-algebra $\mathfrak{R}$, we define $V_{\mathfrak{R}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathfrak{R}$ and we write

  (0.0.2)
  $$\mathrm{Sp}(g,\mathfrak{R}) := \{h \in \mathrm{GL}(V_{\mathfrak{R}}) \mid \psi(hu, hv) = \psi(u,v) \text{ for all } u, v \in V_{\mathfrak{R}}\}.$$

  Since $\det M = \pm 1$ for all $M \in \mathrm{Sp}(g,\mathbb{Z})$, $\mathrm{Sp}(g,\mathbb{Z})$ is a subgroup of $\mathrm{Sp}(g,\mathbb{Q})$.

## 1. Hodge Bundles on Siegel Varieties

Let $\Gamma$ be a neat arithmetic subgroup of $\mathrm{Sp}(g,\mathbb{Q})$. Let $V_{\mathbb{Q}}$ be the fixed rational symplectic vector space as in

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the introduction of notations.

By Borel's embedding theorem, the Siegel space $\mathfrak{H}_g$ can be realized as a bounded domain parameterizing weight one polarized Hodge structures (cf. Proposition A.2 in A2). Moreover, there is a natural variation of Hodge structures on the Siegel space $\mathfrak{H}_g$:

**Corollary** (Cf. [Del79]). *Gluing Hodge structures on $\mathfrak{H}_g$ altogether, the local system $\mathbb{V} := V_{\mathbb{Q}} \times \mathfrak{H}_g$ admits a homogenous variation of polarized rational Hodge structures of weight one on $\mathfrak{H}_g$.*

Let $o$ be an arbitrary fixed base point in $\mathcal{A}_{g,\Gamma}$. Since $\mathfrak{H}_g$ is simply connected, the fundamental group of $\mathcal{A}_{g,\Gamma}$ has $\pi_1(\mathcal{A}_{g,\Gamma}, o) \cong \Gamma$. Then, there is a natural local system $\mathbb{V}_{g,\Gamma} := V_{\mathbb{Q}} \times_\Gamma \mathfrak{H}_g$ on $\mathcal{A}_{g,\Gamma}$ given by the fundamental representation $\rho : \pi_1(\mathcal{A}_{g,\Gamma}, o) \to \mathrm{GSp}(V,\psi)(\mathbb{Q})$. Actually the $\mathbb{Q}$-local system $\mathbb{V}_{g,\Gamma}$ admits a variation of polarized rational Hodge structures of weight one on $\mathcal{A}_{g,\Gamma} := \Gamma\backslash\mathfrak{H}_g$ by using the arguments in Section 4 of [Zuc81]. More precisely, in our previous paper (cf. Proposition 1.8 in Section 1 of [YZ2014]) we obtain:

1. The local system $\mathbb{V}_{g,\Gamma}$ admits a variation of polarized rational Hodge structures on $\mathcal{A}_\Gamma$, and the associated period map attached to this PVHS is

(1.0.1) $$\iota_\Gamma : \mathcal{A}_{g,\Gamma} \xrightarrow{\cong} \Gamma\backslash\mathfrak{S}_g.$$

2. Let $\widetilde{\mathcal{A}}_{g,\Gamma}$ be an arbitrary smooth compactification of $\mathcal{A}_{g,\Gamma}$ with simple normal crossing divisor $D_\infty := \widetilde{\mathcal{A}}_{g,\Gamma} \setminus \mathcal{A}_{g,\Gamma}$. Around the boundary divisor $D_\infty$, all local monodromies of any rational PVHS $\widetilde{V}$ on $\mathcal{A}_{g,\Gamma}$ are unipotent.

Now, we fix this rational PVHS $\mathbb{V}_{g,\Gamma}$ throughout this paper.

### 1.1 Construction of Hodge Bundles on Siegel Varieties

Most materials in this subsection are taken from [Sch73], [Sim90] and [Zuo00].

We note that $\mathbb{H} := \mathbb{V}_{g,\Gamma} \otimes \mathbb{C}$ is a flat complex vector bundle on the $\mathcal{A}_{g,\Gamma}$ with a flat connection $\mathbb{D}$. There is a filtration of $C^\infty$ vector bundles over $\mathcal{A}_{g,\Gamma}$ $0 = \mathbb{F}^2 \subset \mathbb{F}^1 \subset \mathbb{F}^0 = \mathbb{H}$, whose fibers at each point $\tau \in \mathcal{A}_{g,\Gamma}$ gives a Hodge filtration isomorphic to $F_\tau^\bullet := (0 \subset F_\tau^1 \subset V_{\mathbb{C}})$. The vector bundle $\mathbb{H}$ admits a positive Hermitian metric $H$ induced by the polarization $\psi$ of the Hodge structures as follows:

(1.0.2) $$< u, \bar{v} >_H := \psi(C_\tau u, \bar{v}) \; \forall u, v \in \mathbb{H}_\tau,$$

where each $C_\tau$ is the Weil operator on the $F_\tau^\bullet$. We usually call this metric $H$ the **Hodge metric** on $\mathbb{H}$. Let $\mathbb{H}^{p,q} := \mathbb{F}^p \cap \overline{\mathbb{F}^q}$. The smooth decomposition $\mathbb{H} = \bigoplus \mathbb{H}^{p,q}$ is orthogonal with respect to the Hodge metric $H$.

Let $\mathbb{D}^{0,1}$ be the $(0,1)$-part of the flat connection $\mathbb{D}$ and $\mathbb{D}^{1,0}$ the $(1,0)$-part of $\mathbb{D}$. The $\mathbb{D}^{0,1}$ gives a holomorphic structure on $\mathbb{H}$, so that $\mathcal{H} := (\mathbb{H}, \mathbb{D}^{0,1})$ is the corresponding holomorphic bundle. The $\mathbb{D}^{1,0}$ guarantees $\mathcal{H}$ has an integrable holomorphic connection $\mathbb{D}^{1,0} : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_{\mathcal{A}_{g,\Gamma}}$. It is known that all sub bundles $\mathbb{F}^p$'s admit the holomorphic structure $\mathbb{D}^{0,1}$ naturally, so that we have the corresponding holomorphic sub bundles $\mathcal{F}^p$. Moreover, we have the Griffiths transversality:

(1.0.3) $$\mathbb{D}^{1,0} : \mathcal{F}^p \longrightarrow \mathcal{F}^{p-1} \otimes \Omega^1_{\mathcal{A}_{g,\Gamma}}, \forall p.$$

Define $E^{p,2-p} := \mathcal{F}^p/\mathcal{F}^{p+1}$ $\forall p$. We know that each holomorphic vector bundle $E^{p,q}$ is $C^\infty$-isomorphic to the vector bundle $\mathbb{H}^{p,q}$. We set $E := \mathrm{Gr}(\mathcal{H}) = \bigoplus_p E^{p,n-p}$. The flat connection $\mathbb{D}$ on $\mathcal{H}$ actually induces a global holomorphic structure $\overline{\partial}$ on $E$ such that each $E^{p,q}$ is a holomorphic sub bundle od $E$. We write:

(1.0.4) $$E^{p,q} = (\mathbb{H}^{p,q}, \overline{\partial}), \text{ and } E = (\bigoplus \mathbb{H}^{p,q}, \overline{\partial}).$$

The holomorphic vector bundles $E$ and $E^{p,q}$'s are endowed natural Hermitian metrics induced by $H$. For convenience, we still call these Hermitian metrics the Hodge metrics and still write these Hermitian metrics as $H$.

Let $\mathbb{T}(\mathcal{A}_{g,\Gamma})$ be the real tangent bundle of $\mathcal{A}_{g,\Gamma}$. According to $\pm\sqrt{-1}$-eigenvalues of the complex structure $J$ on $\mathbb{T}(\mathcal{A}_{g,\Gamma})$, there is a $C^\infty$ decomposition $\mathbb{T}(\mathcal{A}_{g,\Gamma}) \otimes \mathbb{C} = \mathbb{T}^{1,0}(\mathcal{A}_{g,\Gamma}) \oplus \mathbb{T}^{0,1}(\mathcal{A}_{g,\Gamma})$. The real tangent bundle $\mathbb{T}(\mathcal{A}_{g,\Gamma})$ undertakes the holomorphic tangent bundle $\mathcal{T}_{\mathcal{A}_{g,\Gamma}} := (\Omega^1_{\mathcal{A}_{g,\Gamma}})^\vee$ in sense that

$$\mathbb{T}^{1,0}(\mathcal{A}_{g,\Gamma}) \xrightarrow[C^\infty]{\cong} \mathcal{T}_{\mathcal{A}_{g,\Gamma}}, \; \mathbb{T}^{0,1}(\mathcal{A}_{g,\Gamma}) \xrightarrow[C^\infty]{\cong} \overline{\mathcal{T}_{\mathcal{A}_{g,\Gamma}}}.$$

Let $(p,q)$ be a pair of integers. For any local holomorphic vector filed $\overrightarrow{X}$ of $\mathcal{T}_{\mathcal{A}_{g,\Gamma}}$, there is a local $\mathcal{O}_{\mathcal{A}_{g,\Gamma}}$-linear morphism $\theta^{p,q}(\overrightarrow{X}) : E^{p,q} \to E^{p-1,q+1}$ by the Griffiths transversality 1.0.3. Then we get an $\mathcal{O}_{\mathcal{A}_{g,\Gamma}}$-linear morphism $\theta^{p,q} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_{\mathcal{A}_{g,\Gamma}}$, and so we get the adjoint map $\theta_H^{p-1,q+1*} : E^{p-1,q+1} \to E^{p,q} \otimes \overline{\Omega^1_{\mathcal{A}_{g,\Gamma}}}$ of $\theta^{p,q}$ given by $< \theta^{p,q}(s), \bar{t} >_H = < s, \overline{\theta_H^{p-1,q+1*}(t)} >_H$, where $s$ (resp. $t$) is a local section of $E^{p,q}$ (resp. $E^{p-1,q+1}$). Clearly $\theta^{p,q*}$ can be regarded as an $\overline{\mathcal{O}_{\mathcal{A}_{g,\Gamma}}}$-linear morphism. The **Higgs field** $\theta$ on $E$ is defined as follows:

$$\theta = \bigoplus_{p,q} \theta^{p,q} : \bigoplus_{p,q} E^{p,q} \longrightarrow \bigoplus_{p,q} E^{p,q} \otimes \Omega^1_{\mathcal{A}_{g,\Gamma}}.$$

Respectively, the adjoint morphism of $\theta$ is defined to be $\theta_H^* := \bigoplus_{p,q} \theta_H^{p,q*}$.

**Remark.** Let $A^1$ be the dual of the sheaf of $C^\infty$ germs of $\mathbb{T}(\mathcal{A}_{g,\Gamma})$. Then there is a $C^\infty$ splitting $A^1 = A^{1,0} \oplus A^{0,1}$ where $A^{1,0}$ (resp. $A^{0,1}$) is the dual of the sheaf of $C^\infty$

germs of $\mathcal{T}_{\mathcal{A}_{g,\Gamma}}$ (resp. $\overline{\mathcal{T}_{\mathcal{A}_{g,\Gamma}}}$). We can extend $\theta$ and $\theta_H^*$ naturally as $C^\infty$ morphisms

$$\begin{aligned} \theta : C^\infty(E) &\longrightarrow C^\infty(E)\otimes A^{1,0}, \\ \theta_H^* : C^\infty(E) &\longrightarrow C^\infty(E)\otimes A^{0,1}, \end{aligned}$$

where $C^\infty(E)$ is the sheaf of $C^\infty$ germs of $E$.

Let $\nabla_H$ be the unique Chern connection on $(E,H)$. Thus, the connection $\nabla_H$ is compatible with the Hodge metric, and its $(0,1)$-part has $\nabla_H^{0,1}=\overline{\partial}$. Define $\partial := \nabla_H^{1,0}$. We immediately obtain $\partial^2=\overline{\partial}^2=0$, and get the Chern curvature form

$$\Theta(E,H) := \nabla_H\circ\nabla_H = (\nabla_H^2)^{1,1}.$$

**Lemma 1.1.** *We have:*

$$\begin{aligned} \overline{\partial}(\theta) &:= \overline{\partial}\circ\theta+\theta\circ\overline{\partial}=0, \\ \partial(\theta_h^*) &:= \partial\circ\theta_h^*+\theta_h^*\circ\partial=0. \end{aligned}$$

*Proof.* One can find these two equalities in [Sim88] & [Sim90]. Here we give a direct proof.

The morphism $\theta$ is naturally holomorphic by the definition,so that the first equality is automatically true. Now, we begin to prove the second equality.

It is sufficient to prove the equality at an arbitrary point $p$. Let $(U,p)$ be a local coordinate neighborhood of $p$. Let $\{e_\alpha\}$ be a local holomorphic basis of $E$. We then get a local holomorphic basis $\{e^\alpha\}$ of $E^\vee|_U := \mathrm{Hom}(E|_U,\mathcal{O}_U)$ as follows: For each $\alpha$, let $e^\alpha$ be the dual of $e_\alpha$, i.e., $e^\alpha\in E^\vee|_U$ such that $e^\alpha(e_\beta)=\begin{cases}1, & \beta=\alpha;\\ 0, & \beta\neq\alpha.\end{cases}$ We call the local holomorphic basis $\{e^\alpha\}$ of $E^\vee|_U$ as a local dual base of $\{e_\alpha\}$. Let $\{l_1,\cdots,l_m\}$ be a local holomorphic basis of $\mathcal{T}_{\mathcal{A}_{g,\Gamma}}$ and $\{\phi_1,\cdots,\phi_m\}\subset\Omega^1_{\mathcal{A}_{g,\Gamma}}$ its local holomorphic dual basis. Locally, we can write

$$\theta=\sum_{i=1}^m A^i\phi_i,\ \ \theta_h^*=\sum_i B^i\overline{\phi_i}$$

where $A^i := A^{i,\alpha}_\beta e_\alpha\otimes e^\beta$ and

$$\begin{aligned} B^i &:= B^{i,\alpha}_\beta e_\alpha\otimes e^\beta \text{ with} \\ B^{i,\beta}_\alpha &:= \sum_{\gamma,\delta} H_{\alpha\bar{\gamma}}\overline{A^{i;\gamma}_\delta}H^{\bar{\delta}\beta}, H_{\alpha\bar{\gamma}} := <e_\alpha,\overline{e_\gamma}>_H. \end{aligned}$$

Form the first equality in the lemma, we get

$$(1.1.1)\quad 0=\overline{\partial}\theta=\sum_{i=1}^m\sum_{j=1}^n A^i_{;\bar{j}}\bar{\phi}_j\wedge\phi_i=\sum_{i=1}^m\sum_{j=1}^n A^{i,\alpha}_{\beta;\bar{j}}e_\alpha\otimes e^\beta\bar{\phi}_j\wedge\phi_i$$

where $A^i_{;\bar{j}}$'s for all $j$ are covariant partial derivations of the tensor $A^i$, and so we obtain

$$A^{i,\alpha}_{\beta;\bar{j}}=0 \ \text{ on } \ U\ \forall i,j,\alpha,\beta.$$

On the other hand, we compute that

$$(1.1.2)\quad \partial\theta_h^*=\sum_{i=1}^m\sum_{j=1}^m B^i_{;j}\phi_j\wedge\overline{\phi_i}=\sum_{i=1}^m\sum_{j=1}^m B^{i,\alpha}_{\beta;j}e_\alpha\otimes e^\beta\phi_j\wedge\overline{\phi_i},$$

where $B^i_{;j}$'s for all $j$ are covariant partial derivations of the tensor $B^i$. It is well-known that one can contract the neighborhood $(U,p)$ sufficiently small to get a special holomorphic local basis $\{e_\alpha\}$ of $E$ over $U$ such that $H(p)=\mathrm{Id}$, $dH(p)=0$ under the frame $\{e_\alpha\}$. Then, at the point $p$, We have:

$$B^{i,\alpha}_{\beta;j}(p)=\overline{A^{i,\alpha}_{\beta;\bar{j}}(p)}=0,\ \forall i,j\alpha,\beta.$$

Thus $\partial\theta_h^*=0$ at the point $p$. □

**Corollary 1.2.** *Let $(E,H)$ be Hermitian vector bundle in 1.0.4. We have:*

$$\begin{aligned} \Theta(E,H) &= -(\theta\wedge\theta_h^*+\theta_H^*\wedge\theta), \\ \theta\wedge\theta &= -\partial(\theta)=0, \\ \theta_H^*\wedge\theta_H^* &= -\overline{\partial}(\theta_H^*)=0. \end{aligned}$$

*Proof.* It is known the flat connection $\mathbb{D}$ on $\mathbb{H}$ has the following decomposition

$$\mathbb{D}=\nabla_H+\theta+\theta_H^*.$$

Since $\mathbb{D}^2=0$, we can finish the proof by the lemma 1.1. □

Attached to the PVHS $\mathbb{V}_{g,\Gamma}$, we finally obtain the associated **Hodge bundle** $(E,\theta,H)$ on $\mathcal{A}_{g,\Gamma}$, i.e., a holomorphic system

$$(1.2.1)\qquad (E=\oplus E^{p,q},\theta=\oplus\theta^{p,q})$$

with a Hermitian metric $H$ satisfying the following properties:

- $E^{p,q}$ are orthogonal to each other under the metric $H$;
- $\theta\wedge\theta=0$;
- $\theta^{p,q}: E^{p,q}\longrightarrow E^{p-1,q+1}\otimes\Omega^1_{\mathcal{A}_{g,\Gamma}}$.

The dual local system $\mathbb{V}^\vee_{g,\Gamma}=V^\vee_{\mathbb{Q}}\times_\Gamma\mathfrak{H}_g$ admits a polarized rational VHS of weight $-1$ on $\mathcal{A}_{g,\Gamma}$, its associated Hodge bundles is $(E^\vee=\bigoplus_{p+q=1}E^{\vee-p,-q},\theta_\vee)$ with

$$\begin{aligned} E^{\vee-p,-q} &= (E^{p,q})^\vee=E^{q,p}, \\ \theta_\vee^{-p,-q} &= -\theta^{q,p}: E^{\vee-p,-q}\longrightarrow E^{\vee-p-1,-q+1}\otimes\Omega^1_{\mathcal{A}_{g,\Gamma}}. \end{aligned}$$

Similarly, the local system $\mathrm{End}(\mathbb{V}) := \mathrm{End}(V_{\mathbb{Q}})\times_\Gamma\mathfrak{H}_g$ admits a polarized rational VHS of weight 0 on $\mathcal{A}_{g,\Gamma}$, its associated Hodge bundle is $(\mathrm{End}(E),\theta^{end})$ with

$$\mathrm{End}(E)=\bigoplus_{(p-p')+(q-q')=0}E^{p,q}\otimes E^{\vee-p',-q'}$$

and the Higgs field $\theta^{end}:\mathrm{End}(E)\to\mathrm{End}(E)\otimes\Omega^1_{\mathcal{A}_{g,\Gamma}}$ given by

$$\theta^{end}(u\otimes v^\vee)=\theta(v)\otimes v^\vee+u\otimes\theta_\vee(v^\vee).$$

We notes that $\mathrm{End}(E)$ has a holomorphic sub bundle

$$\begin{aligned}\mathrm{End}(E)^{-1,1} &= \bigoplus_{p+q=1} E^{p,q}\otimes E^{q-1,p+1},\\ &= (E^{0,1})^{\otimes 2}.\end{aligned}$$

We still use $H$ to denote the induced Hermitian metric on $E^\vee$ and $\mathrm{End}(E)$. Throughout this section, we now fix the Hermitian bundles $(E,H)$, $(\mathrm{End}(E),H)$, and $(E^{pq},H)$'s, $(\mathrm{End}(E)^{p,q},H)$'s.

### 1.2 Degeneration of Canonical Metrics on Siegel Varieties

Let $\widetilde{\mathcal{A}}_{g,\Gamma}$ be a smooth compactification of $\mathcal{A}_{g,\Gamma}$ such that the divisor $D_\infty = \widetilde{\mathcal{A}}_{g,\Gamma}\setminus\mathcal{A}_{g,\Gamma}$ is simple normal crossing. Since any local monodromy of $\mathbb{V}_{g,\Gamma}$ around $D_\infty$ is unipotent, the Hodge bundle $(E,\theta)$ has a Deligne's canonical extension $(\overline{E}=\oplus\overline{E^{p,q}},\overline{\theta}=\oplus\overline{\theta^{p,q}})$ with $\overline{\theta^{p,q}}:\overline{E^{p,q}}\to\overline{E^{p-1,q+1}}\otimes\Omega^1_{\widetilde{\mathcal{A}}_{g,\Gamma}}(\log D_\infty)$. Deligne's extension of $(End(E),\theta^{end})$ is $(End(\overline{E}),\overline{\theta^{end}})$. The morphism $\overline{\theta^{1,0}}:\overline{E^{1,0}}\to\overline{E^{0,1}}\otimes\Omega^1_{\widetilde{\mathcal{A}}_{g,\Gamma}}(\log D_\infty)$ represents the global section $\overline{\theta^{1,0}}\in H^0(\widetilde{\mathcal{A}}_{g,\Gamma},\overline{E^{0,1}}^{\otimes 2}\otimes\Omega^1_{\widetilde{\mathcal{A}}_{g,\Gamma}}(\log D_\infty))$. Then, we obtain a sheaf morphism

$$\rho:\mathcal{T}_{\widetilde{\mathcal{A}}_{g,\Gamma}}(-\log D_\infty)\longrightarrow\overline{E^{0,1}}^{\otimes 2}. \tag{1.2.2}$$

Define the restriction map $\rho_0:=\rho|_{\mathcal{A}_{g,\Gamma}}$.

**Lemma 1.3.** *The holomorphic tangent bundle $\mathcal{T}_{\mathcal{A}_{g,\Gamma}}$ of $\mathcal{A}_{g,\Gamma}$ is a holomorphic sub bundle of $(E^{0,1})^{\otimes 2}$. Moreover, the morphism $\rho_0:\mathcal{T}_{\mathcal{A}_{g,\Gamma}}\longrightarrow(E^{0,1})^{\otimes 2}$ is an inclusion of vector bundles.*

*Proof.* We know that the vector bundles $E^{1,0}$, $E^{0,1}$, $\mathcal{O}_{\mathcal{A}_{g,\Gamma}}$, $\mathcal{T}_{\mathcal{A}_{g,\Gamma}}$, $\Omega^1_{\mathcal{A}_{g,\Gamma}}$ are all $\mathrm{Sp}(g,\mathbb{R})$-homogenous, and the morphism $\theta^{1,0}:E^{1,0}\longrightarrow E^{0,1}\otimes\Omega^1_{\mathcal{A}_{g,\Gamma}}$ is a $\mathrm{Sp}(g,\mathbb{R})$-equivariant morphism. Thus, the morphism $\rho_0:\mathcal{T}_{\mathcal{A}_{g,\Gamma}}\longrightarrow(E^{0,1})^{\otimes 2}$ is $\mathrm{Sp}(g,\mathbb{R})$-equivariant. We verify the inclusion at the base point $o\in\mathcal{A}_{g,\Gamma}$: At point $o$, we have $E^{1,0}|_o=H^{1,0}_o, E^{0,1}|_o=H^{0,1}_o$ and $\mathcal{T}_{\widetilde{\mathcal{A}}_{g,\Gamma},o}\subset\mathrm{Hom}(H^{1,0}_o,H^{0,1}_o)=(H^{0,1}_o)^{\otimes 2}$ by Borel's embedding. The construction of the Hodge bundle $(E,\theta)$ shows that the inclusion $\mathcal{T}_{\widetilde{\mathcal{A}}_{g,\Gamma},o}\subset\mathrm{Hom}(H^{1,0}_o,H^{0,1}_o)=(H^{0,1}_o)^{\otimes 2}$ is just the morphism $\rho_0$ at the point $o$. □

We now introduce an induced $\mathrm{Sp}(g,\mathbb{R})$-invariant positive Hermitian metric $H$ (**Hodge metric**) on $\mathcal{A}_{g,\Gamma}$ by the following inclusion

$$\rho_0:\mathcal{T}_{\mathcal{A}_{g,\Gamma}}\xrightarrow{\subset}\mathrm{End}(E)^{-1,1}\subset\mathrm{End}(E).$$

Let $\{l_1,\cdots,l_m\}$ be a holomorphic basis of $\mathcal{T}_{\mathcal{A}_{g,\Gamma}}$ on a local neighborhood $(U,z)$ of $\mathcal{A}_{g,\Gamma}$, and $\{\phi_1,\cdots,\phi_m\}$ be the dual holomorphic basis of $\Omega^1_{\mathcal{A}_{g,\Gamma}}$ over $U$. We define

$$H(l_i,\overline{l_j}):=<\rho_0(l_i),\overline{\rho_0(l_j)}>_H. \tag{1.3.1}$$

Since $\rho_0$ can be linearly extended to a morphism of sheaves of $C^\infty$ germs as well as $\theta$ does, we then obtain a metric $H$ on $\mathcal{A}_{g,\Gamma}$. The Kähler form of $H$ on $U$ can be written locally as

$$\omega_H=\sum_{i,j=1}^m H(l_i,\overline{l_j})\phi_i\wedge\overline{\phi_j}. \tag{1.3.2}$$

**Theorem 1.4.** *Let $\Gamma$ be a neat arithmetic subgroup of $\mathrm{Sp}(g,\mathbb{Z})$.*

*The induced Hodge metric $H$ on the Siegel variety $\mathcal{A}_{g,\Gamma}=\Gamma\backslash\mathfrak{H}_g$ is same as the canonical Bergman metric. Moreover, the Chern connection of $(\mathcal{T}_{\mathcal{A}_{g,\Gamma}},H)$ is compatible with the Levi-Civita connection of the Riemannian manifold $(\mathcal{A}_{g,\Gamma},H)$.*

*Proof.* Notation as in the proof of the lemma 1.1. Since the Hodge metric $H$ on $\mathcal{A}_{g,\Gamma}$ is $\mathrm{Sp}(g,\mathbb{R})$-invariant and $\mathrm{Sp}(g,\mathbb{R})$ is a simple group, it is sufficient to show that $H$ is Kähler.

Let $p$ be an arbitrary point on $\mathcal{A}_{g,\Gamma}$. Let $U$ be a suitable neighborhood of $p$ such that we can choose a local holomorphic coordinates $(z_1,\cdots,z_m)$ satisfying

$$\mathcal{T}_{\mathcal{A}_{g,\Gamma}}|_U=\mathrm{span}\{\frac{\partial}{\partial z_1},\cdots,\frac{\partial}{\partial z_m}\}.$$

Let $\{e_\alpha\}$ be a local holomorphic basis of $E$ and $\{e^\alpha\}$ the local dual holomorphic basis of $E^\vee$.

All calculation below are locally over $U$.

We write $\theta=\sum_{i=1}^k A^i dz_i$, where $A^i=\sum_{\alpha,\beta}A^{i,\alpha}_\beta e_\alpha\otimes e^\beta\in\mathrm{End}(E)$. The Kähler form is then

$$\omega_H:=\sum_{i,j=1}^k H(l_i,\overline{l_j})dz_i\wedge\overline{dz_j}=\sum_{i,j=1}^k<A^i,\overline{A^j}>_H dz_i\wedge\overline{dz_j}.$$

Thus, we have that

$$\begin{aligned}d\omega_H &= \sum_{i,j}(d<A^i,\overline{A^j}>_H)\wedge dz_i\wedge\overline{dz_j}\\ &= \sum_{i,j=1}^m(<\nabla_H A^i,\overline{A^j}>_H+<A^i,\overline{\nabla_H A^j}>)\wedge dz_i\wedge\overline{dz_j},\end{aligned}$$

where $\nabla_H$ is the Chern connection on $(\mathrm{End}(E),H)$. For each $i=1,\cdots,m$, We have:

$$\begin{aligned}\nabla_H A^i &= \overline{\partial}A^i+\partial A^i=\partial A^i\\ &= \sum_{k=1}^n A^{i,\alpha}_{\beta;k}e_\alpha\otimes e^\beta\phi_k.\end{aligned}$$

Since $\partial(\theta)=0$ by the corollary 1.2, we have

$$A^{i,\delta}_{\alpha;j}=A^{j,\delta}_{\alpha;i}\ \forall i,j,\alpha,\delta. \tag{1.4.1}$$

Contract the neighborhood $(U,p)$ sufficiently small, we can choose a special holomorphic basis $\{e_\alpha\}$ of $E$ over $U$ such that $H(p)=\mathrm{Id}$, $dH(p)=0$ under the frame

$\{e_\alpha\}$. At the point $p$, we calculate

$$\begin{aligned}
&(d^{1,0}\omega_H)(p)\\
&=\sum_{i,j,l}\sum_{\alpha,\beta} < A^{i,\alpha}_{\delta;l} e_\alpha\otimes e^\delta, \overline{A^{j,\tau}_{\beta} e_\tau\otimes e^\beta} >_H dz_l\wedge dz_i\wedge \overline{dz_j}\\
&=\sum_{l,i,j=1}^{m}\sum_{\alpha,\beta} A^{i,\alpha}_{\beta;l}\overline{A^{j,\alpha}_{\beta}} dz_l\wedge dz_i\wedge\overline{dz_j}\\
&=\sum_{j=1}^{m}\sum_{\alpha,\beta}\overline{A^{j,\alpha}_{\beta}}(\sum_{i,l=1}^{k} A^{i,\alpha}_{\beta;l} dz_l\wedge dz_i)\wedge\overline{dz_j}\\
&=0.
\end{aligned}$$

Similarly, $d^{0,1}\omega_H=0$ at the point $p$.

The rest is obvious. □

**Remark 1.5.** If the holomorphic tangent bundle of some complex manifold $M$ can be embedded into a harmonic bundle (e.g. some Higgs bundles with Hermitian metric of trivial Chern form) on $M$, one can endow a Kähler metric on $M$ by the same method in above Theorem 1.4. This Kähler metric is still called **Hodge metric** and it is easy to get this Kähler metric has semi-negative holomorphic bisectional curvature by an argument in Lemma 2.2 of [Zuo00].

We still use $H$ to represent the dual metric of $\Omega^1_{\mathcal{A}_{g,\Gamma}}$ induced by $(\mathcal{T}_{\mathcal{A}_{g,\Gamma}},H)$. We write $\Theta(\mathcal{T}_{\mathcal{A}_{g,\Gamma}},H)$ (resp. $\Theta(\Omega^1_{\mathcal{A}_{g,\Gamma}},H)$) as the Chern curvature form of the vector bundle $\mathcal{T}_{\mathcal{A}_{g,\Gamma}}$ (resp. $\Omega^1_{\mathcal{A}_{g,\Gamma}}$). As the canonical Bergman metric on $\mathcal{A}_{g,\Gamma}$ is Kähler-Einstein, there is $\frac{-1}{2\pi\sqrt{-1}}\mathrm{Trace}_H(\Theta(\mathcal{T}_{\mathcal{A}_{g,\Gamma}},H))=-\lambda\omega_H$, where $\lambda$ is a positive constant. Without lost of generality, we always assume $\lambda=1$ for convenience.

**Theorem 1.6.** *Let $\Gamma$ be a neat arithmetic subgroup of $\mathrm{Sp}(g,\mathbb{Z})$. Let $\widetilde{\mathcal{A}}_{g,\Gamma}$ be an arbitrary smooth compactification(not necessary smooth toroidal compactification) of the Siegel variety $\mathcal{A}_{g,\Gamma}:=\Gamma\backslash\mathfrak{H}_g$ such that $D_\infty=\widetilde{\mathcal{A}}_{g,\Gamma}\setminus\mathcal{A}_{g,\Gamma}$ is a simple normal crossing divisor. We have:*

1. *The canonical Bergman metric $H_{\mathrm{can}}$ of $\mathcal{A}_{g,\Gamma}$ is bounded by the logarithmic degeneration along the boundary divisor $D_\infty$ in sense of the following description:*

   *Let $p$ be a point in $\widetilde{\mathcal{A}}_{g,\Gamma}$ with a coordinate chart $(U,(z_1,\cdots,z_n))(n=g(g+1)/2)$ such that*

   $$\begin{aligned}U\cap\mathcal{A}_{g,\Gamma}=\{(z_1,\cdots,z_l,\cdots z_n)\mid 0<|z_i|<1(i=1,\cdots l),\\ |z_j|<1(i=l+1,\cdots,n)\ \}.\end{aligned}$$

   *There holds*

   $$\frac{1}{C}(\prod_{i=1}^{l}-\log|z_i|)^{-M}\le\|\frac{\partial}{\partial z_j}\|_{H_{\mathrm{can}}}\le C(\prod_{i=1}^{l}-\log|z_i|)^{M}\ \forall j$$

   *in the coordinate chart $\{(z_1,\cdots,z_l,\cdots,z_n)\ |0<|z_i|<r(i=1,\cdots,l),|z_j|<r(j=l+1,\cdots,n)\}$ of $p$ for a suitable $r>0$, where $C,M$ are positive constants depending on $r$.*

2. *The Kähler form $\omega_{\mathrm{can}}$ becomes a closed positive current $[\omega_{\mathrm{can}}]$ on $\widetilde{\mathcal{A}}_{g,\Gamma}$.*
3. *The line bundle*

   $$\omega_{\widetilde{\mathcal{A}}_{g,\Gamma}}(D_\infty)=\bigwedge^{\dim\mathcal{A}_{g,\Gamma}}\Omega^1_{\widetilde{\mathcal{A}}_{g,\Gamma}}(\log D_\infty)$$

   *is pseudo-effective on $\widetilde{\mathcal{A}}_{g,\Gamma}$. Precisely, there is an equality $c_1(\Omega^1_{\widetilde{\mathcal{A}}_{g,\Gamma}}(D_\infty))=[\omega_{\mathrm{can}}]$.*
4. *The line bundle $\omega_{\widetilde{\mathcal{A}}_{g,\Gamma}}(D_\infty)$ is big on $\widetilde{\mathcal{A}}_{g,\Gamma}$.*

**Remark.** It is well known that Bergman metric on locally symmetric has Poincaré growth on $D_\infty$. Our result is strong than this classic result in literature.

Before proving the theorem 1.6, we review the theory of degeneration of Hodge metrics on any polarized variation of Hodge structures over a quasi Kähler manifold.

Let $X$ be an open Kähler manifold of complex dimension $m$. Let $\overline{X}$ be one smooth compactification of $X$ such that the boundary $D:=\overline{X}-X$ is a simple normal crossing divisor. Let $j:X\xrightarrow{\subset}\overline{X}$ be the open embedding. Let $\mathbb{V}$ be an arbitrary polarized variation of real Hodge structures over $X$ such that all monodromies around $D$ are unipotent. Denote by $\mathcal{V}=\mathbb{V}\otimes\mathcal{O}_X$. Consequently, we have a Hodge filtration

$$\mathcal{V}=\mathcal{F}^0\supset\mathcal{F}^1\supset\cdots\supset\mathcal{F}^w\supset 0$$

corresponding to the VHS $\mathbb{V}$, where $w$ is the weight of the VHS $\mathbb{V}$.

Let $(\triangle_1,z)\subset\overline{X}$ be a **special coordinate neighborhood**, i.e., a coordinate neighborhood isomorphic to the polycylinder $\Delta^m$ ($\Delta:=\{z\in\mathbb{C}\,|\,|z|\le1\}$) such that

$$\begin{aligned}X\cap\triangle_1&\cong\{z=(z_1,\cdots,z_l,\cdots,z_m)\in\Delta^m\mid z_1\ne0,\cdots,z_l\ne0\}\\&=(\Delta^*)^l\times\Delta^{m-l}.\end{aligned}$$

We then have $\triangle_1\cap D_\infty\cong\{(z_1,\cdots,z_l,\cdots,z_m)\mid z_1\cdots z_l=0\}$. For any real number $0<\varepsilon<1$, let $\triangle_\varepsilon$ be a scaling neighborhood of $\overline{X}$, i.e.,

$$\begin{aligned}&\triangle_\varepsilon\subset\triangle_1\text{ and}\\&\triangle_\varepsilon\cong\{(z_1,\cdots,z_l,\cdots,z_m)\in\Delta^m\mid|z_\alpha|\le\varepsilon\text{ for }\alpha=1,\cdots,l\}.\end{aligned}$$

Let $\gamma_\alpha$ be a local monodromy around $z_\alpha=0$ in $\triangle_1$ for $\alpha=1,\cdots l$. Denote by

$$N_\alpha=\log\gamma_\alpha:=\sum_{j\ge1}(-1)^{j+1}\frac{(\gamma_\alpha-1)^j}{j},\ \forall\alpha,$$

then each $N_\alpha$ is nilpotent. Let $(v.)$ be a flat multi-valued basis of $\mathcal{V}$ over $\triangle_1\cap X$. The formula

$$(\widetilde{v.})(z):=\exp(\frac{-1}{2\pi\sqrt{-1}}\sum_{\alpha=1}^{l}\log z_\alpha N_\alpha)(v.)(z)$$

gives a single-valued basis of $\mathcal{V}$. Deligne's canonical extension $\overline{\mathcal{V}}$ of $\mathcal{V}$ to $\triangle_1$ is generated by $(\widetilde{v.})$ (cf. [Sch73]).

The construction of $\overline{\mathcal{V}}$ is independent of the choice of $z_i's$ and $(v_\cdot)$. For any holomorphic sub bundle $\mathcal{N}$ of $\mathcal{V}$, Deligne's extension of $\mathcal{N}$ is defined to be $\overline{\mathcal{N}} := \overline{\mathcal{V}} \cap j_*\mathcal{N}$. Then, we have extension of the filtration

$$\overline{\mathcal{V}} = \overline{\mathcal{F}}^0 \supset \overline{\mathcal{F}}^1 \supset \cdots \supset \overline{\mathcal{F}}^w \supset 0,$$

which is also a filtration of locally free sheaves.

Let $N$ be a linear combination of $N_\alpha$. Then $N$ defines a weight flat filtration $W_\bullet(N)$ of $\mathbb{V}_\mathbb{C}$ [Sch73] by

$$0 \subset \cdots \subset W_{i-1}(N) \subset W_i(N) \subset W_{i+1}(N) \subset \cdots \subset \mathbb{V}_\mathbb{C}.$$

Denote by $W_\bullet^j := W_\bullet(\sum_{\alpha=1}^{j} N_\alpha)$ for $j = 1, \cdots, l$. We can choose a multi-valued flat multi-grading

$$\mathbb{V}_\mathbb{C} = \sum_{\beta_1,\cdots,\beta_l} \mathbb{V}_{\beta_1,\cdots,\beta_l}$$

such that

$$\bigcap_{j=1}^{l} W_{\beta_j}^j = \sum_{k_j \le \beta_j} \mathbb{V}_{k_1,\cdots,k_l}.$$

Let $h$ be the Hodge metric on the PVHS $\mathcal{V}$. In a special coordinate neighborhood $\triangle_1$, let $v$ be a nonzero local multi-valued flat section of $\mathbb{V}_{k_1,\cdots,k_l}$, then $(\widetilde{v})(z) := \exp(\frac{-\sum_{\alpha=1}^{l} \log z_\alpha N_\alpha}{2\pi\sqrt{-1}}) v(z)$ is a local single-valued section of $\overline{\mathcal{V}}$. There holds a norm estimate (Theorem 5.21 in [CKS86])

$$||\widetilde{v}(z)||_h \le C'''(\frac{-\log|z_1|}{-\log|z_2|})^{k_1/2}(\frac{-\log|z_2|}{-\log|z_3|})^{k_1/2}\cdots(-\log|z_l|)^{k_l/2}$$

on the region

$$\Xi(N_1,\cdots,N_l) := \{(z_1,\cdots,z_l,\cdots,z_m) \in (\Delta^*)^l \times \Delta^{m-l} \mid |z_1| \le |z_2| \le \cdots \le |z_l| \le \varepsilon\}$$

for some small $\varepsilon > 0$, where $C'''$ is a positive constant dependent on the ordering of $\{N_1, N_2, \cdots, N_l\}$ and $\varepsilon$. Since the number of the ordering of $\{N_1, \cdots, N_l\}$ is finite, for any flat multi-valued local section $v$ of $\mathbb{V}$ there exist positive constants $C''(\varepsilon)$ and $M''$ such that

$$(1.6.1) \qquad ||\widetilde{v}(z)||_h \le C''(\varepsilon)(\prod_{\alpha=1}^{l} -\log|z_\alpha|)^{M''}$$

in the domain $\{(z_1,\cdots,z_l,\cdots,z_m) \,|0 < |z_i| < \varepsilon(i = 1,\cdots,l), |z_j| < \varepsilon(j = l+1,\cdots,m)\}$.

Moreover, since the dual $\mathbb{V}^\vee$ is also a polarized real variation of Hodge structures, we then have that for any flat multi-valued local section $v$ of $\mathbb{V}$ there holds

(1.6.2)

$$\frac{1}{C_1'}(\prod_{\alpha=1}^{l} -\log|z_\alpha|)^{-M'} \le ||\widetilde{v}(z)||_h \le C_1'(\prod_{\alpha=1}^{l} -\log|z_\alpha|)^{M'}$$

in the domain $\{(z_1,\cdots,z_l,\cdots,z_m) \,|0 < |z_i| < \varepsilon(i = 1,\cdots,l), |z_j| < \varepsilon(j = l+1,\cdots,m)\}$ for some suitable $\varepsilon > 0$, where $C_1'$ and $M'$ are positive constants.

**Proposition 1.7.** *Let $X$ be an open Kähler manifold of dimension $m$ and $\overline{X}$ a smooth compactification of $X$ such that the boundary $D := \overline{X} - X$ is a simple normal crossing divisor. Let $(\triangle_1, z = (z_1,\cdots,z_l,\cdots,z_m)) \subset \overline{X}$ be an arbitrary special coordinate neighborhood in which $D$ is given by $\prod_{i=1}^{l} z_i = 0$.*

*Let $\mathbb{V}$ be a polarized real VHS on $X$ such that all local monodromies of $\mathbb{V}$ around the simple normal crossing boundary divisor are unipotent, and $h$ the Hodge metric on $\mathcal{V} = \mathbb{V} \otimes \mathcal{O}_X$. Let $\mathcal{N}$ be an arbitrary holomorphic sub bundle of $\mathcal{V}$ and $\overline{\mathcal{N}}$ its Deligne's extension.*

*We have*

$$\Gamma(\triangle_\varepsilon, \overline{\mathcal{N}}) = \{s \in \Gamma(\triangle_\varepsilon \cap X, \mathcal{N}) \mid ||s||_h \le C(\sum_{\alpha=1}^{l} -\log|z_\alpha|)^M \textit{ for some constants } M, C\},$$

*where $\Delta_\varepsilon$ is a scaling neighborhood of $\overline{X}$ for a sufficient small $\varepsilon > 0$.*

*Proof.* Let $(v_\cdot)$ be a local flat multi-valued basis of $\mathcal{V}$ over $\triangle_1 \cap X$, and so we have a local basis $(\widetilde{v}_\cdot)(z) = \exp(\frac{-1}{2\pi\sqrt{-1}}\sum_{\alpha=1}^{l} \log z_\alpha N_\alpha)(v_\cdot)(z)$ of $\overline{\mathcal{V}}$. Denote by $h_{ij} =< \widetilde{v_i}, \widetilde{v_j} >_H$. According to the estimate 1.6.2, there are positive constants $C, M$ such that

$$(1.7.1) \qquad |h_{ij}|, \det(h_{ij}), (\det(h_{ij}))^{-1} \le C(\sum_{\alpha=1}^{l} \log|z_\alpha|)^{2M}$$

in a suitable neighborhood $\triangle_\varepsilon$. One can use Proposition 1.3 in [Mum77] to finish the proof. □

*Proof of the theorem 1.6.*

1. By the lemma 1.3 and the theorem 1.4, we can realize the holomorphic tangent bundle $\mathcal{T}_{\mathcal{A}_{g,\Gamma}}$ as a holomorphic sub bundle of an Hodge bundle given by some PVHS, and the induced Hodge metric $H$ on the Siegel variety $\mathcal{A}_{g,\Gamma} = \Gamma\backslash\mathfrak{H}_g$ is same as the canonical Bergman metric. We then finish the first statement by using the estimate 1.6.2.
2. By the statement (1), we have $\int_{\mathcal{A}_{g,\Gamma}} |\omega_{\text{can}} \wedge \xi| < \infty$ for any smooth $(2(\dim \mathcal{A}_{g,\Gamma}) - 2)$-form $\xi$ on $\widetilde{\mathcal{A}}_{g,\Gamma}$, and so the form $\omega_{\text{can}}$ on $\mathcal{A}_{g,\Gamma}$ defines a current $[\omega_{\text{can}}]$ on $\widetilde{\mathcal{A}}_{g,\Gamma}$ as follows:

$$< [\omega_{\text{can}}], \phi >= \int_{\widetilde{\mathcal{A}}_{g,\Gamma}} [\omega_{\text{can}}] \wedge \phi := \int_{\mathcal{A}_{g,\Gamma}} \omega_{\text{can}} \wedge \phi,$$

where $\phi$ is a smooth $(2\dim(\mathcal{A}_{g,\Gamma}) - 2)$ form on $\widetilde{\mathcal{A}}_{g,\Gamma}$.

We begin to show that $d[\omega_{\text{can}}] = [d\omega_{\text{can}}] = 0$. Let $\xi$ be any smooth $(2\dim(\mathcal{A}_{g,\Gamma}) - 3)$-form on $\widetilde{\mathcal{A}}_{g,\Gamma}$. Let $T_\delta$

be a tube neighborhood of $D_{\infty,n}=\widetilde{\mathcal{A}}_{g,\Gamma}-\mathcal{A}_{g,\Gamma}$ with radius $\delta$ and $M_\delta:=\widetilde{\mathcal{A}}_{g,\Gamma}\setminus T_\delta$. Then $\partial T_\delta=-\partial M_\delta$. By definition, $<d[\omega_{\rm can}],\xi>:=-\int_{\mathcal{A}_{g,\Gamma}}\omega_{\rm can}\wedge d\xi$. On the other hand, we have that

$$\begin{aligned} 0 &= <[d\omega_{\rm can}],\xi>:=\int_{\mathcal{A}_{g,\Gamma}}d\omega_{\rm can}\wedge\xi\\ &= -\int_{\mathcal{A}_{g,\Gamma}}\omega_{\rm can}\wedge d\xi+\int_{\mathcal{A}_{g,\Gamma}}d(\omega_{\rm can}\wedge\xi)\\ &= -\int_{\mathcal{A}_{g,\Gamma}}\omega_{\rm can}\wedge d\xi+\lim_{\delta\to 0}\int_{M_\delta}d(\omega_{\rm can}\wedge\xi)\end{aligned}$$

(by Stoke's theorem)

$$\begin{aligned} &= -\int_{\mathcal{A}_{g,\Gamma}}\omega_{\rm can}\wedge d\xi+\lim_{\delta\to 0}\int_{\partial M_\delta}\omega_{\rm can}\wedge\xi\\ &= -\int_{\mathcal{A}_{g,\Gamma}}\omega_{\rm can}\wedge d\xi-\lim_{\delta\to 0}\int_{\partial T_\delta}\omega_{\rm can}\wedge\xi\\ &= -\int_{\mathcal{A}_{g,\Gamma}}\omega_{\rm can}\wedge d\xi.\end{aligned}$$

Here we use that $\omega_H$ has Poincaré growth on $D_\infty$ to obtain $\lim\limits_{\delta\to 0}\int_{\partial T_\delta}\omega_{\rm can}\wedge\xi=0$.

3. Since $[\omega_{\rm can}]$ is a positive closed current, it is a cohomology class on $\widetilde{\mathcal{A}}_{g,\Gamma}$ of type $(1,1)$. To prove that $[\omega_{\rm can}]$ represents the first Chern class $c_1(\Omega^1_{\widetilde{\mathcal{A}}_{g,\Gamma}}(\log D_\infty))$, we only need to show the following equality

$$<[\omega_{\rm can}],\eta>=<c_1(\Omega^1_{\widetilde{\mathcal{A}}_{g,\Gamma}}(\log D_\infty)),\eta>$$

for any closed smooth $(2\dim(\mathcal{A}_{g,\Gamma})-2)$-form $\eta$ on $\widetilde{\mathcal{A}}_{g,\Gamma}$.
Let $\eta$ be an arbitrary closed smooth $(2\dim(\mathcal{A}_{g,\Gamma})-2)$-form on $\widetilde{\mathcal{A}}_{g,\Gamma}$. Let $\widetilde{H}$ be an arbitrary Hermitian metric on the bundle $\Omega^1_{\widetilde{\mathcal{A}}_{g,\Gamma}}(\log D_\infty)$. We have

$$\begin{aligned} &<c_1(\Omega^1_{\widetilde{\mathcal{A}}_{g,\Gamma}}(\log D_\infty)),\eta>\\ &:=\frac{-1}{2\pi\sqrt{-1}}\int_{\widetilde{\mathcal{A}}_{g,\Gamma}}\mathrm{Trace}_{\widetilde{H}}(\Theta(\Omega^1_{\widetilde{\mathcal{A}}_{g,\Gamma}}(\log D_\infty),\widetilde{H}))\wedge\eta\\ &=\frac{-1}{2\pi\sqrt{-1}}\int_{\widetilde{\mathcal{A}}_{g,\Gamma}}\partial\overline{\partial}\log(\det\widetilde{H})\wedge\eta\\ &=\frac{-1}{2\pi\sqrt{-1}}\int_{\mathcal{A}_{g,\Gamma}}\partial\overline{\partial}\log(\det\widetilde{H})\wedge\eta\end{aligned}$$

where $\Theta(\Omega^1_{\widetilde{\mathcal{A}}_{g,\Gamma}}(\log D_\infty),\widetilde{H})$ is the Chern form of $(\Omega^1_{\widetilde{\mathcal{A}}_{g,\Gamma}}(\log D),\widetilde{H})$, and

$$\begin{aligned} <[\omega_{\rm can}],\eta>&:=\int_{\mathcal{A}_{g,\Gamma}}\omega_{\rm can}\wedge\eta\\ &=\frac{-1}{2\pi\sqrt{-1}}\int_{\mathcal{A}_{g,\Gamma}}\mathrm{Trace}_{H_{\rm can}}\Theta(\Omega^1_{\mathcal{A}_{g,\Gamma}},H_{\rm can})\wedge\eta\\ &=\frac{-1}{2\pi\sqrt{-1}}\int_{\mathcal{A}_{g,\Gamma}}\overline{\partial}\partial\log(\det H_{\rm can})\wedge\eta.\end{aligned}$$

Thus, it is sufficient to show that

$$\lim_{\delta\to 0}\int_{M_\delta}\overline{\partial}\partial\log(\frac{\det H_{\rm can}}{\det\widetilde{H}})\wedge\eta=0.$$

We note that $\zeta:=\partial\log\det H_{\rm can}-\partial\log\det\widetilde{H}$ is a global $(1,0)$-form on $\mathcal{A}_{g,\Gamma}$, we then get

$$\begin{aligned}\int_{M_\delta}\overline{\partial}\partial\log(\frac{\det H_{\rm can}}{\det\widetilde{H}})\wedge\eta &= \int_{M_\delta}d\zeta\wedge\eta\\ &= -\int_{\partial T_\delta}\zeta\wedge\eta.\end{aligned}$$

As an application of the theorem 1.4, we obtain that the $(1,0)$-form $\zeta$ near the boundary divisor $D_\infty$ is nearly bounded in sense of Kollár (cf. [Kol87]) by Proposition 5.22 in [CKS86]. Thus, we have

$$\lim_{\delta\to 0}\int_{\partial T_\delta}\zeta\wedge\eta=0.$$

4. Since the metric connection form of any Hodge metric and its curvature form are both nearly bounded around the boundary divisor $D_\infty$ (cf. Proposition 5.7 [Kol87]), we have:

$$\begin{aligned} &C_1(\Omega^1_{\widetilde{\mathcal{A}}_{g,\Gamma}}(\log D_\infty))^{\dim\mathcal{A}_{g,\Gamma}}\\ &=(\frac{-1}{2\pi\sqrt{-1}})^{\dim\mathcal{A}_{g,\Gamma}}\int_{\mathcal{A}_{g,\Gamma}}\mathrm{Trace}_H(\Theta(\Omega^1_{\mathcal{A}_{g,\Gamma}},H))^{\dim\mathcal{A}_{g,\Gamma}}\\ &>0.\end{aligned}$$

Since $\omega_{\widetilde{\mathcal{A}}_{g,\Gamma}}(D_\infty)$ is a numerically effective line bundle on $\widetilde{\mathcal{A}}_{g,\Gamma}$, we obtain that $\omega_{\widetilde{\mathcal{A}}_{g,\Gamma}}(D_\infty)$ is a big line bundle by Siu's numerical criterion in [Siu93]. □

**Remark.** We must point out that the statement (4) of Theorem 1.6 is first proven in [Mum77] by using some calculations depending on a smooth toroidal compactification $\widetilde{\mathcal{A}}_{g,\Gamma}$, and it can also be proven by Siu-Yau's result on compactification (cf. Lemma 6 in [SY82]) or by Zuo's result on positivity (cf. Theorem 0.1 in [Zuo00]).

**Lemma 1.8** (Moeller-Viehweg-Zuo cf. [MVZ07]). *Let $\Gamma$ be a neat arithmetic subgroup of $\mathrm{Sp}(g,\mathbb{Z})$. Let $\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}$ be a smooth toroidal compactification of the Siegel variety $\mathcal{A}_{g,\Gamma}:=\Gamma\backslash\mathfrak{H}_g$ such that $D_\infty:=\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}\setminus\mathcal{A}_{g,\Gamma}$ is simple normal crossing. Let $(\mathfrak{L},\theta,h)$ be a homogenous Hodge bundle induced by PVHS on $\mathcal{A}_{g,\Gamma}$ and $\mathcal{N}$ any homogenous sub bundle of $\mathfrak{L}$.*

*Deligne's canonical extension of the bundle $\mathcal{N}$ to $\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}$ coincides with the Mumford good extension (cf. [Mum77]) of $\mathcal{N}$ to $\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}$ by the Hodge metric $h$.*

*Proof.* It is a direct consequence of the estimates 1.6.1, 1.6.2 and the proposition 1.7. □

**Lemma 1.9.** *Let $\Gamma$ be a neat arithmetic subgroup of $\mathrm{Sp}(g,\mathbb{Z})$. Let $\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}$ be a smooth toroidal compactification* *of the Siegel variety $\mathcal{A}_{g,\Gamma} := \Gamma\backslash\mathfrak{H}_g$ such that $D_\infty := \overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}\setminus \mathcal{A}_{g,\Gamma}$ is simple normal crossing.*

*We have the following identifications*

$$\mathcal{T}_{\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}}(-\log D_\infty) = \mathrm{Sym}^2(\overline{E^{0,1}}),$$

*and*

$$\omega_{\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}}(D_\infty) = \bigwedge^{\dim_{\mathbb{C}}\mathcal{A}_{g,\Gamma}} \Omega^1_{\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}}(\log D_\infty) = (\det\overline{E^{1,0}})^{g+1}.$$

*Moreover, the line bundle $\omega_{\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}}(D_\infty)$ is semi-positive on the compactification $\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}$.*

*Proof.* We know that there is an inclusion $\mathcal{T}_{\mathcal{A}_{g,\Gamma}} \xrightarrow{\subset} (E^{0,1})^{\otimes 2}$. Since the Higgs field has the property $\theta\wedge\theta = 0$, the holomorphic sub bundle $\mathrm{Sym}^2(E^{0,1})$ of $(E^{0,1})^{\otimes 2}$ must contain the bundle $\mathcal{T}_{\mathcal{A}_{g,\Gamma}}$. According to $\mathrm{rank}_{\mathbb{C}}\mathcal{T}_{\mathcal{A}_{g,\Gamma}} = \mathrm{rank}_{\mathbb{C}}\mathrm{Sym}^2(E^{0,1}) = g(g+1)/2$, we obtain $\mathcal{T}_{\mathcal{A}_{g,\Gamma}} = \mathrm{Sym}^2(E^{0,1})$.

The holomorphic vector bundle $\mathrm{Sym}^2(\overline{E^{0,1}})$ on $\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}$ is Deligne's extension of $\mathrm{Sym}^2(E^{0,1})$. Using the proposition 1.8 in the next subsection, $\mathrm{Sym}^2(\overline{E^{0,1}})$ is also the unique Mumford's good extension of $\mathrm{Sym}^2(E^{0,1})$ by the Hodge metric $H$. Shown in Proposition 3.4 [Mum77] $\mathcal{T}_{\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}}(-\log D_\infty)$ is the unique Mumford's good extension of $\mathcal{T}_{\mathcal{A}_{g,\Gamma}}$ by the metric $H$. Therefore,

$$\mathcal{T}_{\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}}(-\log D_\infty) \cong \mathrm{Sym}^2(\overline{E^{0,1}}).$$

Thus, we obtain that $\omega_{\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}}(D_\infty) = (\det\overline{E^{1,0}})^{g+1}$, so that $\omega_{\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}}(D_\infty)$ is semi-positive by Kawamata's positivity package in [Kawa81]. □

**Remark.** We can also get the semi-positivity of $\omega_{\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}}(D_\infty)$ by an argument of Mumford: it is shown in [Mum77] that the sheaf $\omega_{\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}}(D_\infty)$ is the pull back of an ample line on the Satake-Baily-Borel compactification $\mathcal{A}^*_{g,\Gamma} := \Gamma\backslash\mathfrak{H}^*_g$.

**Theorem 1.10.** *Let $\Gamma\subset \mathrm{Sp}(g,\mathbb{Z})$ be a neat arithmetic subgroup. Let $\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}$ be a smooth toroidal compactification of the Siegel variety $\mathcal{A}_{g,\Gamma} := \Gamma\backslash\mathfrak{H}_g$ such that $D_\infty := \overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}\setminus\mathcal{A}_{g,\Gamma}$ is a simple normal crossing divisor.*

*The logarithmic tangent bundle $\mathcal{T}_{\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}}(-\log D_\infty)$ is a stable vector bundle with respect to the polarization $K_{\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}} + D_\infty$.*

*Proof.* Let $\mathcal{E}$ be Deligne's canonical extension of the Hodge bundle $(E^{\otimes 2}, H)$, and $\overline{E^{0,1}}$ be Deligne's canonical extension of $E^{0,1}$. By the lemma 1.3 and the lemma 1.9, the logarithmic tangent bundle $\mathcal{T}_{\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}}(-\log D_\infty) = \mathrm{Sym}^2(\overline{E^{0,1}})$, and so $\mathcal{T}_{\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}}(-\log D_\infty)$ is a holomorphic sub bundle of $\mathcal{E} := \overline{E}^{\otimes 2}$.

Let $\mathcal{G}$ be an arbitrary sub bundle of $\mathcal{E}$ and $\widetilde{H}$ an arbitrary Hermitian metric on $\mathcal{G}$. Let $\mathcal{G}_0 := \mathcal{G}|_{\mathcal{A}_{g,\Gamma}}$. We know $\mathcal{G}$ is just Deligne canonical extension of $\mathcal{G}_0$. The degree of $\mathcal{G}$ with respect to the polarization $K_{\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}} + D_\infty$ *is* is

$$\begin{aligned}\deg\mathcal{G} &:= \; < c_1(\mathcal{G}), \bigwedge^{\dim\mathcal{A}_{g,\Gamma}-1} c_1(K_{\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}} + D_\infty) > \\ &= \; < c_1(\mathcal{G}), [\omega^{\dim\mathcal{A}_{g,\Gamma}-1}] >\end{aligned}$$

by (3) of the theorem 1.6 and Kollár's argument of 5.18 in [Kol87]. Let $\eta := \omega^{\dim\mathcal{A}_{g,\Gamma}-1}$. Similar calculation as (3) of the theorem 1.6, we have that

$$\begin{aligned}\deg\mathcal{G} &= \int_{\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}} \mathrm{Trace}_{\widetilde{H}}(\Theta(\mathcal{G},\widetilde{H}))\wedge[\eta] \\ &= \frac{-1}{2\pi\sqrt{-1}}\int_{\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}} \partial\overline{\partial}\log(\det\widetilde{H})\wedge[\eta] \\ &= \frac{-1}{2\pi\sqrt{-1}}\int_{\mathcal{A}_{g,\Gamma}} \partial\overline{\partial}\log(\det\widetilde{H})\wedge\eta \\ &= \frac{-1}{2\pi\sqrt{-1}}\int_{\mathcal{A}_{g,\Gamma}} \partial\overline{\partial}\log(\det H)\wedge\eta \\ &\quad + \frac{1}{2\pi\sqrt{-1}}\int_{\mathcal{A}_{g,\Gamma}} \overline{\partial}\partial\log(\frac{\det H}{\det\widetilde{H}})\wedge\eta \\ &= \frac{-1}{2\pi\sqrt{-1}}\int_{\mathcal{A}_{g,\Gamma}} \partial\overline{\partial}\log(\det H)\wedge\eta \\ &= \int_{\mathcal{A}_{g,\Gamma}} \mathrm{Trace}_H(\Theta(\mathcal{G}_0,H))\wedge\omega_{\rm can}^{\dim\mathcal{A}_{g,\Gamma}-1}\end{aligned}$$

Since the canonical Bergman metric is Kähler-Einstein, this essential property implies that the logarithmic tangent bundle $\mathcal{T}_{\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}}(-\log D_\infty)$ is a poly-stable vector bundle with respect to $[\omega]$.

On the other hand, $\mathcal{A}_{g,\Gamma}$ is simple, then we obtain that the logarithmic tangent bundle $\mathcal{T}_{\overline{\mathcal{A}}^{\rm tor}_{g,\Gamma}}(-\log D_\infty)$ can not be decomposed into a direct sum by the argument in the third paragraph of Page 272 in [Yau87] and the argument of Page 478–478 in [Yau93]. □

## 2. Some Applications on Siegel Varieties

All definitions and notations related to toroidal compactifications of Siegel varieties can be found in [AMRT], [Chai], [FC] and [YZ2014]. We do not recite these definitions and notations in this section again, and use them freely.

Let $\mathfrak{F}_0$ be the standard minimal cusp of the Siegel space $\mathfrak{H}_g$. Let $\Sigma_{\mathfrak{F}_0} := \{\sigma_\alpha^{\mathfrak{F}_0}\}$ be a suitable $\mathrm{GL}(g,\mathbb{Z})$-admissible polyhedral decomposition of $C(\mathfrak{F}_0)$ regular with respect to $\mathrm{Sp}(g,\mathbb{Z})$ such that the induced symmetric $\mathrm{Sp}(g,\mathbb{Z})$-admissible family $\{\Sigma_{\mathfrak{F}}\}_{\mathfrak{F}}$ of polyhedral decompositions is projective.

For any positive integer $l$, let $\overline{\mathcal{A}}_{g,l}$ be the symmetric toroidal compactification of the Siegel variety $\mathcal{A}_{g,l} :=$

$\Gamma_g(l)\backslash\mathfrak{H}_g$ constructed by $\{\Sigma_{\mathfrak{F}}\}_{\mathfrak{F}}$, and let

$$D_{\infty,l} := \overline{\mathcal{A}}_{g,l} - \mathcal{A}_{g,l}$$

the boundary divisor. For convenience, we write $\mathcal{A}_g$ for $\mathcal{A}_{g,1}$.

For any positive integer $l$, we sketch a key-step in the construction of the symmetric compactification $\overline{\mathcal{A}}_{g,l}$ as follows:

Let $\mathfrak{F}$ be an arbitrary cusp of depth $k$. $L_{\mathfrak{F}}(l) := \Gamma(l)\cap U^{\mathfrak{F}}(\mathbb{Q})$ is a full lattice in the vector space $U^{\mathfrak{F}}(\mathbb{C})$, and its dual is $M_{\mathfrak{F}}(l) := \mathrm{Hom}_{\mathbb{Z}}(L_{\mathfrak{F}}(l),\mathbb{Z})$. Explicitly, let $\{\zeta_\alpha\}_1^{k(k+1)/2}$ be a lattice basis of $L_{\mathfrak{F}} := \mathrm{Sp}(g,\mathbb{Z})\cap U^{\mathfrak{F}}(\mathbb{Q})$ and $\{\delta_\alpha\}_1^{k(k+1)/2}$ the associated dual basis of $M_{\mathfrak{F}} := \mathrm{Hom}_{\mathbb{Z}}(L_{\mathfrak{F}},\mathbb{Z})$; then $\{\zeta_\alpha^l := l\zeta_\alpha\}_1^{k(k+1)/2}$ is a lattice basis of $L_{\mathfrak{F}}(l)$, and $\{\delta_\alpha^l := \frac{\delta_\alpha}{l}\}_1^{k(k+1)/2}$ is the dual basis of $M_{\mathfrak{F}}(l)$. For any cone $\sigma\in\Sigma_{\mathfrak{F}}$, we get a toroidal variety $X_\sigma(l) := \mathrm{Spec}\mathbb{C}[\sigma^\vee\cap M_{\mathfrak{F}}(l)]$; we then have

$$\widetilde{\Delta}_{\mathfrak{F},\sigma}(l) := \text{ the interior of the closure of } \frac{\mathfrak{H}_g}{\Gamma(l)\cap U^{\mathfrak{F}}(\mathbb{Q})}$$
$$\text{in } X_\sigma(l)\times_{T_{\mathfrak{F}}(l)}\frac{D(\mathfrak{F})}{\Gamma(l)\cap U^{\mathfrak{F}}(\mathbb{Q})}$$

where $T_{\mathfrak{F}}(l) := \mathrm{Spec}\mathbb{C}[M_{\mathfrak{F}}(l)]$ is a torus; gluing all $\widetilde{\Delta}_{\mathfrak{F},\sigma}(l)$ as $\sigma$ runs through $\Sigma_{\mathfrak{F}}$, we obtain an analytic variety $Z'_{\mathfrak{F}}(l)$ and an open morphism $\pi'_{\mathfrak{F}}(l): Z'_{\mathfrak{F}}(l)\to\overline{\mathcal{A}}_{g,l}$. As in Section 2 of [YZ2014], we define

$$Z_{\mathfrak{F}}(l) := \frac{Z'_{\mathfrak{F}}(l)}{\Gamma(l)\cap\mathcal{N}(\mathfrak{F})/\Gamma(l)\cap U^{\mathfrak{F}}(\mathbb{R})}.$$

Let $n,m$ be two positive integers with $m|n$. We are going to construct a natural morphism $\overline{\lambda}_{n,m}:\overline{\mathcal{A}}_{g,n}\to\overline{\mathcal{A}}_{g,m}$. Given a cusp $\mathfrak{F}$ and a cone $\sigma\in\Sigma_{\mathfrak{F}}$, the inclusion of the algebras $\mathbb{C}[\sigma^\vee\cap M_{\mathfrak{F}}(m)]\xrightarrow{\subset}\mathbb{C}[\sigma^\vee\cap M_{\mathfrak{F}}(n)]$ induces a finite surjective morphism $\lambda^\sigma: X_\sigma(n)\longrightarrow X_\sigma(m)$. Therefore, we have an analytic surjective morphism

$$\lambda_{\mathfrak{F}}^\sigma:\widetilde{\Delta}_{\mathfrak{F},\sigma}(n)\longrightarrow\widetilde{\Delta}_{\mathfrak{F},\sigma}(m),$$

such that any $\tau\prec\sigma$ there holds a commutative diagram

$$\begin{array}{ccc}\widetilde{\Delta}_{\mathfrak{F},\tau}(n) & \xrightarrow[\text{open embedding}]{\subset} & \widetilde{\Delta}_{\mathfrak{F},\sigma}(n)\\ \lambda_{\mathfrak{F}}^\tau\downarrow & & \downarrow\lambda_{\mathfrak{F}}^\sigma\\ \widetilde{\Delta}_{\mathfrak{F},\tau}(m) & \xrightarrow[\text{open embedding}]{\subset} & \widetilde{\Delta}_{\mathfrak{F},\sigma}(m)\end{array},$$

and so we obtain a morphism $\lambda'_{\mathfrak{F}}: Z'_{\mathfrak{F}}(n)\to Z'_{\mathfrak{F}}(m)$ by gluing all $\lambda_{\mathfrak{F}}^\sigma\ \forall\sigma\in\Sigma_{\mathfrak{F}}$. Since $\Gamma_g(n)$ is a normal subgroup of $\Gamma_g(m)$, the morphism $\lambda'_{\mathfrak{F}}$ reduces to the morphism

$$\lambda_{\mathfrak{F}}: Z_{\mathfrak{F}}(n)\to Z_{\mathfrak{F}}(m).$$

It can be verified straightforwardly that $\lambda_{\mathfrak{F}}$'s are compatible with the morphisms $\Pi_{\mathfrak{F}_1,\mathfrak{F}_2}$'s and the action of $\Gamma$. Therefore, we have a global morphism

$$\overline{\lambda}_{n,m}:\overline{\mathcal{A}}_{g,n}\to\overline{\mathcal{A}}_{g,m}.$$

Let $\sigma$ be an arbitrary topo-dimensional cone in $\Sigma_{\mathfrak{F}_0}$. Consider the inclusion

$$0\longrightarrow\mathbb{C}[\sigma^\vee\cap M_{\mathfrak{F}_0}(m)]\xrightarrow{\subset}\mathbb{Q}(\mathbb{C}[\sigma^\vee\cap M_{\mathfrak{F}_0}(n)])$$

where $\mathbb{Q}(\mathbb{C}[\sigma^\vee\cap M_{\mathfrak{F}_0}(n)])$ is the quotient field of the integral domain $\mathbb{C}[\sigma^\vee\cap M_{\mathfrak{F}_0}(n)]$. The algebra $\mathbb{C}[\sigma^\vee\cap M_{\mathfrak{F}_0}(n)]$ is indeed the integral closure of $\mathbb{C}[\sigma^\vee\cap M_{\mathfrak{F}_0}(m)]$ in $\mathbb{Q}(\mathbb{C}[\sigma^\vee\cap M_{\mathfrak{F}_0}(n)])$. Then, the compactification $\overline{\mathcal{A}}_{g,n}$ is a normalization of the morphism $\mathcal{A}_{g,n}\to\overline{\mathcal{A}}_{g,m}$ and so the morphism $\mathcal{A}_{g,n}\to\overline{\mathcal{A}}_{g,m}$ factors through the morphism $\overline{\lambda}_{n,m}:\overline{\mathcal{A}}_{g,n}\to\overline{\mathcal{A}}_{g,m}$ (cf. [FC]). Thus, we obtain the following commutative diagram of morphisms

$$\begin{array}{ccc}\overline{\mathcal{A}}_{g,n} & \xrightarrow{\overline{\lambda}_{n,m}} & \overline{\mathcal{A}}_{g,m}\\ & \overline{\lambda}_{n,1}\searrow & \downarrow\overline{\lambda}_{m,1}\\ & & \overline{\mathcal{A}}_{g,1}.\end{array}$$

**Lemma 2.1.** *Let $n,m\geq 1$ be two positive integers with $m|n$. Let $\Sigma_{\mathfrak{F}_0} := \{\sigma_\alpha^{\mathfrak{F}_0}\}$ be a $\overline{\Gamma_{\mathfrak{F}_0}}$(or $\mathrm{GL}(g,\mathbb{Z})$)-admissible polyhedral decomposition of $C(\mathfrak{F}_0)$ regular with respect to $\mathrm{Sp}(g,\mathbb{Z})$.*

*Let $\overline{\mathcal{A}}_{g,n}$ (resp. $\overline{\mathcal{A}}_{g,m}$) be the symmetric toroidal compactification of $\mathcal{A}_{g,n}$ (resp. $\mathcal{A}_{g,m}$) constructed by $\Sigma_{\mathfrak{F}_0}$. The morphism $\overline{\lambda}_{n,m}:\overline{\mathcal{A}}_{g,n}\to\overline{\mathcal{A}}_{g,m}$ has the following property:*

$$\overline{\lambda}_{n,m}^* D_{\infty,m} = \frac{n}{m}D_{\infty,n},$$

*where $D_{\infty,m} := \overline{\mathcal{A}}_{g,m}\setminus\mathcal{A}_{g,m}$ and $D_{\infty,n} := \overline{\mathcal{A}}_{g,n}\setminus\mathcal{A}_{g,n}$.*

*Proof.* By the construction of boundary divisors of Siegel varieties from edges of the fan $\Sigma_{\mathfrak{F}_0}$ in Theorem 2.22 of [YZ2014], to study the relation between $D_{\infty,m}$ and $D_{\infty,n}$ is sufficient to study the morphism $\lambda_{\mathfrak{F}_0}^{\sigma_{\max}}:\widetilde{\Delta}_{\mathfrak{F}_0,\sigma_{\max}}(n)\to\widetilde{\Delta}_{\mathfrak{F}_0,\sigma_{\max}}(m)$ for any top-dimensional cone $\sigma_{\max}$ in $\Sigma_{\mathfrak{F}_0}$.

We can choose a basis $\{\zeta_\alpha\}_1^{g(g+1)/2}$ of $L_{\mathfrak{F}_0} := \mathrm{Sp}(g,\mathbb{Z})\cap U^{\mathfrak{F}_0}(\mathbb{Z})$ such that

$$\sigma_{\max} = \{\sum_{\alpha=1}^{g(g+1)/2}\lambda_\alpha\zeta_\alpha \mid \lambda_\alpha\in\mathbb{R}_{\geq 0},\ \alpha=1,\cdots,g(g+1)/2\}.$$

Let $\{\delta_\alpha\}_1^{g(g+1)/2}$ be the dual basis of $\{\zeta_\alpha\}_1^{g(g+1)/2}$. Then

$$\sigma_{\max}^\vee = \{\sum_{\alpha=1}^{g(g+1)/2}\lambda_\alpha\delta_\alpha \mid \lambda_\alpha\in\mathbb{R}_{\geq 0},\ \alpha=1,\cdots,g(g+1)/2\}.$$

Since the inclusion $0\longrightarrow\mathbb{C}[\sigma^\vee\cap M_{\mathfrak{F}_0}(m)]\xrightarrow{\subset}\mathbb{C}[\sigma^\vee\cap M_{\mathfrak{F}_0}(n)]$ is of the following type

$$\begin{aligned}&0\longrightarrow\mathbb{C}[x_1,\cdots x_i,\cdots x_{g(g+1)/2}]\\ &\xrightarrow{\subset}\mathbb{C}[\sqrt[\frac{n}{m}]{x_1},\cdots\sqrt[\frac{n}{m}]{x_i},\cdots\sqrt[\frac{n}{m}]{x_{g(g+1)/2}}],\end{aligned}$$

we must have $\overline{\lambda}_{n,m}^* D_{\infty,m} = \frac{n}{m}D_{\infty,n}$. □

## 2.1 Spaces of Siegel Cusp Forms

The Siegel space $\mathfrak{H}_g$ has a global holomorphic coordinate system $\tau$. Define a standard Euclidean form $d\mathcal{V}$ on $\mathfrak{H}_g$ to be $d\mathcal{V}_\tau := \bigwedge_{1\le i\le j\le g} d\tau_{ij}$ for $\tau = (\tau_{ij})_{1\le i,j\le g} \in \mathfrak{H}_g$. There is

$$d\mathcal{V}_{M(\tau)} = \det(C\tau+D)^{-(g+1)} d\mathcal{V}_\tau \text{ for } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(g,\mathbb{R}).$$

Let $\omega_{\mathfrak{H}_g}$ be the canonical line bundle on $\mathfrak{H}_g$. For any form $\varphi = f_\varphi \bigwedge_{1\le i\le j\le g} d\tau_{ij}$ in $\Gamma(\mathfrak{H}_g, \omega_{\mathfrak{H}_g}^{\otimes k})$, there is an associated smooth positive $(\frac{g(g+1)}{2}, \frac{g(g+1)}{2})$-form

$$(\varphi\wedge\overline{\varphi})^{1/k} := |f_\varphi|^{2/k} \bigwedge_{1\le i\le j\le g} \frac{\sqrt{-1}}{2\pi} d\tau_{ij}\wedge d\overline{\tau_{ij}}.$$

**Lemma 2.2.** *Let $n\ge 3, k\ge 1, g\ge 2$ be integers. Let $f \in \mathrm{M}_{k(g+1)}(\Gamma_g(n))$ be a modular form. With respect to the correspondence*

$$\begin{aligned} \mathrm{M}_{k(g+1)}(\Gamma_g(n)) &\xrightarrow{\cong} \Gamma(\mathfrak{H}_g, \omega_{\mathfrak{H}_g}^{\otimes k})^{\Gamma_g(n)} \\ &:= \{s\in\Gamma(\mathfrak{H}_g, \omega_{\mathfrak{H}_g}^{\otimes k}) \mid s \text{ is } \Gamma_g(n)\text{-invariant}\} \\ f(\tau) &\longmapsto \varphi_f := f(\tau)(\bigwedge_{1\le i\le j\le g} d\tau_{ij})^{\otimes k}, \end{aligned}$$

*the following two conditions are equivalent:*

*a) $f\in \mathrm{S}_{k(g+1)}(\Gamma_g(n))$;*
*b) the holomorphic form $\varphi_f$ vanishes on all cusps of $\mathfrak{H}_g$.*

*Moreover, if $n\ge 3$ then (a) or (b) is equivalent to the following*
*(c) $\int_{\mathcal{A}_{g,n}} (\varphi_f\wedge\overline{\varphi_f})^{1/k} < \infty$.*

*Proof.* Let $W_g$ be the one dimension isotropic real subspace of $V_{\mathbb{R}}$ generated by $e_g$.
"$(a)\Leftrightarrow(b)$": We only show the case of $n=1$, the others are similar. Suppose $f$ is a cusp form. Then, $f$ vanishes on the cusp $\mathfrak{F}(W_g)$, and so $f$ vanishes on any cusp $\mathfrak{F}$ with $\mathfrak{F}(W_g)\prec\mathfrak{F}$. Since $\varphi_f$ is a $\Gamma_g$-invariant form, $\varphi$ vanishes on all proper cusps of $\mathfrak{H}_g$. The converse part is obvious.

Assume that $n\ge 3$. We begin to show that "$(a)\Leftrightarrow(c)$":

- Suppose $f$ is a cusp form. Then,

$$f(\begin{pmatrix} \tau' & 0 \\ 0 & \sqrt{-1}y \end{pmatrix}) = O(\exp(-\frac{\pi}{2}y)) \text{ for } y>>0,$$

  and so $\int_{\mathcal{A}_{g,n}} (\varphi_f\wedge\overline{\varphi_f})^{1/k} < \infty$.
- Suppose $f$ is not a cusp. Then, there is a $\tau'\in\mathfrak{H}_{g-1}$ such that $\Phi_n(f)(\tau')\ne 0$. Thus, there is a neighborhood $U_{\tau'}$ of fundamental domain such that $\tau'$ is in the closure of $U_{\tau'}$ such that $|f(Z)|\ge c>0$ on $U_{\tau'}$ for some positive constant $c$. Therefore,

$$\int_{\mathcal{A}_{g,n}} (\varphi_f\wedge\overline{\varphi_f})^{1/k} \ge \int_{U_{\tau'}} (\varphi_f\wedge\overline{\varphi_f})^{1/k} = \infty. \qquad \square$$

**Corollary 2.3.** *Let $n\ge 3, k\ge 1, g\ge 2$ be integers. Let $\overline{\mathcal{A}}_{g,n}$ be an arbitrary smooth toroidal compactification of $\mathcal{A}_{g,n}$ with simple normal crossing boundary divisor $D_{\infty,n} := \overline{\mathcal{A}}_{g,n}\setminus\mathcal{A}_{g,n}$.*
*Then, we have:*

$$\Gamma(\overline{\mathcal{A}}_{g,n}, \omega_{\overline{\mathcal{A}}_{g,n}}(D_{\infty,n})^{\otimes k}) \cong \Gamma(\mathcal{A}_{g,n}, \omega_{\mathcal{A}_{g,n}}^{\otimes k}) \cong \mathrm{M}_{k(g+1)}(\Gamma_g(n)),$$
$$\Gamma(\overline{\mathcal{A}}_{g,n}, \omega_{\overline{\mathcal{A}}_{g,n}}(D_{\infty,n})^{\otimes k-1}\otimes\omega_{\overline{\mathcal{A}}_{g,n}}) \cong \mathrm{S}_{k(g+1)}(\Gamma_g(n)).$$

*where $\omega_{\overline{\mathcal{A}}_{g,n}}$ is the canonical line bundle on $\overline{\mathcal{A}}_{g,n}$ and $\omega_{\mathcal{A}_{g,n}}$ is the canonical line bundle on $\mathcal{A}_{g,n}$.*

*Proof.* With [AMRT], Mumford shows in [Mum77] that the canonical line bundle $\omega_{\mathcal{A}_{g,n}}$ extends to an ample line bundle $L_{g,n}$ on $\mathcal{A}_{g,n}^*$ and that the canonical morphism $\overline{\pi}_{g,n}: \overline{\mathcal{A}}_{g,n}\to\mathcal{A}_{g,n}^*$ is proper with $\overline{\pi}_{g,n}^*(L_{g,n}) = \omega_{\overline{\mathcal{A}}_{g,n}}(D_{\infty,n})$.

- Then, $\mathcal{O}_{\mathcal{A}_{g,n}^*} = (\overline{\pi}_{g,n})_*\mathcal{O}_{\overline{\mathcal{A}}_{g,n}}$ and so $(\overline{\pi}_{g,n})_*\omega_{\overline{\mathcal{A}}_{g,n}}(D_{\infty,n})^{\otimes k} = (\overline{\pi}_{g,n})_*(\overline{\pi}_{g,n})^*L^{\otimes k} \cong L^{\otimes k}$. Thus, $\Gamma(\mathcal{A}_{g,n}^*, L_{g,n}^{\otimes k}) \cong \Gamma(\mathcal{A}_{g,n}^*, (\overline{\pi}_{g,n})_*\omega_{\overline{\mathcal{A}}_{g,n}}(D_{\infty,n})^{\otimes k}) \cong \Gamma(\overline{\mathcal{A}}_{g,n}, \omega_{\overline{\mathcal{A}}_{g,n}}(D_{\infty,n})^{\otimes k})$. Let $j: \mathcal{A}_{g,n} \hookrightarrow \mathcal{A}_{g,n}^*$ be open embedding. Since $\mathcal{A}_{g,n}$ is normal and $\mathrm{codim}(\mathcal{A}_{g,n}^*\setminus\mathcal{A}_{g,n}) = g \ge 2$, we then have $j_*\omega_{\mathcal{A}_{g,n}}^{\otimes k} = L^{\otimes k}$. Thus, $\Gamma(\mathcal{A}_{g,n}^*, L^{\otimes k}) \cong \Gamma(\mathcal{A}_{g,n}, \omega_{\mathcal{A}_{g,n}}^{\otimes k})$. That $\Gamma(\mathcal{A}_{g,n}, \omega_{\mathcal{A}_{g,n}}^{\otimes k}) \cong \mathrm{M}_{k(g+1)}(\Gamma_g(n))$ is obvious.
- By the lemma 2.2 and the lemma A.1, we have

$$\mathrm{S}_{k(g+1)}(\Gamma_g(n)) \cong \{s\in\Gamma(\mathcal{A}_{g,n}, \omega_{\mathcal{A}_{g,n}}^{\otimes k}) \mid \int_{\mathcal{A}_{g,n}} (s\wedge\overline{s})^{1/k} < \infty\}.$$

  Shown in Theorem 2.1 of [Sak77], there holds

$$\begin{aligned} &\{s\in\Gamma(\mathcal{A}_{g,n}, \omega_{\mathcal{A}_{g,n}}^{\otimes k}) \mid \int_{\mathcal{A}_{g,n}} (s\wedge\overline{s})^{1/k} < \infty\} \\ &\cong \Gamma(\overline{\mathcal{A}}_{g,n}, \omega_{\overline{\mathcal{A}}_{g,n}}(D_{\infty,n})^{\otimes k-1}\otimes\omega_{\overline{\mathcal{A}}_{g,n}}). \end{aligned} \qquad \square$$

**Remark.** Consider the short sequence

$$\begin{aligned} 0 &\longrightarrow \omega_{\overline{\mathcal{A}}_{g,n}}(D_{\infty,n})^{\otimes k-1}\otimes\omega_{\overline{\mathcal{A}}_{g,n}} \longrightarrow \omega_{\overline{\mathcal{A}}_{g,n}}(D_{\infty,n})^{\otimes k} \\ &\longrightarrow \omega_{\overline{\mathcal{A}}_{g,n}}(D_{\infty,n})^{\otimes k}|_{D_{\infty,n}} \longrightarrow 0, \end{aligned}$$

we have that $s\in\Gamma(\overline{\mathcal{A}}_{g,n}, \omega_{\overline{\mathcal{A}}_{g,n}}(D_{\infty,n})^{\otimes k}) \cong \mathrm{M}_{k(g+1)}(\Gamma_g(n))$ is a cusp form if and only if $s|_{D_{\infty,n}} = 0$. Certainly, if $\overline{\mathcal{A}}_{g,n}$ is projective then this result can also obtained by regarding $\overline{\mathcal{A}}_{g,n}$ as the normalization of the blowing-up of $\mathcal{A}_{g,n}^*$ along the ideal sheaf $\mathcal{J}$ supported on the subscheme $\mathcal{A}_{g,n}^*\setminus\mathcal{A}_{g,n}$ (cf. Chap IV [AMRT]).

Since the logarithmic canonical line bundle $\omega_{\overline{\mathcal{A}}_{g,n}}(D_{\infty,n})$ of any smooth compactification $\overline{\mathcal{A}}_{g,n}$ of the Siegel variety $\mathcal{A}_{g,n}$ is big, there is a finite positive number $N_0$ such that

$$\frac{\dim_{\mathbb{C}}\mathrm{S}_{k(g+1)}(\Gamma_g(n))}{k^{g(g+1)/2}} = \frac{\dim_{\mathbb{C}}\Gamma(\overline{\mathcal{A}}_{g,n}, \omega_{\overline{\mathcal{A}}_{g,n}}(D_{\infty,n})^{\otimes k-1}\otimes\omega_{\overline{\mathcal{A}}_{g,n}})}{k^{g(g+1)/2}} > 0$$

for any integer $k \geq N_0$. Actually, for dimensions of spaces of Siegel cusp forms,we have the following asymptotic formula which is probably well known to experts:

**Theorem 2.4.** *Let $n \geq 3, k \geq 1, g \geq 2$ be integers.*

$$\limsup_{k\to\infty} \frac{\dim_{\mathbb{C}} \mathrm{S}_{k(g+1)}(\Gamma_g(n))}{k^{g(g+1)/2}} = [\Gamma_g(1):\Gamma(n)]\prod_{i=1}^{g}\zeta(1-2i),$$

*where $\zeta(s)$ is the Riemann-Zeta function.*

*Proof.* Let $\overline{\mathcal{A}}_{g,n}$ be an arbitrary smooth toroidal compactification of $\mathcal{A}_{g,n}$ with simple normal crossing boundary divisor $D_{\infty,n} := \overline{\mathcal{A}}_{g,n} \setminus \mathcal{A}_{g,n}$. Let $\omega_{\overline{\mathcal{A}}_{g,n}}$ be the canonical line bundle on $\overline{\mathcal{A}}_{g,n}$ and $\omega_{\mathcal{A}_{g,n}}$ the canonical line bundle on $\mathcal{A}_{g,n}$.

Define $L := \omega_{\overline{\mathcal{A}}_{g,n}}(D_{\infty,n})$. Siegel's lemma says that there exists $C > 0$ such that

$$\dim_{\mathbb{C}} H^0(D_{\infty,n}, L^{\otimes k}|_{D_{\infty,n}}) \leq Ck^{g(g+1)/2-1} \quad \forall k \in \mathbb{Z}_{\geq 0},$$

then by the corollary 2.3 we get:

$$\begin{aligned}\limsup_{k\to\infty} &\frac{\dim_{\mathbb{C}} \mathrm{S}_{k(g+1)}(\Gamma_g(n))}{k^{g(g+1)/2}} \\ &= \limsup_{k\to\infty} \frac{\dim_{\mathbb{C}} H^0(\overline{\mathcal{A}}_{g,n}, L^{\otimes k-1}\otimes \omega_{\overline{\mathcal{A}}_{g,n}})}{k^{g(g+1)/2}} \\ &= \limsup_{k\to\infty} \frac{H^0(\overline{\mathcal{A}}_{g,n}, L^{\otimes k})}{k^{g(g+1)/2}}.\end{aligned}$$

We recall Demailly's holomorphic Morse inequalities in [Dem89]: Let $\widetilde{H}$ be an arbitrary Hermitian metric on the bundle $L$ and $R(L,\widetilde{H})$ the curvature form of the metric connection of the Hermitian line bundle $(L,\widetilde{H})$. For any non-negative integer $q$, let $X(q,\widetilde{H})$ be the set $x$ of $\overline{\mathcal{A}}_{g,n}$ such that $\frac{\sqrt{-1}}{2\pi}R(L,\widetilde{H})_x$ is non degenerate with exact $q$ negative eigenvalues. Set $X(\leq q,\widetilde{H}) := \bigcup_{i=0}^{q} X(i,\widetilde{H})$. For any non-negative integer $q$, we have

$$\begin{aligned}\sum_{j=0}^{q} &\dim_{\mathbb{C}} H^j(\overline{\mathcal{A}}_{g,n}, L^{\otimes k}) \\ &\leq \frac{k^{\frac{g(g+1)}{2}}}{(\frac{g(g+1)}{2})!}\int_{X(\leq q,\widetilde{H})} (-1)^q \bigwedge^{\frac{g(g+1)}{2}} c_1(L,\widetilde{H}) \\ &\quad + o(k^{\frac{g(g+1)}{2}}) \text{ as } k \to \infty\end{aligned}$$

with equality for $q = g(g+1)/2$. In particular, for any non-negative integer $q$, there is the weak More inequalities

$$\begin{aligned}&\dim_{\mathbb{C}} H^q(\overline{\mathcal{A}}_{g,n}, L^{\otimes k}) \\ &\leq \frac{k^{\frac{g(g+1)}{2}}}{(\frac{g(g+1)}{2})!}\int_{X(q,\widetilde{H})} (-1)^q \bigwedge^{\frac{g(g+1)}{2}} c_1(L,\widetilde{H}) \\ &\quad + o(k^{\frac{g(g+1)}{2}}) \text{ as } k \to \infty.\end{aligned}$$

We now use the arguments in section 2.3.3 of [MM] to show that for any integer $q \geq 1$,

$$\text{(2.4.1)} \qquad \dim_{\mathbb{C}} H^q(\overline{\mathcal{A}}_{g,n}, L^{\otimes k}) = o(k^{\frac{g(g+1)}{2}}) \text{ as } k \to \infty.$$

In the lemma 1.9, we obtain that $L$ is a numerically effective(nef) line bundle on $\overline{\mathcal{A}}_{g,n}$. Therefore, for every small $\epsilon > 0$ there is a smooth metric $H_\epsilon$ on $L$ such that $c_1(L,\widetilde{H}) \geq -\epsilon\theta$, where $\theta$ is a given positive $(1,1)$-form on $\overline{\mathcal{A}}_{g,n}$. On $\overline{\mathcal{A}}_{g,n}$, for any positive integer $q$, we have

$$\begin{aligned}0 &\leq \frac{(-1)^q}{(\frac{g(g+1)}{2})!}c_1(L,H_\epsilon)^{g(g+1)/2}\chi_\epsilon \\ &\leq \frac{1}{q!}(\epsilon\theta)^q \wedge \frac{(-1)^q}{(\frac{g(g+1)}{2}-q)!}(c_1(L,H_\epsilon)+\epsilon\theta)^{g(g+1)/2-q} \\ &\leq \frac{1}{q!}(\epsilon\theta)^q \wedge \frac{(-1)^q}{(\frac{g(g+1)}{2}-q)!}(c_1(L,H_\epsilon)+\theta)^{g(g+1)/2-q},\end{aligned}$$

where $\chi_\epsilon$ is the characteristic function of $X(q,H_\epsilon)$. By Demailly's weak More inequalities, we then obtain

$$\begin{aligned}&\dim_{\mathbb{C}} H^q(\overline{\mathcal{A}}_{g,n}, L^{\otimes k}) \\ &\leq \frac{k^{\frac{g(g+1)}{2}}\epsilon^q}{(\frac{q!g(g+1)}{2}-q)!}\int_{\overline{\mathcal{A}}_{g,n}} [\theta]^q([c_1(L)]+[\theta])^{\frac{g(g+1)}{2}-q} \\ &\quad + o(k^{\frac{g(g+1)}{2}}) \text{ as } k \to \infty,\end{aligned}$$

and so we obtain 2.4.1.

Since $L$ is big by (4) of the theorem 1.6, the Morse inequalities for $q = g(g+1)/2$ shows that

$$\limsup_{k\to\infty} \frac{H^0(\overline{\mathcal{A}}_{g,n}, L^{\otimes k})}{k^{g(g+1)/2}} = \frac{1}{(\frac{g(g+1)}{2})!}\int_{\overline{\mathcal{A}}_{g,n}} \bigwedge^{\frac{g(g+1)}{2}} c_1(L,\widetilde{H}).$$

By (4) of the theorem 1.6, we actually get

$$\int_{\overline{\mathcal{A}}_{g,n}} \bigwedge^{\frac{g(g+1)}{2}} c_1(L,\widetilde{H}) = \int_{\mathcal{A}_{g,n}} \bigwedge^{\frac{g(g+1)}{2}} \omega_{\mathrm{can}},$$

where $\omega_{\mathrm{can}}$ is the Kähler form of the canonical Bergman metric $H_{\mathrm{can}}$ on $\mathcal{A}_{g,n}$.

Therefore, we obtain:

$$\begin{aligned}\limsup_{k\to\infty} \frac{H^0(\overline{\mathcal{A}}_{g,n}, L^{\otimes k})}{k^{g(g+1)/2}} &= \mathrm{Vol}(\mathcal{A}_{g,n}) \\ &= [\Gamma_g(1):\Gamma(n)]\mathrm{Vol}(\mathcal{A}_{g,1}) \\ &= [\Gamma_g(1):\Gamma(n)]\prod_{i=1}^{g}\zeta(1-2i).\end{aligned}$$

The last equality for volume can be found in [Har]. □

*Another proof of Theorem 2.4.* We have

$$H^q(\overline{\mathcal{A}}_{g,n}, \omega_{\overline{\mathcal{A}}_{g,n}}(D_{\infty,n})^{\otimes k-1}\otimes \omega_{\overline{\mathcal{A}}_{g,n}}) = 0 \quad \text{for } q \geq 1, k \geq 2$$

by the Kawamata-Viehweg vanishing theorem (cf. [EV]), then we use the Riemann-Roch-Hirzebruch theorem to obtain

$$\lim_{k\to\infty}\frac{\dim_{\mathbb{C}} S_{k(g+1)}(\Gamma_g(n))}{k^{g(g+1)/2}} = \mathrm{Vol}(\mathcal{A}_{g,n})$$
$$= [\Gamma_g(1):\Gamma(n)]\prod_{i=1}^{g}\zeta(1-2i). \quad \square$$

## 2.2 General Type of Siegel Varieties with Suitable Level Structures

We can also get the following result by the theorem 1.6.

**Corollary 2.5** (Mumford cf. [Mum77]). *Let $g \geq 1, n \geq 3$ be two integers. The Siegel variety $\mathcal{A}_{g,n}$ is of logarithmic general type.*

From the covering lemma A.2 and the theorem A.3 in A1, we immediately have:

**Corollary 2.6** (Mumford-Tai's Theorem cf. Chap. IV. [AMRT] and [Mum77]). *Let $g \geq 1, l \geq 3$ be two integers. There is a positive integer $N(g,l)$ such that the Siegel variety $\mathcal{A}_{g,kl}$ is of general type for any integer $k > N(g,l)$.*

Now we describe a relation between the existence of nontrivial cusp forms and the type of manifolds: The existence of a nontrivial Siegel cusp form implies the general type of Siegel varieties with certain level structure. Actually, the spaces of Siegel cusp forms supply the following effective version of Mumford-Tai's theorem:

**Theorem 2.7.** *Let $l$ be an arbitrary positive integer and let $g \geq 2$ be an integer. If*

$$N(g,l) := \min\{k \in \mathbb{Z}_{>0} \mid \dim_{\mathbb{C}} S_{k(g+1)}(\Gamma_g(l)) > 0\}$$

*is a finite number then the Siegel variety $\mathcal{A}_{g,Nl}$ is of general type for any integer $N \geq \max\{\frac{3}{l}, N(g,l)\}$.*

**Remark.** The theorem 2.4 guarantees that $N(g,l)$ is a finite integer if $g \geq 2$ and $l \geq 3$. There are many examples from number theory showing that $N(g,1)$ is finite for some low degree $g$.

**Example 2.8.** By the following list of examples of level one cusp forms for low degree $g$, the Siegel varieties $\mathcal{A}_{g,n}$ below are of general type:

(i) $\mathcal{A}_{2,n}$ for $g=2$ and $n \geq 10$, (ii) $\mathcal{A}_{3,n}$ for $g=3$ and $n \geq 9$, (iii) $\mathcal{A}_{4,n}$ for $g=4$ and $n \geq 8$.

- Case $g=2$: Igusa shows in [Igu64] that there is a cusp form $\chi_{10,2}$ of weight 10 with development

$$\chi_{10}\begin{pmatrix}\tau_1 & z\\ z & \tau_2\end{pmatrix} = (\exp(2\pi\sqrt{-1}\tau_1)\exp(2\pi\sqrt{-1}\tau_2)+\cdots)$$
$$\times(\pi z)^2+\cdots$$

  which vanishes along the "diagonal" $z=0$ with multiplicity 2. So the zero divisor of $\chi_{10,2}$ in $\mathcal{A}_2$ is the divisor of abelian surfaces that are products of elliptic curves with multiplicity 2. Thus, there is a cusp form $\vartheta_2 := \chi_{10,2}^3 \in S_{10(2+1)}(\Gamma_2)$.
- Case $g=3$: Tsuyumine shows in [Tsu86] that the ring of classical modular forms $\oplus M_k(\Gamma_3)$ is generated by 34 elements, and there is a cusp form $\chi_{18,3}$ of weight 18, namely the product of the 36 even theta constants $\theta[\epsilon]$. The zero divisor of $\chi_{18,3}$ on $\mathcal{A}_3$ is the closure of the hyperelliptic locus. Thus, there is a cusp form $\vartheta_3 := \chi_{18,3}^2 \in S_{9(3+1)}(\Gamma_3)$.
- Case $g=4$: Igusa shows in [Igu81] that up to isometry there is only one isomorphism class of even unimodular positive definite quadratic forms in 8 variables, namely $E_8$. In 16 variables there are exactly two such classes, $E_8 \oplus E_8$ and $E_{16}$. To each of these quadratic forms in 16 variables we can associate a Siegel modular form on $\Gamma_4$ by means of a theta series: $\theta_{E_8\oplus E_8}$ and $\theta_{E_{16}}$. The difference $\chi_{8,4} := \theta_{E_8\oplus E_8} - \theta_{E_{16}}$ is a cusp form of weight 8. The zero divisor of $\chi_{8,4}$ on $\mathcal{A}_4$ is the closure of the locus of Jacobians of Riemann surfaces of genus 4 in $A_4$ (cf. [Igu81] and [Poo96]). Thus, there is a cusp form $\vartheta_4 := \chi_{8,4}^5 \in S_{8(4+1)}(\Gamma_4)$.

However these examples for low genus Siegel varieties of general type are not optimal. Actually, one has general type for $g=2, n\geq 4$; $g=3, n\geq 3$; $g=4,5,6, n\geq 2$; $g\geq 7$, $n\geq 1$. The case $\mathcal{A}_{6,1}$ is still open, all other cases of low genus Siegel varieties are known to be rational or unirational. Except for the case $\mathcal{A}_{6,1}$, Hulek has completed the problem of general type for low genus Siegel varieties (cf. Theorem 1.1 [Hul00]).

In Section 3 of [YZ2014], we show that there are some restricted conditions to get a projective smooth toroidal compactification of a Siegel variety with normal crossing boundary divisor. To prove the theorem 2.7, we need a projective smooth compactification of a Siegel variety with normal crossing boundary divisor. For further studies on non locally symmetric varieties, we prefer the following consequence of Hironaka's Main theorem II to directly refining cone decomposition in smooth toroidal compactifications.

**Theorem 2.9** (Hironaka cf. [Hir63]). *Let $C$ be reduced divisor on a nonsingular variety $W$ over a field $k$ of characteristic zero. There exists a sequence of monoidal transformations*

$$\{\pi_j : W_j = Q_{Z_{j-1}}(W_{j-1}) \to W_{j-1} \mid 1 \leq j \leq l\}$$

*($Q_Z(X) \to X$ means the monoidal transform of $X$ with the center $Z$) and reduced divisor $D_j$ on $W_j$ for $1 \leq j \leq l$ such that*

*i. $W_0 = W, D_0 = C$,*

*ii.* $D_j = \pi_j^{-1}(D_{j-1})$ *(Here* $\pi_j^{-1}(D_{j-1})$ *is defined to be* $\pi_j^*(D_{j-1})_{\text{red}}$*),*

*iii.* $Z_j$ *is a nonsingular closed subvariety contained in* $D_j$*,*

*iv.* $W^* := W_l$ *is a nonsingular variety and* $D_\infty := D_l$ *is a simple normal crossing divisor on* $W^*$*.*

*Moreover, the map* $\pi = \pi_l \circ \cdots \circ \pi_1$ *is a proper birational morphism from* $W^*$ *to* $W$ *such that the restriction morphism* $\pi|_{W^*\setminus D_\infty} : W^* \setminus D_\infty \xrightarrow{\cong} W \setminus C$ *is an isomorphism.*

*Proof of Theorem 2.7.* We fix a $\overline{\Gamma_{\mathfrak{F}_0}}$ (or $\mathrm{GL}(g,\mathbb{Z})$)-admissible polyhedral decomposition $\Sigma_{\mathfrak{F}_0}$ of $C(\mathfrak{F}_0)$ regular with respect to $\mathrm{Sp}(g,\mathbb{Z})$ such that the induced symmetric $\mathrm{Sp}(g,\mathbb{Z})$-admissible family $\{\Sigma_{\mathfrak{F}}\}_{\mathfrak{F}}$ of polyhedral decompositions is projective.

Let $N \geq \max\{\frac{3}{l}, N(g,l)\}$ be an integer and define $n := Nl$. Since we fixed the decomposition $\Sigma_{\mathfrak{F}_0}$, we have the following commutative diagram:

$$\begin{array}{ccc} \overline{\mathcal{A}}_{g,n} & \xrightarrow{\overline{\lambda}_{n,l}} & \overline{\mathcal{A}}_{g,l} \\ {\scriptstyle \pi_{g,n}}\downarrow & & \downarrow{\scriptstyle \pi_{g,l}} \\ \mathcal{A}^*_{g,n} & \xrightarrow{\lambda^*_{n,l}} & \mathcal{A}^*_g \end{array}$$

The lemma 2.1 shows that there is

$$\overline{\lambda}^*_{n,l} D_{\infty,l} = N D_{\infty,n}$$

where $D_{\infty,l} := \overline{\mathcal{A}}_{g,l} \setminus \mathcal{A}_{g,l}$ and $D_{\infty,n} := \overline{\mathcal{A}}_{g,n} \setminus \mathcal{A}_{g,n}$.

The components of the boundary divisor $D_{\infty,n} = \overline{\mathcal{A}}_{g,n} \setminus \mathcal{A}_{g,n}$ may have self-intersections. However, Hironaka's results on resolution of singularities show that there exists a smooth compactification $\widetilde{\mathcal{A}}_{g,n}$ of $\mathcal{A}_{g,n}$ and a proper birational morphism

$$\nu_n : \widetilde{\mathcal{A}}_{g,n} \longrightarrow \overline{\mathcal{A}}_{g,n} \tag{2.9.1}$$

such that

- $\widetilde{D}_{\infty,n} = \nu^*(D_{\infty,n})_{\text{red}}$ is a simple normal crossing divisor, $\widetilde{\mathcal{A}}_{g,n} - \widetilde{D}_{\infty,n} = \mathcal{A}_{g,n}$, and the restricted morphism $\nu_n|_{\mathcal{A}_{g,n}}$ is the identity morphism;
- furthermore, write $D_{\infty,n} = \sum_{i=1}^{l} D_i$ and let $\widetilde{D}_i$ be the strict transform of $D_i$, then

  $$\nu_n^*(D_{\infty,n}) = \sum_{i=1}^{l} \widetilde{D}_i + E, \quad \text{and} \quad \widetilde{D}_{\infty,n} = \sum_{i=1}^{l} \widetilde{D}_i + E_{\text{red}}$$

  (Here $E_{\text{red}}$ is the exceptional divisor of $\nu_n$).

  Then we get:

$$\begin{aligned} \nu_n^*(D_{\infty,n}) &= \sum_{i=1}^{l} \widetilde{D}_i + E = \sum_{i=1}^{l} \widetilde{D}_i + E_{\text{red}} + (E - E_{\text{red}}) \\ &= \widetilde{D}_{\infty,n} + (E - E_{\text{red}}). \end{aligned}$$

Let $k_0 := N(g,l)$ and let $f \in S_{k_0(g+1)}(\Gamma_g(l))$ be a non trivial cusp form. Let $\theta_f := f(d\mathcal{V})^{\otimes k} \in \Gamma(\mathcal{A}_{g,n}, \omega^{\otimes k}_{\mathcal{A}_g})$ where $d\mathcal{V}$ is the standard Euclidean form on the Siegel space $\mathfrak{H}_g$. By the lemma 2.2, we have $\int_{\mathcal{A}_{g,n}} (\theta_f \wedge \overline{\theta_f})^{1/k} < \infty$. Theorem 2.1 in [Sak77] says that $\theta_f$ defines a $k$-ple $\frac{g(g+1)}{2}$-form on $\widetilde{\mathcal{A}}_{g,n}$ with at most $(k-1)$-ple poles along $\widetilde{D}_{\infty,n}$, i.e.,

$$\vartheta_g \in H^0(\widetilde{\mathcal{A}}_{g,n}, \omega^{\otimes k_0}_{\widetilde{\mathcal{A}}_{g,n}} \otimes \mathcal{O}_{\widetilde{\mathcal{A}}_{g,n}}((k_0 - 1)\widetilde{D}_{\infty,n})).$$

Let $D_g$ be the Zariski closure in $\widetilde{\mathcal{A}}_{g,n}$ of the zero divisor of $f$ on $\mathcal{A}_{g,n}$, and let $m_f$ be the vanishing order of $f$ at the cusp $\mathfrak{F}(W_g)$, where $W_g$ is the one dimensional isotropic real subspace of $V_{\mathbb{R}}$ generated by the vector $e_g$. Since $\overline{\mathcal{A}}_{g,n}$ can be regarded as the normalization of the blowing-up of $\mathcal{A}^*_{g,n}$ along the ideal sheaf $\mathcal{J}$ supported on the subscheme $\mathcal{A}^*_{g,n} \setminus \mathcal{A}_{g,n}$, we have

$$\begin{aligned} \mathrm{div}(\vartheta_g) &= N m_f \nu_n^*(D_{\infty,n}) + D_g \\ &= N m_f \widetilde{D}_{\infty,n} + N m_f (E - E_{\text{red}}) + D_g. \end{aligned}$$

Thus, we get

$$\begin{aligned} &\omega^{\otimes k_0}_{\widetilde{\mathcal{A}}_{g,n}} \otimes \mathcal{O}_{\widetilde{\mathcal{A}}_{g,n}}((k_0 - 1)\widetilde{D}_{\infty,n}) \\ &= \mathcal{O}_{\widetilde{\mathcal{A}}_{g,n}}(\mathrm{div}(\vartheta_g)) \\ &= \mathcal{O}_{\widetilde{\mathcal{A}}_{g,n}}(N m_f \widetilde{D}_{\infty,n} + N m_f (E - E_{\text{red}}) + D_g), \end{aligned}$$

and

$$\omega^{\otimes k_0}_{\widetilde{\mathcal{A}}_{g,n}} = \mathcal{O}_{\widetilde{\mathcal{A}}_{g,n}}((N m_f - k_0 + 1)\widetilde{D}_{\infty,n} + N m_f (E - E_{\text{red}}) + D_g).$$

Since $D_{\infty,n}$, $(E - E_{\text{red}})$ and $D_g$ are all effective divisors on $\overline{\mathcal{A}}_{g,n}$, we have that

$$\begin{aligned} \omega_{\widetilde{\mathcal{A}}_{g,n}}(\widetilde{D}_{\infty,n})^{\otimes k_0} &\subset \mathcal{O}_{\widetilde{\mathcal{A}}_{g,n}}(h(\widetilde{D}_{\infty,n} + (E - E_{\text{red}}) + D_g)) \\ &\subset \omega^{\otimes h k_0}_{\widetilde{\mathcal{A}}_{g,n}} \quad \text{for } \forall h > N m_f. \end{aligned}$$

Therefore $\omega_{\widetilde{\mathcal{A}}_{g,n}}$ becomes a big line bundle on $\widetilde{\mathcal{A}}_{g,n}$ by the corollary 2.5. □

There is extensive work on the relationship between the existence of special modular forms and the geometry of moduli spaces of abelian varieties. The principal of using the existence of low weight cusp forms to study the Kodaira dimension of moduli spaces of polarized abelian varieties is first used in [GS96] and [Grit95], our theorem 2.7 provides a different version of this principal. The principal is also efficient for studying moduli spaces of $K3$ surfaces and moduli spaces of irreducible symplectic manifolds (cf. [Kon93] & [GHS11]); Gritsenko,Hulek and Sankaran have proven that the moduli spaces of polarized K3 surfaces are general type (cf. [GHS07]).

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## Appendix

### A.1 On General Type Varieties

We will show that one can obtain a variety of general type from any variety of logarithmic general type by covering method. This subsection is parallel to part of work of Mumford in Section 4 of [Mum77], but our result is a little generalization and our technique is difficult from [Mum77].

Let $X$ be a complex manifold which is a Zariski open set of a compact complex manifold $\overline{X}$ such that such that the boundary $D := \overline{X} - X$ is divisor with at most simple normal crossing. Let $L$ be a holomorphic line bundle on $\overline{X}$. For any positive integer $m$, let $\Phi_{mL}$ be a meromorphic map define by a basis of $H^0(\overline{X}, mL)$. The $L$-dimension of $\overline{X}$ is defined to be

$$\kappa(L,\overline{X}) := \begin{cases} \max_{m\in N(L,X)}\{\dim_{\mathbb{C}}(\Phi_{mL}(\overline{X}))\}, & \text{if } N(L,\overline{X}) \neq \emptyset; \\ -\infty, & \text{if } N(L,\overline{X}) = \emptyset, \end{cases}$$

where $N(L,\overline{X}) := \{m > 0 \mid \dim_{\mathbb{C}} H^0(\overline{X}, mL) > 0\}$. We call $\kappa(X) := \kappa(\mathcal{O}_{\overline{X}}(K_{\overline{X}}), \overline{X})$ **Kodaira dimension** of $X$, and $\overline{\kappa}(X) := \kappa(\mathcal{O}_{\overline{X}}(K_{\overline{X}} + D), \overline{X})$ **logarithmic Kodaira dimension** of $X$. $X$ is said to be of **general type** (resp. **logarithmic general type**) if $\kappa(X)$ (resp. $\overline{\kappa}(X)$) equals to $\dim X$. All definitions above are independent of the choice of smooth compactification of $X$ (cf. [Iitaka77]).

**Lemma A.1.** *Let $(\overline{X}, D)$ be a compact complex manifold with a simple normal crossing divisor $D$. Let $m, l$ be two arbitrary positive integers, we then have an isomorphism*
(A.1.1)

$$H^0(\overline{X}, \mathcal{O}_{\overline{X}}(mK_{\overline{X}} + lD)) \cong \left\{ \begin{matrix} m\text{-ple } n\text{-form on } \overline{X} \text{ with at most} \\ l\text{-plepolesalong } D \end{matrix} \right\}.$$

*Proof.* We have a system of coordinates charts $\{(U_\alpha, (z_1^\alpha, \cdots, z_n^\alpha))\}_\alpha$ on $\overline{X}$ satisfying $\overline{X} = \bigcup_\alpha U_\alpha$. Let $\sigma$ be a holomorphic section of $\mathcal{O}_{\overline{X}}(D)$ defining $D$. We can write $\sigma = \{\sigma_\alpha\}_\alpha$ such that $(\sigma_\alpha) = D \cap U_\alpha$ with the rule $\sigma_\alpha = \delta_{\alpha\beta}\sigma_\beta\ \forall \alpha, \beta$, where every $\delta_{\alpha\beta}$ is a transition function of the line bundle $\mathcal{O}_{\overline{X}}(D)$.

Let $\varphi \in H^0(\overline{X}, \mathcal{O}(mK_{\overline{X}} + lD))$ be a global holomorphic section. We write $\varphi = \{\varphi_\alpha\}_\alpha$ such that

$$\varphi_\alpha = k_{\alpha\beta}^m \delta_{\alpha\beta}^l \varphi_\beta \text{ on } U_\alpha \cap U_\beta,$$

where every $k_{\alpha\beta} = \det(\frac{\partial z_i^\beta}{\partial z_j^\alpha})$ is a transition function of the canonical line bundle $O_{\overline{X}}(K_{\overline{X}})$. Then, we obtain the corresponding $m$-ple $n$-form $\omega = \{\omega_\alpha\}_\alpha$ on $\overline{X}$ as follows:

$$\omega_\alpha := \frac{\varphi_\alpha}{\sigma_\alpha^l}(dz_1^\alpha \wedge \cdots \wedge dz_n^\alpha)^m \text{ in } U_\alpha.$$

Conversely, let $\omega$ be a $m$-ple $n$-form on $\overline{X}$ with at most $l$-ple poles along $D$. Since $U_\alpha \cap D = \{z_{i_1}^\alpha \cdots z_{i_t}^\alpha = 0\}$, we have

$$\omega_\alpha := \omega|_{U_\alpha} = \frac{f_\alpha(dz_1^\alpha \wedge \cdots \wedge dz_n^\alpha)^m}{(z_{i_1}^\alpha)^{s_1} \cdots (z_{i_t}^\alpha)^{s_t}} \text{ on } U_\alpha,$$

where every $s_i$ in an positive integer with $s_i \leq l$. Then, we can write $\omega = \{\omega_\alpha\}_\alpha$ where

$$\omega_\alpha := \omega|_{U_\alpha} = \frac{\varphi_\alpha(dz_1^\alpha \wedge \cdots \wedge dz_n^\alpha)^m}{\sigma_\alpha^l} \text{ on } U_\alpha.$$

Since $\omega_\alpha = \omega_\beta$ on $U_\alpha \cap U_\beta$, then $\varphi = \{\varphi_\alpha\}_\alpha$ defines a global section in $H^0(\overline{X}, \mathcal{O}(mK_{\overline{X}} + lD))$ with $\mathrm{div}(\varphi) \sim mK_{\overline{X}} + lD$. □

**Lemma A.2** (Kawamata cf. [EV] & [Kawa81]). *Let $X$ be a $n$-dimensional quasi-projective nonsingular variety and let $D = \sum_{i=1}^r D_i$ be a simple normal crossing divisor on $X$. Let $d_1, \cdots, d_r$ be positive integers. There exists a quasi-projective nonsingular variety $Z$ and a finite surjective morphism $\gamma: Z \to X$ such that*

*i.* $\gamma^* D_i = N_j(\gamma^* D_i)_{red}$ *for* $i = 1, \cdots, r$;
*ii.* $\gamma^* D$ *is a simple normal crossing divisor.*

**Theorem A.3.** *Let $X$ be a complex non-singular quasi-projective variety of logarithmic general type. There is a nonsingular quasi-projective variety $Y$ of general type with a finite surjective morphism $f: Y \to X$.*

*Proof.* Let $n = \dim_{\mathbb{C}} X$. Let $\overline{X}$ be a projective smooth compactification of $X$ with a simple normal crossing boundary divisor $B$. Since $\overline{\kappa}(Y) := \kappa(K_{\overline{X}} + B, \overline{X}) = n$, it is shown by Sakai in Proposition 2.2 of [Sak77] that for some integer $N > 0$ there are meromorphic differentials

$$\eta_0, \cdots, \eta_n \in H^0(\overline{X}, \mathcal{O}_{\overline{X}}(NK_{\overline{X}} + (N-1)B))$$

such that $\{\eta_i/\eta_0, \cdots, \eta_n/\eta_0\}$ is a transcendence base of the function field $\mathbb{C}(X)$.

Let $d$ be an integer more than $N$. We use Kawamata's covering trick by setting $d = N_1 = N_2 = \cdots$ in the lemma A.2 to get a projective manifold $\overline{Y}_d$ and a finite surjective morphism $f: \overline{Y}_d \to \overline{X}$ such that $f^*(B) = dD_d$, where $D_d$ is a simple normal crossing divisor on $\overline{Y}_d$.

We begin to show that $\overline{Y}_d$ is of general type.

For any $p \in \overline{Y}_d$, we choose a local system of regular coordinates $(z_1, \cdots, z_n)$ in the polycylindrical neighborhood $U_p := \{|z_1| \leq r_p, \cdots, |z_n| \leq r_p\}$ of $p$ such that if $p \in D_d$ then the equation $z_1 \cdots z_s = 0$ defines $D_d$ around $p$. For $q = \pi_d(p)$, we choose a local system of regular coordinates $(w_1,\cdots,w_n)$ in the polycylindrical neighborhood $W_q := \{|w_1|\le r_q,\cdots,|w_n|\le r_q\}$ of $q$ such that if $q\in B$ then the equation $w_1\cdots w_t = 0$ defines $B$ around $q$.

By definition, we have $(\overline{f})^{-1}(B)\subset D_d$ and $f^*w_i = \prod_j z_j^{n_{ij}}\epsilon_i$ with $n_{ij}\ge 0$, where $\epsilon_i$ is an unit around $p$. Thus we have $f^*\frac{dw_i}{w_i} = \sum_j n_{ij}\frac{dz_j}{z_j} + \frac{d\epsilon_i}{\epsilon_i} \in \Omega^1(\log D_d)$ around the point $p$.

Let $\omega\in\{\eta_0,\cdots,\eta_n\}$ be an element. Since

$$H^0(\overline{X},\mathcal{O}_{\overline{X}}(NK_{\overline{X}}+(N-1)B)) \cong \left\{\begin{array}{c} N\text{-ple } n\text{-form on } \overline{X} \text{ with} \\ \text{at most } (N-1)\text{-ple poles along } B\end{array}\right\},$$

we can write

$$\omega = g(w)\frac{(dw_1\wedge\cdots dw_n)^N}{w_1^{s_1}\cdots w_t^{s_t}} \text{ on } W_q$$

where $g(w)$ is a holomorphic function on $W_q$ and $s_1,\cdots,s_t$ are integers in $[0,N-1]$. Around the point $q$, we then have

$$\omega = h(w)(\prod_{i=1}^t w_i)(\frac{dw_1\wedge\cdots dw_n}{w_1\cdots w_t})^N \text{ on } W_q$$

where $h(w)$ is a holomorphic function on $W_q$. Since $f^*(B) = dD_d$, we get that

$$f^*(\prod_{i=1}^t w_i) = (\prod_{j=1}^s z_i)^d\cdot\varepsilon \text{ around } p$$

where $\varepsilon$ is a unit around $p$, and we get that

$$f^*(\omega) = k(z)(\prod_{j=1}^s z_i)^d(\frac{dz_1\wedge\cdots dz_n}{z_1\cdots z_s})^N \text{ around } p$$

where $k(z)$ is a holomorphic function around $p$. Thus each $f^*(\eta_i)$ is regular on $\overline{Y}_d$. The lemma A.1 says that all $f^*(\eta_1),\cdots,f^*(\eta_n)$ are in $H^0(\overline{X},\mathcal{O}_{\overline{X}}(NK_{\overline{X}}))$. Therefore, $\{f^*(\frac{\eta_1}{\eta_0}),\cdots,f^*(\frac{\eta_n}{\eta_0})\}$ is a transcendence base of the function field $\mathbb{C}(\overline{Y}_d)$ and $\overline{Y}_d$ is of general type. □

### A.2 On Siegel Modular Forms

Some materials related to the Satake-Baily-Borel compactification of a Siegel variety are taken from [BB66].

- Denote congruent groups by

(A.3.1)

$$\Gamma_g(1) := \mathrm{Sp}(g,\mathbb{Z}),$$
$$\Gamma_g(n) := \{\gamma\in\mathrm{Sp}(g,\mathbb{Z}) \mid \gamma\equiv I_{2g} \mod n\}\ \forall n\ge 2.$$

Obviously, each $\Gamma_g(n)$ is a normal subgroup of $\mathrm{Sp}(g,\mathbb{Z})$ with finite index. For convenience, we write $\Gamma_g$ for $\Gamma_g(1)$.

- A subgroup $\Gamma\subset\mathrm{Sp}(g,\mathbb{Q})$ is said to be **arithmetic** if $\rho(\Gamma)$ is commensurable with $\rho(\mathrm{Sp}(g,\mathbb{Q}))\cap\mathrm{GL}(n,\mathbb{Z})$ for some embedding $\rho:\mathrm{Sp}(g,\mathbb{Q})\xrightarrow{\hookrightarrow}\mathrm{GL}(n,\mathbb{Q})$. By a result of Borel, a subgroup $\Gamma\subset\mathrm{Sp}(g,\mathbb{Q})$ arithmetic if and only if $\rho'(\Gamma)$ is commensurable with $\rho'(\mathrm{Sp}(g,\mathbb{Q}))\cap\mathrm{GL}(n',\mathbb{Z})$ for every embedding $\rho':\mathrm{Sp}(g,\mathbb{Q})\xrightarrow{\hookrightarrow}\mathrm{GL}(n')$. We note that a subgroup $\Gamma\subset\mathrm{Sp}(g,\mathbb{Z})$ is arithmetic if and only if $[\mathrm{Sp}(g,\mathbb{Z}):\Gamma]<\infty$.
- Let $k'$ be a subfield of $\mathbb{C}$. An automorphism $\alpha$ of a $k'$-vector space is defined to be **neat** (or **torsion free**) if its eigenvalues in $\mathbb{C}$ generate a torsion free subgroup of $\mathbb{C}$. An element $h\in\mathrm{Sp}(g,\mathbb{Q})$ is said to be **neat** (or **torsion free**) if $\rho(h)$ is neat for one faithful representation $\rho$ of $\mathrm{Sp}(g,\mathbb{Q})$. A subgroup $\Gamma\subset\mathrm{Sp}(g,\mathbb{R})$ is **neat** if all elements of $\Gamma$ are torsion free. It is known that if $n\ge 3$ then $\Gamma_g(n)$ is a neat arithmetic subgroup of $\mathrm{Sp}(g,\mathbb{Q})$.

The Siegel space $\mathfrak{H}_g$ of degree $g$ is a complex manifold defined to be the set of all symmetric matrices over $\mathbb{C}$ of degree $g$ whose imaginary parts are positive defined. The action of $\mathrm{Sp}(g,\mathbb{R})$ on $\mathfrak{H}_g$ is defined as

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix}\bullet\tau := \frac{A\tau+B}{C\tau+D}.$$

It is known that the simple real Lie group $\mathrm{Sp}(g,\mathbb{R})$ acts on $\mathfrak{H}_g$ transitively. A **Siegel variety** is defined to be $\mathcal{A}_{g,\Gamma} := \Gamma\backslash\mathfrak{H}_g$, where $\Gamma$ is an arithmetic subgroup of $\mathrm{Sp}(g,\mathbb{Q})$.

- A Siegel variety $\mathcal{A}_{g,\Gamma}$ is a normal quasi-project variety.
- Any neat arithmetic subgroup $\Gamma$ of $\mathrm{Sp}(g,\mathbb{Q})$ acts freely on the Siegel Space $\mathfrak{H}_g$, then the induced $\mathcal{A}_{g,\Gamma}$ is a regular quasi-projective complex variety of dimension $g(g+1)/2$.
- A **Siegel variety of degree** $g$ **with level** $n$ is defined to be

$$\mathcal{A}_{g,n} := \Gamma_g(n)\backslash\mathfrak{H}_g.$$

Thus, the Siegel varieties $\mathcal{A}_{g,n}$ $(n\ge 3)$ are quasi-projective complex manifolds.

The Siegel space $\mathfrak{H}_g$ can be realized as a bounded domain parameterizing weight one polarized Hodge structures:

**Proposition** (Borel's embedding cf. [Sat] & [Del73]). *Define*

$$\mathfrak{S}_g = \mathfrak{S}(V_{\mathbb{R}},\psi) := \{F^1\in\mathrm{Grass}(g,V_{\mathbb{C}}) \mid \psi(F^1,F^1)=0,\ \sqrt{-1}\psi(F^1,\overline{F^1})>0\}.$$

*The map* $\imath:\mathfrak{H}_g\xrightarrow{\cong}\mathfrak{S}_g\ \tau\mapsto F^1_\tau$ *identifies the Siegel space* $\mathfrak{H}_g$ *with the period domain* $\mathfrak{S}_g$*, where* $F^1_\tau :=$ *the subspace of* $V_{\mathbb{C}}$ *spanned by the column vectors of* $\binom{\tau}{I_g}$*. Moreover, the map* $\imath$ *is biholomorphic.*

We set

$$\overline{\mathfrak{S}_g} := \{F^1 \in \mathrm{Grass}(g, V_{\mathbb{C}}) \mid \psi(F^1, F^1) = 0, \sqrt{-1}\psi(F^1, \overline{F^1}) \geq 0\},$$

$$\partial\mathfrak{S}_g := \{F^1 \in \check{\mathfrak{S}}_g \mid \sqrt{-1}\psi(F^1, \overline{F^1}) \geq 0, \psi(\cdot, \bar{\cdot}) \text{ is degenerate on } F^1\}.$$
$$= \{F^1 \in \overline{\mathfrak{S}_g} \mid F^1 \cap \overline{F^1} \text{ is a non trivial isotropic space}\}$$

A **boundary component** of the Siegel space $\mathfrak{S}_g = \mathfrak{S}(V, \psi)$ is a subset in $\partial\mathfrak{S}_g$ given by

$$\mathfrak{F}(W_{\mathbb{R}}) := \{F^1 \in \overline{\mathfrak{S}_g} \mid F^1 \cap \overline{F^1} = W_{\mathbb{R}} \otimes \mathbb{C} \text{ where } W_{\mathbb{R}} \text{ is an isotopic real subspace of } V_{\mathbb{R}}\}.$$

A boundary component $\mathfrak{F}(W)$ of the Siegel space $\mathfrak{S}_g$ is rational (i.e., $\mathfrak{F}(W)$ is a cusp) if and only if $W = W_{\mathbb{Q}} \otimes \mathbb{R}$, where $W_{\mathbb{Q}}$ is an isotropic subspace of $(V_{\mathbb{Q}}, \psi)$.

Define $\mathfrak{H}_g^* := \bigcup_{\mathrm{cusp}\,\mathfrak{F}} \mathfrak{F}$. The set $\mathfrak{H}_g^*$ is stable under the action of $\mathrm{Sp}(g, \mathbb{Q})$. Actually, the set $\mathfrak{H}_g^*$ is a disjoint union of locally closed $\mathrm{Sp}(g, \mathbb{Q})$-orbits $\mathfrak{H}_g^* = \mathcal{O}_0 \mathring{\bigcup} \mathcal{O}_1 \mathring{\bigcup} \cdots \mathring{\bigcup} \mathcal{O}_g$, and each orbit $\mathcal{O}_r$ is a set of disjoint union rational boundary components with same rank, i.e.,

$$\mathcal{O}_{g-r} := \mathring{\bigcup_{\mathfrak{F}(W) \text{ with } \dim_{\mathbb{R}} W = r}} \mathfrak{F}(W).$$

In particular, $\mathcal{O}_g = \mathfrak{F}(\{0\}) = \mathfrak{H}_g$. Let $\Gamma$ be an arbitrary arithmetic subgroup of $\mathrm{Sp}(g, \mathbb{Q})$. Let $\mathcal{A}_{g,\Gamma}^* := \Gamma \backslash \mathfrak{H}_g^*$ be the **Satake-Baily-Borel compactification** of $\mathcal{A}_{g,\Gamma}$. It is known that the analytic variety $\mathcal{A}_{g,\Gamma}^*$ has an algebraic structure as a normal projective complex variety.

Let $l$ be an arbitrary positive integer. It is known that there is $\mathfrak{F}(W) \cong \mathfrak{H}_{g-r}$ for any cusp with $\dim_{\mathbb{R}} W = r$. Therefore, the quotient $\Gamma_g(l) \backslash \mathcal{O}_{g-r}$ is a disjoint union of $[\mathrm{Sp}(g, \mathbb{Z}) : \Gamma_g(l)]$ locally closed subsets, and each disjoint component of $\Gamma_g(l) \backslash \mathcal{O}_r$ is canonically isomorphic to the Siegel variety $\mathcal{A}_{g-r,l}$. The Satake-Baily-Borel compactification $\mathcal{A}_{g,l}^*$ of the Siegel variety $\mathcal{A}_{g,l}$ is

$$\mathcal{A}_{g,l}^* := \Gamma_g(l) \backslash \mathfrak{H}_g^*$$
$$= (\Gamma_g(l) \backslash \mathcal{O}_0) \mathring{\bigcup} (\Gamma_g(l) \backslash \mathcal{O}_1) \mathring{\bigcup} \cdots \mathring{\bigcup} (\Gamma_g(l) \backslash \mathcal{O}_g).$$

In particular, we have $\mathcal{A}_g^* = \mathcal{A}_g \mathring{\bigcup} \mathcal{A}_{g-1} \bigcup \cdots \mathring{\bigcup} \mathcal{A}_0$. The construction of Satake-Baily-Borel compactifications shows there is a natural morphism

$$\lambda_{n,m}^* : \mathcal{A}_{g,n}^* \longrightarrow \mathcal{A}_{g,m}^* \tag{A.3.2}$$

for any two positive integers $m, n$ with $m|n$.

**Definition A.4** (Cf. [Fre]). Let $k \geq 1, n \geq 1, g \geq 2$ be integers. A complex-valued function on $\mathfrak{H}_g$ is called a **Siegel modular form of weight $k$, degree $g$ and level $n$** if the following conditions are satisfied:

- $f : \mathfrak{H}_g \to \mathbb{C}$ is a holomorphic function;
- $f(\tau) = (f|\gamma)(\tau) := \det(C\tau + D)^k f(\gamma(\tau))$ $\forall \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(n)$.

Denote by

$$\mathrm{M}_k(\Gamma_g(n)) := \{\text{Siegel modular forms of weight } k, \text{ degree } g \text{ and level } n\}.$$

Recall some important properties of Siegel modular forms (cf. [Fre]): Let $f \in \mathrm{M}_k(\Gamma_g(n))(g \geq 2)$. The Siegel modular form $f$ has an expansion of the form

$$f(\tau) = \sum_{2A \in \mathrm{Sym}_g(\mathbb{Z}), A \geq 0} c(A) \exp(\frac{\sqrt{-1}\pi}{n} \mathrm{Tr}(A\tau)) \tag{A.4.1}$$

where $c(A)$ are constant coefficients. The series A.4.1 converges absolutely on $\mathfrak{H}_g$ and uniformly on each subset of $\mathfrak{H}_g$ of the form $W_\epsilon^g = \{X + \sqrt{-1}Y \in \mathfrak{H}_g \mid Y \geq \epsilon I_{2g}\}$ with $\epsilon > 0$. In particular, $f$ is bounded on each subsets. All coefficients $c(A)$ for $2A \in \mathrm{Sym}_g(\mathbb{Z})$ and $A \geq 0$ satisfy

$$c({}^tVAV) = (\det(V))^k \exp(-\frac{\sqrt{-1}\pi}{n} \mathrm{Tr}(AVU)) c(A) \tag{A.4.2}$$

for any $M \in \Gamma_g(n)$ of the form $M = M(V, U) = \begin{pmatrix} V^{-1} & U \\ 0 & {}^tV \end{pmatrix}$. The series A.4.1 is called the Fourier expansion of $f$, and any $c(A)$ with $2A \in \mathrm{Sym}_g(\mathbb{Z})$ and $A \geq 0$ is a Fourier coefficient of $f$.

Let $n$ be an arbitrary positive integer. Define $\tau_t = \begin{pmatrix} \tau' & 0 \\ 0 & \sqrt{-1}t \end{pmatrix}$ in $\mathfrak{H}_g$ with $\tau' \in \mathfrak{H}_{g-1}, t > 0$. By Proposition 1.3 in Section 1 [YZ2014], $\lim_{t \to \infty} \tau_t$ corresponds to the following element in $\partial\overline{\mathfrak{S}_g}$

$$F_{\tau', \infty} := \text{the subspace of } V_{\mathbb{C}} \text{ spanned by the column vectors of } \begin{pmatrix} \tau' & 0 \\ 0 & 1 \\ I_{g-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $W_g$ be the one dimension isotropic real subspace of $V_{\mathbb{R}}$ generated by $e_g$. $F_{\tau',\infty}$ is actually in the cusp $\mathfrak{F}(W_g)$. The properties A.4.1 and A.4.2 guarantee that any modular $f \in \mathrm{M}_k(\Gamma_g(n))$ can extend to be a holomorphic function on

$$\mathfrak{H}_g \bigcup \bigcup_{\text{Cusp } \mathfrak{F} \text{ with } \mathfrak{F}(W_g) \preceq \mathfrak{F}} \mathfrak{F},$$

we then have

$$f(F_{\tau',\infty}) = \lim_{t \to \infty} f(\begin{pmatrix} \tau' & 0 \\ 0 & \sqrt{-1}t \end{pmatrix}) := \Phi_n(f)(\tau')$$
$$= \sum_{2A' \in \mathrm{Sym}_{g-1}(\mathbb{Z}), A' \geq 0} c(\begin{pmatrix} A' & 0 \\ 0 & 0 \end{pmatrix}) \exp(\sqrt{-1}\pi \mathrm{Tr}(A'\tau')).$$

Therefore, for $g \geq 2$ we can define the **Siegel operators** $\Phi_n : \mathrm{M}_k(\Gamma_g(n)) \to \mathrm{M}_k(\Gamma_{g-1}(n))$ by sending $f$ to $\Phi_n(f)$ for all positive integer $n$. For any integers $n \geq 1, k \geq 1, g \geq 2$,

$$S_k(\Gamma_g(n)) := \{f \in M_k(\Gamma_g(n)) \mid \Phi_n(f|\gamma) = 0 \text{ for } \forall \gamma \in \Gamma_g\}.$$

is a set of all **Siegel cusp forms** of weight $k$, degree $g$ and level $n$.

## References


[AMRT] Ash, A.; Mumford, D.; Rapoport, M.; Tai, Y.: *Smooth compactifications of locally symmetric varieties.* Second edition. With the collaboration of Peter Scholze. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2010.

[BB66] Baily, W.L.; Borel, A.: *Compactification of arithmetic quotients of bounded symmetric domains.* Ann. of Math. **84** (1966), 442–528.

[Chai] Chai, C.-L.: *Siegel moduli schemes and their compactifications over* $\mathbb{C}$. Arithmetic geometry (Storrs, Conn., 1984), 231–251, Springer, New York, 1986.

[CK87] Cattani, E.; Kaplan, A.: *Degenerating variations of Hodge structure.* Actes du Colloque de Théorie de Hodge (Luminy, 1987). Astérisque No. 179–180 (1989), 9, 67–96.

[CKS86] Cattani, E.; Kaplan, A.; Schmid, W.: *Degeneration of Hodge structures.* Ann. Math., **38** (1986), 457–535.

[Del72] Deligne, P.: *Théorie de Hodge II.* I.H.É.S. Publ. Math. **40** (1971), 5–57.

[Del73] Deligne, P.: *Espaces hermitiens symétriques.* Notes from a seminar at IHES, Spring 1973. English version: *Deligne' Notes on Symmetric Hermitian Spaces.* Notes taken by Milne, J.S. at website http://www.jmilne.org/math/.

[Del79] Deligne, P.: *Variétés de Shimura: interpretation modulaire, et techniques de construction de modéles canoniqes.* Proc. Symp. Pure Math., A.M.S. 33, part 2, 1979, pp. 247–290.

[Dem89] Demailly, J.-P.: *Holomorphic Morse inequalities.* Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), Proc. Sympos. Pure Math., vol. 52, Amer. Math. Soc., Providence, RI, 1991, pp. 93–114.

[EGH10] Erdenberger, C.; Grushevsky, S.; Hulek, K.: *Some intersection numbers of divisors on toroidal compactifications of Ag.* J. Algebraic Geom. **19** (2010), no. 1, 99–132.

[EV] Esnault, H.; Viehweg, E.: *Lectures on vanishing theorems.* DMV Seminar, **20**. Birkhäuser Verlag, Basel, 1992. vi+164 pp.

[FC] Faltings, G.; Chai, C.-L: *Degeneration of abelian varieties. With an appendix by David Mumford.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 22. Springer-Verlag, Berlin, 1990.

[Fre] Freitag, E.: *Siegelsche Modulfunktionen.* Grundlwhrem der Mathematischen Wissenschaften 254. Springer-Verlag, Berlin, 1983.

[GS96] Gritsenko, V.A.; Sankaran, G.K.: *Moduli of abelian surfaces with a (1,p2) polarisation (English summary).* Izv. Ross. Akad. Nauk Ser. Mat. **60** (1996), no. 5, 19–26; translation in Izv. Math. **60** (1996), no. 5, 893–900.

[GHS07] Gritsenko, V.A.; Hulek, K.; Sankaran, G.K.: *The Kodaira dimension of the moduli of K3 surfaces.* Invent. Math. **169** (2007), no. 3, 519–567.

[GHS11] Gritsenko, V.A.; Hulek, K.; Sankaran, G.K: *Moduli Spaces of Polarized Symplectic O'Grady Varieties and Brocherds Products.* J. Differential Geom. **88** (2011), Number 1, 61–85.

[Grit95] Gritsenko, V.: *Irrationality of the moduli spaces of polarized abelian surfaces (English summary). With an appendix by the author and K. Hulek.* Abelian varieties (Egloffstein, 1993), 63–84, de Gruyter, Berlin, 1995.

[Gri] *Topics in transcendental algebraic geometry.* Proceedings of a seminar held at the Institute for Advanced Study, Princeton, N.J., during the academic year 1981/1982. Edited by Phillip Griffiths. Annals of Mathematics Studies, 106. Princeton University Press, Princeton, NJ, 1984.

[Har] Harder, G.: *Eisensteinkohomologie und die Konstruktion gemischter Motive.* Lecture Notes in Mathematics, 1562. Springer-Verlag, Berlin, 1993.

[Hir63] Hironaka, H.: *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II.* Ann. of Math. (2) **79** (1964), 109–203; ibid. (2) **79** (1964), 205–326.

[Hul00] Hulek, K.: *Nef divisors on moduli spaces of abelian varieties.* Complex analysis and algebraic geometry, 255–274, de Gruyter, Berlin, 2000.

[Igu64] Igusa, J.: *On Siegel modular forms of genus two.* Amer. J. Math. **84** (1962), 175–200.

[Igu67] Igusa, J.: *A desingularization problem in the theory of Siegel modular functions.* Math. Ann. **168** (1967), 228–260.

[Igu81] Igusa, J.: *Schottky's invariant and quadratic forms.* Christoffel Symposium, Birkhäuser Verlag, 1981, 352–362.

[Iitaka77] Iitaka, S.: *On logarithmic Kodaira dimension of algebraic varieties.* Complex analysis and algebraic geometry, pp. 175–189. Iwanami Shoten, Tokyo, 1977.

[Kawa81] Kawamata, Y.: *Characterization of abelian varieties.* Compositio Math. **43** (1981), no. 2, 253–276.

[Kol87] Kollár, J.: *Subadditivity of the Kodaira Dimension: Fibres of general type.* Algebraic Geometry, Sendai, 1985. Advanced Studies in Pure Mathematics **10** (1987), 361–398.

[Kon93] Kondo, S.: *On the Kodaira dimension of the moduli space of K3 surfaces.* Compos. Math. **89** (1993), 251–299.

[MM] Ma, X.-N.; Marinescu, G.: *Holomorphic Morse Inequalities and Bergman kernels.* Progress in Mathematics, 254. Birkhäuser Verlag, Basel, 2007.

[MVZ07] Moeller, M.; Viehweg, E.; Zuo, K.: *Stability of Hodge bundles and a numerical characterization of Shimura varieties.* J. Differential Geom. **92**(1) (2012), 71–151.

[Mil] Milne, J.S.: *Alegraic Groups, Lie Groups, and their Arithmetic Subgroups.* Version 2.00, 2010. Unpublished book at website http://www.jmilne.org/math/.

[Mum77] Mumford, D.: *Hirzebruch's proportionality theorem in the non-compact case.* Inventiones math. **42** (1977), 239–272.

[Mum83] Mumford, D.: *On the Kodaira dimension of the Siegel modular variety.* Algebraic geometry—open problems (Ravello, 1982), 348–375, Lecture Notes in Math., 997, Springer, Berlin, 1983.

[Poo96] Poor, C.: *Schottky's form and the hyperelliptic locus.* Proc. Amer. Math. Soc. **124** (1996), no. 7, 1987–1991.

[Sak77] Sakai, F.: *Kodaira dimensions of complements of divisors.* Complex analysis and algebraic geometry, pp. 239–257. Iwanami Shoten, Tokyo, 1977.

[Sat] Satake, I.: *Algebraic structures of symmetric domains*. Kanô Memorial Lectures, 4. Iwanami Shoten, Tokyo; Princeton University Press, Princeton, N.J., 1980.

[Sch73] Schmid, W.: *Variation of Hodge Structure: The singularities of the period mapping*. Invent. math. **22** (1973), 211–319.

[Sim88] Simpson, C.: *Constructing variations of Hodge Structure using Yang-Mills theory and applications to uniformization*. Journal of the AMS **1** (1988), 867–918.

[Sim90] Simpson, C.: *Harmonic bundles on noncompact curves*. Journal of the AMS **3** (1990), 713–770.

[SY82] Siu, Y.-T.; Yau, S.-T.: *Compactification of negatively curved complete Kähler manifolds of finite volume*. Seminar on Differential Geometry, pp. 363–380, Ann. of Math. Stud., 102, Princeton Univ. Press, Princeton, N.J., 1982.

[Siu93] Siu, Y.-T.: *An effective Matsusaka big theorem*. Ann. Inst. Fourier (Grenoble) **43** (1993), no. 5, 1387–1405.

[Tai82] Tai, Y.: *On the Kodaira dimension of the moduli space of abelian varieties*. Invent. Math. **68** (1982), no. 3, 425–439.

[TY86] Tian, G.; Yau, S.-T.: *Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry*. Mathematical aspects of string theory (San Diego, Calif., 1986), 574–628, Adv. Ser. Math. Phys., 1, World Sci. Publishing, Singapore, 1987.

[Tsu86] Tsuyumine, S.: *On Siegel modular forms of degree three*. Amer. J. Math. **108** (1986), no. 4, 755–862.

[vdG08] van der Geer, G.: *Siegel modular forms and their applications*. (English summary) The 1–2–3 of modular forms, 181–245, Universitext, Springer, Berlin, 2008.

[Wang93] Wang, W.: *On the smooth compactification of Siegel spaces*. J. Differential Geom. **38** (1993), no. 2, 351–386.

[Yau78] Yau, S.-T.: *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*. Comm. Pure Appl. Math. **31** (1978), 339–411.

[Yau87] Yau, S.-T.: *Uniformization of geometric structures*. The mathematical heritage of Hermann Weyl (Durham, NC, 1987), 265–274, Proc. Sympos. Pure Math., 48, Amer. Math. Soc., Providence, RI, 1988.

[Yau93] Yau, S.-T.: *A splitting theorem and an algebraic geometric characterization of locally Hermitian symmetric spaces*. Comm. Anal. Geom. **1** (1993), no. 3–4, 473–486.

[YZ2014] Yau, S.-T.; Zhang, Y.: *The Geometry on Smooth Toroidal Compactifications of Siegel varieties*. American Journal of Mathematics **136** (2014), no. 4, 859–941.

[Zuc81] Zucker, S.: *Locally homogeneous variations of Hodge structure*. Enseign. Math. (2) **27** (1981), no. 3–4, 243–276 (1982).

[Zuo00] Zuo, K.: *On the negativity of kernels of Kodaira-Spencer maps on Hodge bundles and applications*. Asian J. of Math. **4** (2000), 279–302.