Steven Zelditch

Dr. Steven Zelditch received in 1975 from Harvard University his bachelor's degree in mathematics and in 1981 from the University of California, Berkeley his PhD under Alan Weinstein. From 1981 to 1985 Zelditch was Ritt Assistant Professor at Columbia University. At Johns Hopkins University he was from 1985 to 1989 an assistant Professor, from 1989 to 1992 an associate professor, and from 1992 to 2010 a professor. In 2010 he moved to Northwestern University, where he is currently Wayne and Elizabeth Jones Professor of Mathematics. He has done research on the spectral and scattering theory of the Laplace operator on Riemannian manifolds and especially the asymptotic and distribution of its eigenfunctions and on the inverse spectral problem.

In 2002 he was an invited speaker at the International Congress of Mathematicians in Beijing. He was elected a Fellow of the American Mathematical Society in 2012. In 2013, he won the Stefan Bergman Prize from the American Mathematical Society.

Comments on S.-T. Yau

S.-T. Yau came to prominence when I was a grad student at UC Berkeley around 1980. At that time, S. S. Chern was the figure-head of geometry at Berkeley, and I was a PhD student of Alan Weinstein, a former student of Chern. Geometry went through a phase transition at Berkeley when Yau (among others) put nonlinear PDE at the forefront of geometry. Previously, one might say that ODE's such as the geodesic equation were more in the forefront; e.g. Weinstein's thesis was about geodesics (Cut loci and conjugate loci). At around the same time, the role of pseudo-differential and Fourier integral operator theory as the rigorous framework for the classical limit of quantum mechanics became established. It is a very different use of (mostly linear) PDE in geometric analysis and classical dynamics.

Granted Yau's leadership in directing geometers and mathematical physicists to nonlinear PDE, one might imagine that he might not be supportive of work in a linear PDE area such as quantum mechanics and spectral geometry. Leaders in a field of mathematics are not always open to ideas coming from other branches where they do not specialize. But I have seen no trace of this 'closed' attitude in Yau's mathematics. It is obvious that quantum mechanics and field theory are the physics of the 20th–21st century, and my perception is that Yau is interested in every aspect of 'Geometry and Physics', including quantum mechanics and its classical limit. This includes the eigenvalue problem $\Delta\varphi = -\lambda\varphi$ on a Riemannian manifold (M,g).

Yau's breadth of interests is reflected by the problems in his problem lists. One of his most famous problems is to prove the Yau conjecture on nodal sets: namely if (M,g) is a compact C^∞ Riemannian manifold of dimension n, and φ_λ is a Δ-eigenfunction of eigenvalue $-\lambda$, then the surface measure $\mathcal{H}^{n-1}(Z_{\varphi_\lambda})$ of the nodal set of φ_λ satisfies

$$c_g\sqrt{\lambda} \le \mathcal{H}^{n-1}(Z_{\varphi_\lambda}) \le C_g\sqrt{\lambda}$$

for some positive constants $c_g, C_g > 0$. Both the upper and lower bound were proved by Donnelly-Fefferman in the late 1980s in the real analytic case. The C^∞ case remained open until 2016, when Malinnikova-Logunov proved the sharp lower bound and gave a polynomial upper bound. It is still open to prove (or dis-prove) the sharp upper bound.

A related problem on Yau's lists is to bound (from above or below) the number of critical points of an eigenfunction in terms of the eigenvalue. Since critical point sets may be hypersurfaces (e.g. for zonal spherical harmonics) it is better to ask for bounds on the number of critical values. This number is finite if the metric is real analytic. To date, no one has obtained an upper bound in this or any case. Jakobson-Nadirashvili constructed metrics on the 2-torus for which there exists a sequence of eigenfunctions with eigenvalue tending to infinity and with a bounded number of critical points. Hence there does not exist an interesting lower bound, at least on a 2-torus. One may try to add curvature assumptions or genericity hypotheses. Yau has recently put another critical point problem on the 2015 ICCM problem list asking about the number of critical points of the Green's function on certain manifolds. The Green's function is a solution of $\Delta G(x,y) = \delta_y(x) - \frac{1}{V}$ where V is the volume of (M,g). More generally one consider the Green's functions $G_\lambda(x,y)$ solving $(\Delta+\lambda)G_\lambda(x,y) = \delta_y(x) - \frac{1}{V}$. I worked on this and related critical point problems off and on for Neumann Green's functions of bounded plane domains. At one point I mentioned a few problems and results to Yau but expressed some reservations on the grounds that I did not see any applications. His answer was, "so what, it's interesting".

This answer might apply to the problem of Yau that I spent the most time on. The problem was to characterize metrics g on S^2 whose Laplacian has the same eigenvalue multiplicities as the standard Laplacian. Recall that the standard one has eigenvalues $\lambda_N := N(N+1)$ with multiplicities $m_N = 2N+1$. The problem is to forget the eigenvalues and only require that the multiplicity of the Nth distinct eigenvalue is $2N+1$. I like this problem because it is elementary to state, and it seems almost certain that the only such metric is the standard one. It is a test of the tools of inverse spectral theory to see if they can resolve the problem. On the first day I considered it, I proved that such a metric g must be a Zoll metric, i.e. one all of whose geodesics are closed. Moreover, there do not exist exceptionally short closed geodesics, such as on a Moebius band. For such Zoll metrics, the eigenvalues come in clusters of width 1 around the standard eigenvalues. Under the multiplicity assumption, each cluster is a single eigenvalue, so that I called them 'maximally degenerate Laplacians'. Moreover, I proved that the Nth eigenvalue was $N(N+1)+O(N^{-\infty})$. I also proved that g is standard if it is a Zoll surface of revolution[1]. At that point, I got stuck: I could not get rid of the small error and prove that both the eigenvalues and the multiplicities were those of the standard Laplacian. I calculated what are known as the sub-principal and sub-principal symbols of the Laplacian, which had to be zero in this case. The first is well-known to be zero, and the second is given by integrals of certain types of polynomials over closed geodesics; the polynomials being in the curvature (and its derivatives) and the Jacobi fields (and their derivatives). These integrals all had to vanish, but I could not see how that implied that the metric was round. I therefore tried out a different approach in a subsequent article, showing that the spectral projections for such clusters had nice asymptotic expansions, and on the diagonal were constants plus rapidly decaying errors. In addition, I proved that if one pulls back the Euclidean metric on the eigenspace under the eigenmap embedding, one gets the Zoll metric plus a rapidly decaying error. At that point, I recalled (from the book of Boutet de Monvel-Guillemin) that Zoll metrics are analogous circle bundles of holomorphic line bundles over Kähler manifolds. The Zoll spectral projections are analogous to Bergman kernels on Kähler manifolds. I therefore decided to prove the analogous result in the line bundle setting. The proof is quite different from the Zoll case, but using the so-called Boutet-Sjostrand parametrix for the Bergman kernel of a strictly pseudo-convex domain, it is actually easier than in the Zoll case. Thus, I came to prove what is sometimes called the TYZ expansion after Tian-Yau-myself. I was unaware of Kähler geometry at the time and did not know that Yau had also posed the problem of studying asymptotics of Bergman kernels and Bergman metrics, nor that Tian had written a thesis with Yau on the problem. This was explained to me by my Hopkins colleague B. Shiffman, who taught me some complex geometry in the process. The TYZ expansion of course became better known that the Zoll Kodaira-type embedding theorem. To me, the moral of the story is that work on an interesting, but apparently un-important problem, can have unexpected and 'important' applications in apparently unrelated fields. It may be a dangerous to work on 'interesting, un-important problems' for too much of the time, but Yau's phrase "so what, it's interesting" seems to me a healthy and profitable view for some of the time. Naturally, one sees a lot more of the "it's interesting and it's important" in Yau's own work.

[1] I later learned that M. Engman had proved this result by a different method.