# Carlos Simpson

Dr. Carlos Simpson received his PhD in 1987 from Harvard University under the supervision of Wilfried Schmid. He became a professor at the University of Toulouse III and then at the University of Nice. He is the research director of the Centre National de la Recherche Scientifique. His research interests include algebraic geometry.

Simpson was an invited speaker at the International Congress of Mathematicians in 1990 at Kyoto. In 2015 he received the Sophie Germain Prize.

## Reflections on $c_1^2 = 3c_2$ and Other Things

> *"Fortune has offered me the opportunity to rest upon someone* else's *laurels!"*
>
> —Prof. Eugene Calabi

Professor Yau has always had a bit of a playful 'bad boy' streak. A few years ago after the "Current Developments in Mathematics" conference at Harvard, we went out to dinner at a Chinese restaurant. We ordered a whole fish. Halfway through, Professor Yau explained to us that fishermen consider it to be bad luck to turn over a fish, and then promptly turned it over to serve the other half.

Quite willing to dispense with formality if necessary, in this spirit he started his own publishing company.

On the subject of publishing: it is important for today's students to realize that Professor Yau published upwards of 25 major mathematical papers *before* the paper earning his Fields medal.

One of the minor miracles to come out of Professor Yau's solution of the Calabi conjecture [16] was the *uniformization theorem*: for surfaces, any compact surface of general type with $c_1^2 = 3c_2$ is a quotient of the unit two-ball by a cocompact lattice. We get a similar statement in all dimensions.

In my thesis, I was able to give an alternative proof of uniformization for surfaces using the construction of complex variations of Hodge structure. This was a generalization of the Donaldson-Uhlenbeck-Yau theorem on existence of Hermitian-Einstein (Yang-Mills) metrics on stable complex vector bundles. The optimal version of my proof relied on the Uhlenbeck-Yau theorem of regularity of $L_1^2$ subsheaves. Although, for surfaces, it would have been possible to get away with just Donaldson's method.

We had been studying the Donaldson-Uhlenbeck-Yau theorem in a study group directed by Professor Siu in 1986–1987. Other participants in this group included Tom Mrowka, Alan Nadel and Hajime Tsuji. We wanted to understand the proof of the regularity theorem, but it seemed difficult. When I asked further about this, Professor Siu said that the passage from dimension 1 to higher dimensions followed from Shiffman's Hartogs-style theorem on separately almost-everywhere meromorphic functions [12]. So, the mysterious case was dimension 1. Thanks to discussions with all the people in our study group, we were pretty good at Sobolev spaces back then. Those techniques seemed sufficient to get the one-dimensional case by an iterative solution of the $\overline{\partial}$-problem inside the weak subbundle. That was never published.

An interesting detail is that if you go back and look at Shiffman's paper, it relies on a paper by Hadamard from the late 19th century [4] giving a formula for the locations of poles and hence the radius of meromorphy. Hadamard gave a sort of "proof by example of the first cases" that wouldn't meet today's standards. I don't know where the first full proof appears, but references are made to complex analysis texts in the intervening time. Parlett's report [8] discusses progress on obtaining practical implementations for Hadamard's formula.

Dan Popovici recently published a new proof of the regularity theorem [9], giving a nice argument for the dimension 1 case and refering to Shiffman for the higher dimensional case.

At the time Professor Yau joined the mathematics faculty at Harvard, I had already defended my thesis and gone on to my first job at Princeton. Coming back to visit, I noticed that he had many students under an intense supervision program. Professor Yau was very protective of his students.

He was also good friends with my own thesis supervisor Wilfried Schmid. Professor Yau gave a beautiful speech at Wilfried's 70th birthday conference, including some very interesting comments on the 'Math wars' containing important details of how he followed Professor Schmid's contributions to mathematics education in Massachusetts.

We can see that Professor Yau has always been interested and involved in education at all levels.

Let us look some more at projective surfaces of general type with $c_1^2 = 3c_2$, quotients of the two-ball by Yau's theorem.

These constitute the first case where Mostow-Margulis rigidity does not hold in its strong form: there are cocompact lattices $\Gamma \subset SU(2,1)$ that are not arithmetic subgroups. Examples were discovered by Deligne and Mostow [2].

Particular among these surfaces are the "fake projective planes": surfaces of general type with the same Betti numbers as $\mathbb{P}^2$, automatically having $c_1^2 = 3c_2$. The first examples were given by Mumford [7].

Klingler showed that the lattices corresponding to fake projective planes are always arithmetic [5]. Using Klingler's theorem, Prasad, Yeung, Cartwright, Steger and others [10, 1] have now arrived at a complete classification of fake projective planes: there are 100.

For more general two-ball quotients, in recent work with Langer [6] and making crucial use of the recent advance by Esnault and Groechenig [3], we have been able to show that the tautological representation of $\Gamma$ comes from geometry, that is to say it is the monodromy representation of a family of varieties. We were inspired by Klingler's technique, and one can imagine getting some kind of vast generalization of the results of Prasad-Yeung-Cartwright-Steger:

**Question.** *Could we arrive at a classification of all compact two-ball quotients?*

One of the most salient aspects of the uniformization theorem is topological: a property of the Chern numbers $c_1^2 = 3c_2$ implies that the surface has a large fundamental group.

Professor Siu once pointed out to me the following folklore question: do all the surfaces between the lines $c_1^2 = 3c_2$ and $c_1^2 = 2c_2$ have infinite fundamental group? We now know that this is not the case. Roulleau and Urzúa have shown [11] that there are simply connected surfaces with $c_1^2/c_2$ arbitrarily close to 3.

I believe that a possible formulation of the philosophy behind Professor Siu's question, would be the following kind of statement. We might expect that a surface having $c_1^2/c_2 \sim 3$, should look pretty non-simply connected "at first sight". In other words, creatures moving around on that surface would think that it is probably not simply connected.

A more mathematical way of formulating this idea would be to look at *isoperimetric inequalities*. These measure the relationship between the length of a curve and the area of a bounding disk, assuming the curve is contractible. We could say that a simply connected surface "looks non-simply connected", if it has a bad isoperimetric inequality. In other words, if there are relatively short loops whose minimal bounding disk has a very large area.

**Question.** *Could we show in quantitative terms that simply connected surfaces whose Chern numbers lie near the line $c_1^2 = 3c_2$, tend to have bad isoperimetric inequalities?*

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