Vincent Moncrief

Dr. Vincent Moncrief received his PhD from the University of Maryland College Park in 1972. After that he worked subsequently at the University of California Berkeley and at the University of Utah. He is currently a professor at Yale University. His research interests include general relativity and mathematical physics.

Was the Big Bang a Thurston Earthquake in $2+1$ Dimensional Einstein Gravity?

Though I have known Shing-Tung Yau for many years one of my most extensive and memorable interactions with him involved our efforts to deal with the question raised by the title. More precisely stated though we were addressing the issue of characterizing the initial singularities of vacuum solutions to the Einstein equations on 3 manifolds of the form $\Sigma \times \mathbb{R}$ where Σ is a compact, connected, orientable surface of genus > 1. This is a problem that had intrigued me, off and on, for a number of years and I succeeded in engaging Yau's interest therein during a joint visit to Caltech about a dozen years ago. We would often meet either on the Caltech campus or at Harvard and spend long hours at the blackboard trying to sort out the needed estimates, taking breaks only to have lunch or dinner.

Since these visits were usually separated by intervals of several years I was often astonished, when we resumed a technical conversation that had broken off years earlier, when Yau would take it up exactly where we had left off, recalling all of the formulas and estimates that we had previously been discussing apparently without any reference to notes. By contrast I always had my voluminous stack of notes at hand, handwritten on pads of yellow 'legal' paper and would immediately begin extending them with new calculations as we pressed ahead. All the geometrical formulas and related ideas seemed to be of such a second nature to Yau, as for example the Schrödinger equation would be to me, that he had no need to be reminded of them.

But what were we trying to do? I had not too long before completed a detailed article on the global behavior of vacuum solutions to the $2+1$ dimensional Einstein equations and, following an invitation from Yau, was eventually to publish it in a special issue of *Surveys in Differential Geometry XII* under the title 'Relativistic Teichmüller Theory: A Hamilton-Jacobi Approach to $2+1$ Dimensional Einstein Gravity'. Lest the reader suppose that I was trying to set a record for 'dropping' famous names in the formulation of an article title I should say that I resisted the temptation to add 'via Monge-Ampère equations' or to include mention of the 'Dirichlet energy' or 'Gauss map' for fear of sounding overly ostentatious even though each of these would have been equally appropriate in view of the content.

In any case this article dealt with the formulation of the 'reduced' Einstein equations as a (non-autonomous) Hamiltonian system defined on the cotangent bundle $T^*\mathfrak{T}(\Sigma)$ of the Teichmüller space $\mathfrak{T}(\Sigma)$ of the surface Σ. These reduced equations were arrived at through imposing a certain rigid, geometrical gauge condition—that of constant-mean-curvature (or CMC) slicing and spatially harmonic (or SH) coordinates within the slices to yield a special case of the CMCSH gauge that Lars Andersson, Arthur Fischer and I have since extended to apply to Einstein's equations in higher dimensions. Only in three dimensions, thanks to the absence of gravitational waves (more precisely to the fact that Ricci flat 3 dimensional spacetimes are flat) does this reduced system simplify to a finite dimensional one. It is worth mentioning that, whereas the problem at hand is only a model for the higher dimensional Einsteinian spacetimes of more direct relevance to physics, the products of the $2+1$ dimensional solutions with lines or circles yield genuine (though of course highly circumscribed) $3+1$ dimensional (but still flat) solutions of Einstein's equations.

In spite of the locally trivial character of the resulting ($2+1$ dimensional, vacuum) spacetimes, their Teichmüller parameters undergo, in the generic case, a rather nontrivial evolution as one sweeps through the leaves of a CMC foliation. One could prove however that every such solution evolved, as the spatial area of the slices tended to infinity, so as to converge to an interior point in the corresponding Teichmüller space and that every point in Teichmüller space was asymptotically achieved in this relatively 'tame' way. The solution curves emanating from any particular point in Teichmüller space (as one followed their evolutions 'backwards' in time) each defined a certain *ray structure* that, though similar in character to a well-known such structure defined by Michael Wolf, was distinct from it. The difficulty was to study the evolution in the 'collapsing' temporal direction—as the area tended to zero and the *big bang* was approached. It was not difficult to show that, aside from the solutions generated from the zero section of the cotangent bundle, and which corresponded to fixed points of the Hamiltonian flow, every solution curve asymptotically ran to the 'boundary' of Teichmüller space as the area of the slices tended to zero.

For each of Wolf's ray structures he had proven that the collection of ideal endpoints of his rays effectively attached the Thurston boundary to the corresponding Teichmüller space and thus this seemed to be the natural conjecture to make as well for the ray structures arising from the 'Einstein flow'. If this were so then the Thurston boundary provided the natural 'space of big bang singularities' for these $2+1$ dimensional Einsteinian spacetimes. While an independent demonstration of this result was already available in a different (so-called 'cosmic time') gauge we wanted to establish it directly in CMCSH gauge since the gauge independence of such a result was far from evident. The idea was thus to show that the conformal geometries of CMC hypersurfaces degenerated in a certain mathematically precise way as those surfaces evolved towards vanishing area.

During our discussions of this problem Yau never failed to be amused by my affinity for engaging in African (hunting) safaris and one of my visits to Harvard for a mathematics conference followed shortly after I had completed such an adventure on the shores of Lake Caborra Bassa in northern Mozambique. I had previously only observed this lake from a great distance—the heights of the Zambesi escarpment in neighboring Zimbabwe in 1997 but finally, in 2007, I had had the chance to visit the lake itself. I had shown Yau the photos of my trip and, during the dinner for the conference, he had shared these with the other visiting mathematicians. Knowing, as I do, that such safari adventures are not as 'socially acceptable' as they once were, I was unconvinced of the wisdom of this gesture but the photo that attracted the most attention was one of myself attempting to pull a $14\frac{1}{2}$ foot crocodile out of the lake and onto a sandbank with a lasso. The crocodile was still alive but on its 'last legs' and not really thinking of biting anybody so that this maneuver was not as dangerous as it might have seemed. I was told in Mozambique though that such crocodiles are a great hazard to inhabitants of the villages bordering the lake and thus that I was doing them a nonnegligible favor in preventing one from pursuing its potentially treacherous activities.

Later there was some brief discussion of possibly donating the resulting 'trophy' to the *Morningside Center of Mathematics* in Beijing so that it could be placed in the foyer of the institute to greet incoming visitors but Yau ultimately declined this suggestion—presumably because it didn't fit very well with the institute's other décor. The crocodile now resides in my own home where it takes up the better part of a downstairs room but serves at least to discourage any attempted break-ins. Hopefully it has not been needed for that purpose in Beijing.

There were two different global characterizations of the 'Einstein flow' discussed in the *Surveys* article alluded to above. The first resulted from realizing that a certain Dirichlet energy function for harmonic maps defined over the surface Σ provided a complete solution to the Hamilton-Jacobi equation for the reduced Einstein equations. The harmonic maps in question were simply the Gauss maps for CMC slices in the associated, flat spacetimes. While in principle this complete solution to the Hamilton-Jacobi equation determines all of the solution curves for the Einstein flow and implicitly determines their asymptotics as well, a more explicit characterization of these curves resulted from solving an associated (parametrized) Monge-Ampère equation.

The solution to this (fully non-linear) elliptic equation depends, parametrically, upon the mean curvature variable τ (which plays the role of 'time') and upon the choice of an arbitrary point of the cotangent bundle, $T^*\mathfrak{T}(\Sigma)$, which can be thought of as an asymptotic data point (in the limit of infinite, 'future' expansion). By exploiting the method of continuity one could prove that every solution of this Monge-Ampère equation extends to a solution for all τ in the interval $(-\infty, 0]$. The limit $\tau \searrow -\infty$ corresponds to a big bang singularity at which the geometric area of Σ tends to zero and for which, generically, the corresponding solution curve runs 'off-the-edge', so to speak, of Teichmüller space. The opposite limit $\tau \nearrow 0$

corresponds to that of infinite cosmological expansion for which the geometric area of Σ blows up but for which the induced conformal geometry always asymptotes to an interior point of Teichmüller space which together with an associated asymptotic "velocity" is determined by the chosen point of $T^*\mathfrak{T}(\Sigma)$. Our expectation was that, by deriving suitable estimates (the 'engines' which drive such geometrical arguments in Yau's vivid expression) for the Monge-Ampère equation, we would be able to characterize the asymptotic behaviors of the solution curves in the $\tau \searrow -\infty$ limit sufficiently well as to show that they were indeed effectively attaching the Thurston boundary to $\mathfrak{T}(\Sigma)$.

We never actually finished this project however but I'd like to think that that only means we haven't actually finished it *yet*. However Yau is 70 now and I will soon be 76 so it's not clear when we will find the time and energy needed to finish the job. I still have all of my yellow 'legal pad' notes of our calculations and sketches of ideas safely set aside in case we get the chance to return to this project. I don't know if Yau ever took any notes of our endeavors but, then again, I'm sure he wouldn't need them.