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On the Donaldson-Uhlenbeck-Yau Correspondence

1. Introduction

I would like to dedicate this paper to Prof. S.-T. Yau on the occasion of his seventieth birthday. As have many mathematicians, I have learned a great deal of geometry and analysis from Yau, and I appreciate his generous spirit and groundbreaking mathematics. It is a privilege to have been and to continue to be a student of Prof. Yau.

This note will address a bit of the history and impact of the Donaldson-Uhlenbeck-Yau correspondence. Of course, this is a very active and wide-ranging field, and due to my limited knowledge and perspective, I will not attempt to be comprehensive in the selection of topics.

Given a holomorphic vector bundle $(E,\bar\partial)$ over a compact Kähler manifold (M,ω) of complex dimension n, a Hermitian metric h on E canonically determines a Chern connection ∇ which preserves both the metric and the complex structure $\bar\partial$ on E. The curvature F^∇ in turn is a $(1,1)$ form with values in $\mathrm{End}(E)$. Let Λ be the contraction operator with respect to the Kähler form

$$\Lambda\colon F^\nabla \mapsto (F^\nabla\wedge\omega^{n-1})/\omega^n.$$

Then ∇ is said to be a Hermitian-Yang-Mills connection if the contracted curvature endomorphism ΛF^∇ is diagonal in that

$$\Lambda F^\nabla = 2\pi i\mu I_E$$

for μ the *slope* of E and I_E the identity endomorphism. We also call h a Hermitian-Yang-Mills metric on E.

The Donaldson-Uhlenbeck-Yau Theorem relates Hermitian-Yang-Mills connections to an algebraic property of the holomorphic vector bundle, Mumford-Takemoto stability. For a holomorphic vector bundle E, the degree is defined by

$$d=\int_M c_1(E)\wedge\omega^{n-1}=\frac{i}{2\pi}\int_M \mathrm{Tr}\,F^\nabla\wedge\omega^{n-1},$$

while the slope $\mu=d/r$ for r the rank of E. This definition also extends to more singular objects than vector bundles, coherent analytic sheaves. A holomorphic vector bundle E over (M,ω) is *Mumford-Takemoto stable* if for every proper nontrivial coherent analytic subsheaf $\mathcal{F}$ of E, the slopes satisfy $\mu(\mathcal{F})<\mu(E)$. E is *polystable* if it is a direct sum of one or more holomorphic subbundles each with the same slope.

Now we are ready to state the Donaldson-Uhlenbeck-Yau Theorem.

Theorem 1. *Let (M,ω) be a compact Kähler manifold, and let E be a holomorphic vector bundle over M. Then E admits a Hermitian-Yang-Mills connection if and only if E is polystable. The Hermitian-Yang-Mills connection is unique. The polystable decomposition of holomorphic subbundles is orthogonal with respect to the associated Hermitian-Yang-Mills metric, and the metric is unique on each of these stable subbundles.*

The easier part of this theorem, that Hermitian-Yang-Mills bundles are polystable, is due independently to Kobayashi and Lübke, while the more difficult converse is due to Donaldson for algebraic surfaces [9] and to Uhlenbeck-Yau for the general case of Kähler manifolds [31].

There is a small issue with terminology. Hermitian-Yang-Mills connections/metrics are also known as Hermitian-Einstein. Hermitian-Yang-Mills connections are so named because of their relations to Yang-Mills theory from physics. On a Riemannian manifold M, Yang-Mills connections for the group $U(r)$ are critical points of the functional

$$\int_M |F^A|^2,$$

where A is a Hermitian connection on a rank-r complex vector bundle over M. Over a Kähler surface, anti-self-dual connections are Yang-Mills, and the Hermitian-Yang-Mills condition is then a generalization of these ASD connections to arbitrary Kähler manifolds.

On the other hand, the Hermitian-Einstein nomenclature reflects the geometry of Einstein metrics in Riemannian geometry, in which the Ricci curvature tensor is a constant multiple of the metric. In particular, Yau's construction of a Kähler-Einstein metric on a complex manifolds with negative or zero first Chern class [34] induces a Hermitian-Einstein metric on the holomorphic tangent bundle. The contracted curvature ΛF^∇ plays the role of the Ricci tensor.

2. Early Developments

Narasimhan-Seshadri first proved a version of the Donaldson-Uhlenbeck-Yau Theorem on compact Riemann surfaces in the 1960s [26]. Even earlier, in the 1950s, Calabi began investigating the geometry and analysis behind Kähler-Einstein metrics [5]. In the 1970s, the construction of Kähler-Einstein metrics on compact Kähler manifolds of negative and zero first Chern class by Yau [34] (and also [3] for the negative case). Moreover, the mathematical community was recognizing the importance of Yang-Mills theory in geometry, which one can see a bit later in the pioneering work of Atiyah-Bott [2].

With these background developments in mind, there was a conjecture known to many experts in the US, the UK and the Soviet Union that the existence of Hermitian-Yang-Mills connections on holomorphic vector bundles over Kähler manifolds should be equivalent to Mumford-Takemoto polystability. In other words, the conjecture was to generalize Narasimhan-Seshadri's result to Kähler manifolds of higher dimension. Given the deep analytic techniques needed to prove this conjecture in general, it is striking and important to note that it was not a conjecture for long: At least in manuscript form, Uhlenbeck-Yau's proof began to circulate a few years before it publication date of 1986, and Donaldson's work on algebraic surfaces was earlier.

Kobayashi and Lübke independently proved that every Hermitian-Yang-Mills vector bundle over a compact Kähler manifold must be polystable [17, 23]. As we will see, the other implication in the Donaldson-Uhlenbeck-Yau Theorem requires significantly more advanced analytic techniques.

We also mention that as an initial effort in this circle of ideas, Donaldson reproved Narasimhan-Seshadri's theorem on Riemann surfaces using techniques of Hermitian-Yang-Mills theory [8].

3. Donaldson-Uhlenbeck-Yau Theorem

The proof that every polystable holomorphic vector bundle E over a compact Kähler manifold (M,ω) admits a Hermitian-Yang-Mills connection is due involves deep techniques in partial differential equations.

Donaldson's approach on an algebraic surface M [9] is to begin with an initial Hermitian metric h_0 on E and then to consider the Yang-Mills flow

$$h_t^{-1}\frac{\partial h_t}{\partial t} = -2i(\Lambda F^h - 2\pi\mu I_E). \tag{1}$$

Crucially, Donaldson identifies this flow as the gradient flow with respect to a natural functional involving secondary Bott-Chern classes. This functional is convex in an appropriate sense, and moreover, there is an adjunction-type formula for a hyperplane section consisting of an algebraic curve $C \subset M$. The minima of Donaldson's functional are Hermitian-Yang-Mills metrics.

Donaldson uses the stability of the bundle E via a theorem of Mehta-Ramanathan in algebraic geometry [24], which allows him to use hyperplane sections to reduce to the case of an algebraic curve $C \subset M$. In this way, Donaldson's functional is proved to be bounded below on M by using Narasimhan-Seshadri's theorem to bound the corresponding functional on C. In this way, Donaldson applies PDE techniques to show that the gradient flow (1) for the convex functional bounded below exists for all time and converges as $t \to \infty$ to a Hermitian-Yang-Mills metric h_∞. In the process of proving this convergence as $t \to \infty$, Donaldson uses Uhlenbeck's weak compactness and removable singularities theorems for L^p-bounded connections [32, 33]. Via these techniques and Hartogs extension, Donaldson shows h_∞ is a Hermitian-Yang-Mills metric on a holomorphic vector bundle E' which is an image under a holomorphic map of E.

Soon thereafter, in an analytic tour de force, Uhlenbeck-Yau proved the DUY correspondence on a general Kähler manifold (M,ω) [31]. The Uhlenbeck-Yau approach differs from Donaldson's in a few important ways.

- Uhlenbeck-Yau solve an elliptic system instead of a parabolic one. (Note Simpson incorporated the

Uhlenbeck-Yau estimates into Donaldson's flow later in his study of Higgs bundles.)

- Uhlenbeck-Yau work directly with the stability condition in producing, via an intricate continuity-method argument, either a Hermitian-Yang-Mills metric (in the polystable case) or a destabilizing subsheaf.
- Uhlenbeck-Yau develop new estimates in Hermitian-Yang-Mills theory related to Yau's C^2 estimates for the complex Monge-Ampére equation.
- Uhlenbeck-Yau develop a new regularity result for weakly meromorphic functions to characterize projections to possibly singular analytic subsheaves of E.

We outline in very broad strokes Uhlenbeck-Yau's proof. First of all, on a holomorphic vector bundle E over a compact Kähler manifold (M,ω), equip E with a background Hermitian metric h_0. Then for another Hermitian metric h, consider the positive-definite Hermitian endomorphism $H = hh_0^{-1}$. Then Uhlenbeck-Yau consider the system for $\epsilon > 0$

$$L_\epsilon(H) = \Lambda F_0 - \Lambda\bar\partial(H^{-1}\partial_0 H) + \epsilon \log H = 0,$$

where F_0 is the curvature of h_0, ∂_0 is the $(1,0)$ part of the Chern connection of h_0 (note for simplicity here, I have written only the case for degree $\mu = 0$). The case $L_0(H) = 0$ is the Hermitian-Yang-Mills equation for H, and so the strategy is to consider the limit as $\epsilon \to 0$. For large ϵ, Uhlenbeck-Yau use an auxiliary continuity method to find a solution to $L_\epsilon(H) = 0$. They then develop detailed and powerful estimates to show both openness and closedness in the continuity method in ϵ for $L_\epsilon(H) = 0$ away from $\epsilon = 0$.

There are two cases to address the limit as $\epsilon \to 0$. If the L^2 norm of $\log H_\epsilon$ remains bounded as $\epsilon \to 0$, then the closedness estimates show that there is a limit H_0 which must solve the Hermitian-Yang-Mills equation. On the other hand, if the L^2 norm of $\log H_\epsilon$ is unbounded, we may rescale H_ϵ so that its supremum norm is 1, and then take a subsequential weak limit to find $\tilde H_0$, whose kernel is a destabilizing subsheaf of E. In particular, the limit

$$\Pi = I_E - \lim_{s\to 0} \tilde H_0^s$$

exists in a weak sense, is a projection operator almost everywhere, and satisfies

$$|(I_E - \Pi)\bar\partial\Pi|^2 = 0$$

in the sense of distributions. (For Π a smooth endomorphism on E, this last equation exactly states that Π is a holomorphic projection to a subbundle.)

By a detailed and powerful analytic argument, Uhlenbeck-Yau show that Π is a projection onto a holomorphic subsheaf, and use the Chern-Weil theory to show the image subsheaf is destabilizing. One of the regularity results Uhlenbeck-Yau develop to address the projection to a subsheaf, on the meromorphicity of separately meromorphic functions, is also due to Shiffman [28].

Later, Donaldson gave a proof in the case of M projective of arbitrary dimension [10], which in some sense simplifies the Uhlenbeck-Yau proof but at the cost of relaying on algebraic geometry results of Mehta-Ramanathan [25]. To this day, the Uhlenbeck-Yau techniques (including the parabolic versions of the estimates developed in Simpson's thesis) remain the only proof which works in all cases of the Donaldson-Uhlenbeck-Yau correspondence.

4. Later Developments

Soon after Uhlenbeck-Yau's groundbreaking paper, Hitchin introduced a generalization (at least on Riemann surfaces) in terms of *Higgs bundles* [15]. The generalization to higher dimensional Kähler manifolds is due to Simpson. On a Kähler manifold (M,ω), a Higgs bundle is a pair (E,Φ), where E is a holomorphic vector bundle over M and $\Phi \in H^0(M, K\otimes \operatorname{End} E)$ is a holomorphic one-form valued endomorphism of E. In this case, one still seeks a metric h on E, but the Higgs field Φ modifies the Chern connection $\nabla = \nabla^h$ of h to form the connection

$$A^{\Phi,h} = \nabla^h + \Phi + \Phi^{*_h},$$

where $*_h$ is the adjoint with respect to the metric h.

Hitchin's case of a Higgs bundle over a Riemann surface (for technical reasons, Hitchin addresses only the case of rank-2 bundles) is a dimension reduction of the self-dual Yang-Mills equations on $\mathbb{R}^4$. The theory of Higgs bundles is a far-reaching achievement in its own right, and though our main focus is more generally on the DUY correspondence, we do note a few of the major consequences of Hitchin's theory here.

- Moduli spaces of Higgs bundles over a closed Riemann surface provide many natural examples of complete hyper-Kähler manifolds (these are Riemannian manifolds with metric g and 3 integrable complex structures I,J,K for which g is Kähler and which satisfy the quaternion relations $I^2 = J^2 = K^2 = IJK = -1$) [15].
- Hitchin and Simpson's theory of Higgs bundles on Riemann surfaces, together with a complementary result of Donaldson, Corlette and also Jost-Yau, can be used to analyze the space of irreducible representations from the fundamental group of a surface of genus at least 2 into a Lie group G. In particular, Hitchin identifies a component of the representation variety, now called

the *Hitchin component*, analogous to Teichmüller space for any split real G [16].

- Hitchin's geometric and analytic techniques (the Hitchin fibration [14]) were adapted by Ngô to prove the Fundamental Lemma in geometric Langlands theory [27].

In light of Hitchin's work, the Donaldson-Uhlenbeck-Yau correspondence may rightly be called the Hitchin-Donaldson-Uhlenbeck-Yau correspondence in the case of Higgs bundles.

Simpson's proof the Donaldson-Uhlenbeck-Yau correspondence for Higgs bundles of arbitrary rank over Kähler manifolds is another major achievement [29]. Simpson incorporated the Uhlenbeck-Yau estimates into Donaldson's flow, as well as the Higgs field. The adaptation of the elliptic Uhlenbeck-Yau estimates to the parabolic setting is not surprising, as it is common for a parabolic problem and the corresponding stationary elliptic problem to share common estimates. The incorporation of the Higgs field into the Uhlenbeck-Yau estimates is a demonstration of their flexibility and power.

On a compact complex manifold of dimension n with an arbitrary Hermitian metric g, there is a conformal modification of g, which is unique up to constant multiples, satisfying the Gauduchon condition $\partial\bar{\partial}(\omega^{n-1}) = 0$ [12, 13]. The Gauduchon condition is sufficient to define the degree of sheaves and thus the Mumford-Takemoto stability conditions. Li-Yau developed the Donaldson-Uhlenbeck-Yau correspondence in this case [18]. Li-Yau-Zheng then used it to find a new proof of Bogomolov's theorem on class VII_0 complex surfaces with $b_2 = 0$ [20, 21]. See also [30]. A striking feature of Li-Yau-Zheng's argument is as these complex surfaces are very far from being projective, they contain no curves, and in fact there are no nontrivial subsheaves which may destabilize the bundle. Thus the stability condition is trivially checked to construct the Hermitian-Yang-Mills connection needed in the proof.

We also mention other versions of the Donaldson-Uhlenbeck-Yau correspondence for vortices over Kähler manifolds [4]; these are formally similar to Higgs bundles. As well, one can consider real special affine manifolds instead of a complex manifold as a base [22]. And there are ways to expand the DUY correspondence in terms of algebraic or tensorial constructions from the case of vector bundles, such as principal bundles [1].

In terms of more recent advances, Li-Yau and later Fu-Yau and others have investigated the Strominger system, which describes supersymmetric string theory with torsion [19, 11]. The Strominger system is a coupled system on a compact complex manifold M and holomorphic vector bundle E satisfying $c_1(M) = c_1(E) = 0$ and $c_2(M) = c_2(E)$. The solution involves Hermitian metrics on both M and E, where the metric g on M is *balanced* in that $d(\omega_g^{n-1}) = 0$ and the metric on E is Hermitian-Yang-Mills with respect to ω_g. This system serves as a non-Kähler analog of a Calabi-Yau manifold in string theory.

The deformed Hermitian-Yang-Mills equation arises in mirror symmetry as a mirror to the special Lagrangian equation. On a holomorphic line bundle over Kähler manifold (M, ω) of dimension n, a connection ∇ is deformed Hermitian-Yang-Mills if there is a constant θ so that

$$(F^{\nabla})^{2,0} = 0, \qquad \mathrm{Im}\left(e^{-i\theta}(\omega + F^{\nabla})^n\right) = 0.$$

Recently Collins-Jacob-Yau have found necessary and sufficient analytic conditions to solve this deformed Hermitian-Yang-Mills equation in the critical phase case [6], and Collins-Yau have further developed algebraic conditions as well [7]. The story is still very far from finished with the geometry and physics of special Lagrangians and the deformed Hermitian-Yang-Mills equation, and this is an example of the vigorous and important research still ongoing within the framework of the Donaldson-Uhlenbeck-Yau correspondence.

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