Alexander Grigor'yan

Dr. Alexander Grigor'yan is a famous mathematician. He is currently a professor in Universität Bielefeld and works on differential geometry and probability theory. His research interests include geometric analysis on Riemannian manifolds, graphs, and metric spaces. He and Shing-Tung Yau obtained great success by applying the technique of differential geometry in graph theory.

Grigor'yan was awarded the Whitehead Prize from the London Mathematical Society in 1997, Moscow Mathematical Society Prize in 1988 and the Gold medal in the International Mathematical Olympiad in 1974.

Yau in My Life

I met Yau personally in 1993, exactly 25 years ago, but I heard about Yau for the first time in 1980, during my first year of PhD study at mechmat[1] of Moscow State University.

Mechmat was one of the largest math departments in the world. It was divided into nearly 30 chairs (divisions) according to the subject. My supervisor was Eugene Landis (from the chair of differential equations) who worked in the area of linear elliptic and parabolic PDEs of the 2nd order. Initially he wanted me to work on the Harnack inequality and Liouville theorems for elliptic PDEs in non-divergent form, but this problem was solved in the same year by Krylov and Safonov [6] (both from the chair of probability theory), so that Landis was thinking of getting another problem for me.

At one of the mechmat seminars, he heard a talk by another PhD student, supervised by I. M. Gel'fand (from the chair of functional analysis). The talk was about PDEs on Riemannian manifolds, and Landis was excited about new prospects that may arise because of interrelation of PDEs with geometry. After the talk, Gel'fand explained that this subject was initiated by S.-T. Yau—"a young Chinese genius from the US", as he said. And he concluded his comments by advising to everybody: "Read Yau!"

Landis forwarded Gel'fand's advice to me. He presumed that Yau was quite young and his list of publications should be rather short. Landis wanted me to see what kind of problems were interesting in PDEs on manifolds and hoped that it would be possible to apply his rather geometric methods in PDEs to the manifold setting.

So, I went to the library to "read Yau". The library of mechmat was a unique one. On top of having subscriptions to all major international journals, every single article from every issue of every journal was catalogued there by the names of all coauthors (and this was long before the advent of MathSciNet!). So, I opened a "Y" drawer and quickly found a thick bunch of library cards with the name "S.-T. Yau" on them. The author S.-T. Yau was obviously a prolific one!

Nowadays MathSciNet gives 50 publications of Yau up to 1980. It was impossible to read all of them, so I looked at a few randomly chosen papers but could not find results of the kind that Landis mentioned.

So, I decided to do something myself. One of the tools of Landis was his multidimensional version of the mean value theorem: existence of a surface separating two concentric spheres such that the flux through this surface of a given function has a certain bound. I extended this lemma to a manifold setting and obtained with its help the following result: if, on a complete manifold M, the volume $V(x,r)$ of a geodesic ball of radius r centered at x admits the estimate $V(x,r) = o(r^2)$ as $r \to \infty$ then M is parabolic, that is, any positive superharmonic function on M is constant. I was quite excited by this result and quickly wrote a paper on that. However, Landis insisted that I should read Yau more carefully to make sure that the result was really new.

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I went to the library again and this time found, indeed, a paper by S. Y. Cheng and S.-T. Yau [1] from 1975 where they proved a better result: if, for some $\in M$,

$$V(x,r) = O\left(r^2\right) \text{ as } r \to \infty \tag{1}$$

then M is parabolic. Of course, I was hugely disappointed: my first really interesting discovery was known already 5 years ago! Besides, the proof was simpler than mine.

Now I appreciated the advice of Gel'fand "read Yau" and read carefully a few papers of him including [1], [9], [14], [15], [16], [17], [19]. These papers of Yau shaped my interest in the subject of PDEs on manifolds and basically determined my future work in mathematics.

However, the result of [1] remained for me most striking. From a probabilistic point of view, the parabolicity of a manifold M is equivalent to the recurrence of Brownian motion on M. Hence, the condition (1) implies recurrence, which, in particular, gives a geometric explanation of a celebrated theorem of Pólya [13] that Brownian motion is recurrent in $\mathbb{R}^n$ if and only if $n \le 2$.

The idea that certain restrictions on the volume growth may affect properties of solutions of PDEs as well as properties of sample paths of stochastic processes on M, kept appearing in my research over years again and again. It is not obvious at all that such a rough geometric characteristic as volume should affect long time behavior of Brownian paths, in particular, recurrence.

Nevertheless, a similar result turned out to be true also for stochastic completeness: if, for some $x \in M$,

$$V(x,r) \le \exp\left(Cr^2\right)$$

then Brownian motion is stochastically complete, that is, its lifetime is ∞ [2].

More recently, I was studying a semi-linear inequality

$$\Delta u + u^\sigma \le 0 \text{ on } M \tag{2}$$

where $\sigma > 1$, and proved that if, for some $x \in M$ and all large r,

$$V(x,r) \le Cr^p \ln^q r,$$

where $p = \frac{2\sigma}{\sigma-1}$ and $q = \frac{1}{\sigma-1}$, then (2) has no positive solution [4]. This result can be regarded as a generalization of [1] since any solution to (2) is superharmonic.

I am sure that the topic with volume growth conditions is not yet exhausted and more results are to be expected.

After completing my PhD in 1982, I switched from elliptic PDEs to parabolic ones, also in the setting of Riemannian manifolds. My purpose was to obtain, under appropriate assumptions, the parabolic Harnack inequality similar to the one of Jürgen Moser [12] in the Euclidean setting. I was inspired, in particular, by the work of S. A. Molchanov [11] who gave from a probabilistic perspective very subtle short time asymptotics of the heat kernel, and by the work of A. K. Gushchin [5] who obtained two sided estimates of the heat kernel with the Neumann boundary condition in unbounded domains in $\mathbb{R}^n$ taking into account the geometry of the domain. I was using the methods of Landis [8] as well as some tools from [7] and came up with a theorem that the volume doubling condition and the scale invariant Poincaré inequality imply the uniform parabolic Harnack inequality for the heat equation and, hence, two sided Gaussian estimates of the heat kernel.

So, it was a good time to read Yau again! In one of my trips to Moscow (at that time I was working in Volgograd State University) I came across a groundbreaking paper of Peter Li and S.-T. Yau [10] of 1986, where they proved the gradient estimates for the heat equation and used them to obtain two-sided Gaussian estimates of the heat kernel on manifolds of non-negative Ricci curvature. This result motivated me to check if my conditions (volume doubling and Poincaré inequality) were satisfied on such manifolds. The answer was positive and I included my proof of that fact into my paper [3] on this subject.

The paper [10] of Li and Yau exerted an enormous impact on geometric analysis. It is the second most cited paper of S.-T. Yau after [18] where he solved Calabi's conjecture. Apart from stimulating further research on elliptic and parabolic PDEs on manifolds, the gradient estimates of Li and Yau motivated Richard Hamilton in his program on Ricci flow that eventually lead to the resolution of the Poincaré conjecture by Grigory Perel'man.

In 1993, I could finally meet Yau in person. I visited CUNY by invitation of Isaac Chavel and wrote Yau a letter about my desire to come to Harvard to meet him. Surprisingly for me, he replied positively and invited me to give a talk at his differential geometry seminar. I was speaking about the result I was most proud of—the aforementioned conditions for the parabolic Harnack inequality, but in the middle of the talk I noticed that Yau closed the eyes! I blamed myself that, perhaps, my talk was too boring so that Yau fell asleep. However, to my surprise and joy, Yau asked at the end a cascade of questions showing his full understanding of my talk. Moreover, he invited me to visit Harvard next year for a few months.

That visit made a great impact not only on my future career but also on my whole family. For my daughter who got her PhD in literature, the Harvard

Library was one of the frequent professional destinations. And after meeting with Yau and his family, it was not by surprise, that my son chose also a career of a mathematician.

When speaking about Yau, it is impossible not to mention his dedication to China, Chinese history, Chinese culture, and last but not least, to the Chinese cuisine. During each of my visits to Harvard, he would invite me to a lunch or dinner at the Chinese restaurant at Harvard Square, where I came to like Chinese food and learned how to use chopsticks. Due to the invitation of Yau, I visited in 1998 for the first time his Alma Mater—the Chinese University of Hong Kong.

Hong Kong has become a most beloved place also for my family. And now my granddaughter is proud to be the best in the family to use chopsticks because she was visiting Hong Kong with my wife and I every year for the past ten years.

Yau's contribution to the rise of mathematics in the world and, in particualar, in China is impossible to overestimate. A number of Institutes and Centers of Mathematics that were arranged under his patronage and leadership in Harvard, Beijing, Hangzhou, Hong Kong, Sanya, Shenzhen, and Taiwan give constant impetus to the fostering of young mathematical talents. And they all have opportunities not only to read Yau but also to listen to Yau!

References

[1] S. Y. Cheng and S.-T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975), 333–354.

[2] A. Grigor'yan, *On stochastically complete manifolds*, (in Russian) DAN SSSR **290** (1986), no. 3, 534–537. Engl. transl.: Soviet Math. Dokl. **34** (1987), no. 2, 310–313.

[3] A. Grigor'yan, *The heat equation on non-compact Riemannian manifolds*, (in Russian) Matem. Sbornik **182** (1991), no. 1, 55–87. Engl. transl.: Math. USSR Sb. **72** (1992), no. 1, 47–77.

[4] A. Grigor'yan and Yuhua Sun, *On non-negative solutions of the inequality $\Delta u + u^{\sigma} \leq 0$ on Riemannian manifolds*, Comm. Pure Appl. Math. **67** (2014), no. 8, 1336–1352.

[5] A. K. Gushchin, *Uniform stabilization of solutions of the second mixed problem for a parabolic equation*, (in Russian) Matem. Sbornik **119(161)** (1982), no. 4, 451–508. Engl. transl.: Math. USSR Sb. **47** (1984), 439–498.

[6] N. V. Krylov and M. V. Safonov, *A certain property of solutions of parabolic equations with measurable coefficients*, (in Russian) Izv. Akad. Nauk SSSR **44** (1980), 81–98. Engl. transl.: Math. USSR Izvestija **16** (1981), 151–164.

[7] O. A. Ladyzhenskaya and N. N. Ural'tseva, Linear and Quasilinear Elliptic Equations, Academic Press, New York and London, 1968.

[8] E. M. Landis, The Second Order Equations of Elliptic and Parabolic Type, (in Russian) Nauka, Moscow, 1971. Engl. transl.: Transl. of Mathematical Monographs 171, AMS publications, 1998.

[9] P. Li and S.-T. Yau, *Eigenvalues of a compact Riemannian manifold*, Proc. Symp. Pure Math. **36** (1980), 205–239.

[10] P. Li and S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), no. 3-4, 153–201.

[11] S. A. Molchanov, *Diffusion processes and Riemannian geometry*, (in Russian) Uspekhi Matem. Nauk **30** (1975), no. 1, 3–59. Engl. transl.: Russian Math. Surveys **30** (1975), no. 1, 1–63.

[12] J. Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. **17** (1964), 101–134.Correction: Comm. Pure Appl. Math. **20** (1967), 231–236.

[13] G. Pólya, *Über eine Aufgabe der Wahrscheinlichkeitstheorie betreffend die Irrfahrt im Straßennetz*, Math. Ann. **84** (1921), 149–160.

[14] P. Yang and S.-T. Yau, *Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **7** (1980), 55–63.

[15] S.-T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228.

[16] S.-T. Yau, *Isoperimetric constant and the first eigenvalue of a compact Riemannian manifold*, Ann. Sci. Ecole Norm. Sup., 4th serie **8** (1975), no. 4, 487–507.

[17] S.-T. Yau, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry*, Indiana Math. J. **25** (1976), 659–670.

[18] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I.*, Comm. Pure Appl. Math. **31** (1978), 339–411.

[19] S.-T. Yau, *On the heat kernel of a complete Riemannian manifold*, J. Math. Pures Appl., ser. 9 **57** (1978), 191–201.