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Arthur E. Fischer Meets Shing-Tung Yau and Others

1. Brief Historical Background of UC Berkeley Mathematics Department, 1969–1972 and of Princeton Mathematics Department, 1965–1969

To set the historical background, I had just graduated from Princeton in June 1969 with a PhD in the *Program in Mathematical Physics*, which was the first PhD in this program that Princeton had ever granted. I was just 23 years old. After a great summer at conferences and traveling in Italy and Greece, I arrived at the University of California, Berkeley as a *Lecturer in Mathematics* in September 1969. This was my first job, ever. I was very excited.

It was in fact my dream job, and not only because my fellow Princeton graduate students were jealous. At Berkeley, I was surrounded by some of the greatest mathematical minds in the world, among whom were Shiing-Shen Chern [49, 50, 52], Shoshichi Kobayashi [28, 29], and Joseph Wolf [48] in differential geometry, Stephen Smale [39, 40] and Morris Hirsch [27] in differential topology, Edwin Spanier [41] in algebraic topology, Rainier Sachs [36] and Abraham Taub [43] (who has a universe named after him) in mathematical relativity, and frequent visitors Raoul Bott [5] and Serge Lang [30, 31].

In order to put my meeting with Yau into a proper mathematical perspective, I have to give a very brief background about my own mathematical work at that time. Those readers familiar with this work, or for other reasons, may skip to Section 2 without loss of generality.

For the four years that I was at Princeton, 1965–1969, Princeton was awash in the burgeoning fields of global analysis, mathematical physics, quantum field theory, and general relativity. In the Spring of 1966, Ralph Abraham gave lectures in the Princeton Physics Department on classical mechanics using the modern language of symplectic geometry, which with the assistance of Jerrold Marsden and with the support and enthusiasm of Arthur Wightman eventually led to the publication of the classic *Foundations of Mechanics* [1]. Richard Palais was at the Princeton Institute of Advanced Study, a world expert in infinite-dimensional manifolds and global and nonlinear analysis [33]. Arthur Wightman, one of the founders of the axiomatic approach to quantum field theory, was promulgating the set of Wightman axioms [42], and John Wheeler was reinvigorating mathematical general relativity throughout the world [44, 45, 46]. Jerrold Marsden was Arthur Wightman's student who in turn was John Wheeler's student. To boot, on top of all this, the accidental discovery of the cosmic microwave background (CMB) electromagnetic radiation in 1964 by the radio astronomers Arno Penzias and Robert Wilson [35] and the interpretation by Robert Dicke, James Peebles, P. G. Roll, and D. T. Wilkinson of this radiation as a signature of the big bang provided essentially confirming evidence of the big bang origin of the universe. This, in turn, provided a huge impetus for the further development of cosmology and thus also experimental general relativity, with discussions of these results being frequently heard in the halls of Princeton's Palmer Physical Laboratory. It was within this extraordinarily stimulating intellectual environment that I was a graduate student in mathematical physics.

My thesis [14] was on *The Theory of Superspace* and my thesis advisor was John Wheeler. *Superspace* is the *space of all isometry classes*, or *geometries*,

$$\mathcal{S} = \mathcal{S}(M) = \frac{\text{Riem}(M)}{\text{Diff}(M)} = \frac{\mathcal{M}}{\mathcal{D}}$$

of Riemannian metrics $\mathcal{M} = \text{Riem}(\mathcal{M})$ on a fixed closed (compact without boundary) manifold M, modulo the group of diffeomorphisms $\mathcal{D} = \text{Diff}(\mathcal{M})$ of M, acting on $\text{Riem}(M)$ by pullback, which can be interpreted as globalized coordinate transformations. These spaces and this action had been under close scrutiny by David Ebin [12] and Richard Palais [32, 33, 34], culminating in the Ebin-Palais Slice Theorem for this action.

Because the isotropy group $\text{Isotropy}(g)$ of this action at a metric g is the isometry group $I_g(M) = \{f \in \mathcal{D} \mid f^*g = g\}$ of the metric, and because non-isometric metrics have isometry groups which may have different dimensions and different numbers of components, superspace is not a manifold, but has singularities. Understanding the structure of these singularities in superspace was an open question at the time. In my thesis I proved, among other things, that superspace has the structure of an infinite dimensional stratified set, with the strata being *manifolds of geometries* (infinite-dimensional in general, but finite dimensional for homogeneous geometries) indexed by the conjugacy class $(I_g(M)) = \{f \circ I_g(M) \circ f^{-1} \mid f \in \mathcal{D}\}$ of the isometry group $I_g(M)$, where the elements of the conjugacy class $(I_g(M))$ are the set of Lie group actions on M that are equivalent to the Lie group action of the Lie group $I_g(M)$ on M.

The strata, in turn, have the crucial stratification property that the strata of the geometries of higher symmetry are completely contained in the boundary of the strata of lower symmetries. Thus a limit of a sequence of geometries of the same symmetry type, and thus in the same strata, must either remain in that strata, or be in a *higher* symmetry strata. As it turns out, the strata structure of superspace is somewhat universal for orbit spaces of many kinds and thus the strata structure is applicable to many other moduli spaces that occur in mathematics and physics.

Superspace $\mathcal{S}$, as well as being of importance in differential geometry, is also of importance in general relativity, inasmuch as it is the configuration space of a reduced Hamiltonian formulation of general relativity, reduced by the group of diffeomorphisms $\mathcal{D}$, and as such, its cotangent bundle $T^*\mathcal{S}$, suitably defined as a cotangent bundle of a stratified space, is the phase space of such a reduced Hamiltonian formulation.

Using these ideas and working with Jerry Marsden [17, 18, 21, 22], we were able to amalgamate, geometrize, and globalize the historical works of Dirac [10, 11], Arnowitt, Deser, and Misner [4], DeWitt [9], and Wheeler [44, 45, 46], into a single unified Hamiltonian formulation of general relativity (here taken in its vacuum form), given by the following two sets of equations,

$$\frac{\partial}{\partial t}\begin{pmatrix} g \\ \pi \end{pmatrix} = \boldsymbol{J} \circ D\Phi(g,\pi)^* \begin{pmatrix} N \\ X \end{pmatrix}$$
$$\Phi(g,\pi) = 0$$

where $\Phi(g,\pi) = \big(\mathcal{H}(g,\pi), 2\delta(g,\pi)\big)$ is the generalized Hamiltonian of the theory, (N,X) is Wheeler's lapse function and shift vector field, respectively, $D\Phi(g,\pi)^*$ is the natural L_2 adjoint of the Fréchet derivative $D\Phi(g,\pi)$ of Φ evaluated at $(g,\pi) \in T^*\mathcal{M}$, the cotangent bundle of $\mathcal{M}$, and where

$$\boldsymbol{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

is the symplectic matrix of the cotangent bundle $T^*\mathcal{M}$. With these definitions, the first set of equations are the Hamiltonian evolution equations of the generalized Hamiltonian Φ and the second set of equations are the constraint equations for general relativity.

In fact, in this formulation, one can see an explicit realization of Wheeler's dream and insight that the evolution equations of general relativity are somewhat redundant inasmuch as they are generated by the constraint equations. Thus all the information of the evolution equation is already encoded in the constraint equations, as is more or less evident from the above formulation.

This formulation for general relativity also led to more general results on symplectic splittings of dynamical systems with symmetry [2, 21, 22]. Later, this theme was further developed by Vincent Moncrief and myself [23, 24, 25, 26] to effect a conformal reduction that enabled us to reduce the Hamiltonian formulation of general relativity with constraints to an unconstrained Hamiltonian system on the cotangent bundle of a Teichmüller-like space of conformal structures on the fixed closed manifold M.

2. My First Meeting with Yau Was Very Inauspicious

Against this historical background, and unbeknownst to me at the time, Shing-Tung Yau had also just arrived in the Fall of 1969 as a first year graduate student in the mathematics department at UC Berkeley. Yau was a student of Chern, and completed his PhD in an unprecedented two year period, which was even more conspicuous at Berkeley than elsewhere since at that time Berkeley was known for its five, six, seven, and even eight-year graduate students.

Thus there we were, together, virtually unknown to each other, thrust into the same incredibly rich mathematical world. What would happen?

The mathematics department at that time, in 1969 and 1970, was crammed into Campbell Hall, the Astronomy building, and was not to occupy its current home on the top four floors of Evans Hall until 1971. In this small volume, low entropy state, there was considerable interaction between faculty, lecturers, graduate students, and visiting faculty. The colloquia were often packed with a lively crowd, a hundred people or so in a theater-like setting, which was a lot of people, at least as compared with the small audiences and tranquil demeanor of the Princeton colloquia.

It was in this low entropy state that I first met Yau in the Fall of 1969, when he was just starting out as a first year graduate student. My first meeting with Yau was very inauspicious. I was in the photocopy room of Campbell Hall in the Fall of 1969, now nearly 50 years ago, and Yau walked in. We vaguely had known of each other's existence, but not much more. Yau was there to photocopy his notes on links between curvature and topology.

I was interested. He told me more. I talked about what I was working on, a global formulation of the Hamiltonian structure of general relativity. It was a hit. These might be related. We both got excited.

Bringing in my background in mathematical physics, relativity, and global analysis, I knew immediately that links between curvature and topology had profound implications for physical models. The Friedmann models in cosmology immediately sprung to mind as exemplary models, where the curvature and topology of the spatial hypersurfaces of spacetime are crucial in determining the fate of the universe (see [15, 16] for more recent applications of the Friedmann models).

Surely these exemplary models were not isolated instances of how curvature and topology play an important role in modeling and interpreting the physical world. I enthusiastically emphasized this point to Yau, that any precise links between curvature and topology that he was developing could in fact be critical in explaining and understanding many aspects of the physical world. Yau was moved, and I was pleasantly surprised. It was an exciting conversation, and stimulated many afterthoughts in me, which I have often returned to in my mind through the years.

Our conversation apparently also made a lasting impression on Professor Yau, and I am indeed honored to see that this conversation and my comments also stayed with Professor Yau throughout his career (see [51] and photo below, where Yau talks about this meeting in the photocopy room).

On looking back, I often marvel how my meeting with Yau, improbable as it was, and inauspicious as it was, meeting in a photocopy room, was incredibly timely. We ran into each other, talked, and we each came away excited and stimulated to go on and to further contemplate what we had learned from each other. It was as though a true harmonic chord had been struck on some universal instrument and then continued to resonate down through the ensuing decades.

During his lecture [51], Professor Yau expounds at 19:50/56:44 on how Arthur Fischer's "insistence" that links between curvature and topology would be "useful" for explaining and understanding the physical world had "stayed" with Yau throughout his career, for now nearly 50 years, 1969–2018.

3. Perhaps I Should Have Spent More Time in the Photocopy Room

As a personal aside, I have to remark that the photocopy room in Campbell Hall turned out to be a very fruitful place for me to have visited. Also in the Fall of 1969, I met (or re-met) Jerry Marsden in this copy room. I knew Jerry at Princeton, but he was a year ahead of me and we had only interacted infrequently. When we met in the photocopy room at Berkeley, I hadn't seen him in over a year, and so I thought I should ask him if he knew who I was. He replied that he thought that I was a student in the calculus class that he had taught the year before at Berkeley. I wasn't, and we had a good laugh about it after I explained that I knew him from Princeton. As with Yau, we talked about our work, his work with David Ebin [13] about extending Arnold's [3] geometrization of hydrodynamics to the group of volume-preserving diffeomorphisms of a closed manifold, and my work on globalizing the Hamiltonian formulation of general relativity. Again, it was a hit, and from this one chance meeting, we went on to write more than 30 papers together in general relativity, partial differential equations, global and nonlinear analysis, group actions, infinite-dimensional manifolds, Riemannian geometry, symplectic geometry, and symplectic splittings of dynamical systems with symmetry, over the ensuing years.

But that is another story, although I have reflected on what I might have achieved in my mathematical career had I spent more time in the photocopy room.

4. An Impromptu Elevator Pitch Leads to a Solution of the Positive Mass Conjecture in General Relativity

Lastly, I have to remark on an indirect link I have with Professor Yau, through Professor Chern, which does not involve the photocopy room in Campbell Hall, but does involve the elevators in Evans Hall,

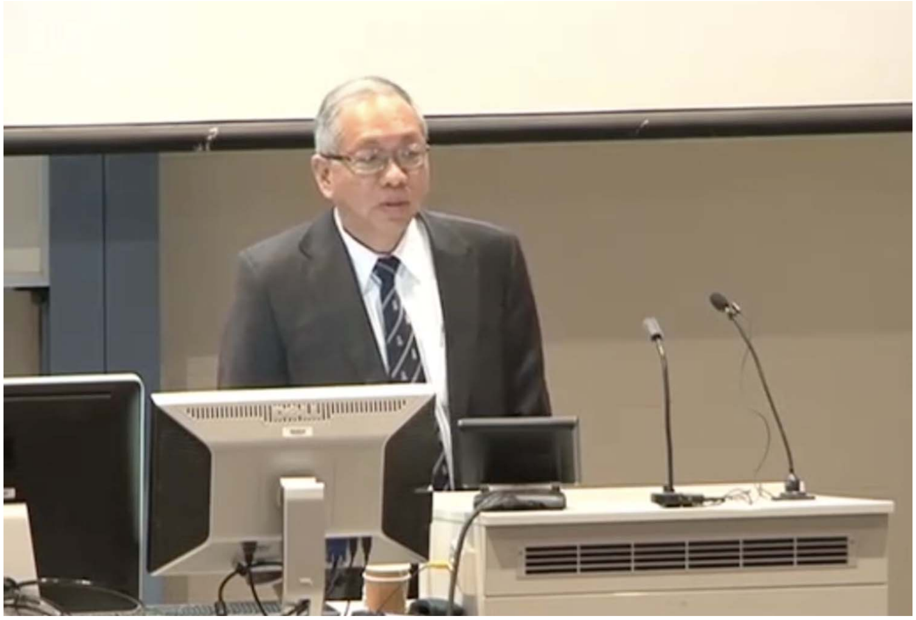

Figure 1. Professor Yau during his worldwide tour and lecture on The shape of inner space: String theory and the geometry of the universe's hidden dimensions given at The Australian National University on 24 November 2010.

the home of the mathematics department since 1971. Evans Hall is a ten story high concrete monolith, with three very busy and very slow elevators that take mathematicians to the top four floors of the building that constitute the Mathematics Department.

Early one cold winter morning in 1971, I was one of those passengers. I hopped on at the ground floor and suddenly several yards away appeared Professor Chern, hurrying toward the elevator. I immediately and instinctively grabbed the elevator door to hold it open for him as he hurried in. It was a good decision.

Professor Chern, as well as being a great mathematician, was also a wonderful human being, and in particular, was always very welcoming to new faculty. In the elevator, Chern immediately inquired how my work was going and what I was working on.

I had to think quickly; what was I working on that Professor Chern might be interested in? At that time, Jerry Marsden and I were working on problems both in Riemannian geometry and general relativity. For example, in Riemannian geometry we were interested in the structure of various subspaces of Riemannian metrics on a fixed closed manifold M. For example, one of my favorites is that for ρ a scalar function on M not equal to a positive constant or zero, the subspace

$$\mathcal{M}_\rho = \{g \in \mathrm{Riem}(M) \,|\, R(g) = \rho\}$$

where $R(g)$ denotes the scalar curvature of g, is a smooth (C^∞) closed infinite-dimensional submanifold of $\mathrm{Riem}(M)$ [19]. These results in turn led us to the new concept of *linearization stability* [20, 21], which in turn turned out to be very useful in studying nonlinear partial differential equations in general and general relativity in particular. As part of this program, we applied these techniques to study the positive mass conjecture in general relativity, at that time an important and open problem. Using these new methods, we were able to show that the mass functional was a positive definite functional around flat space, which enabled us to prove a local version of the positive mass conjecture, although our results were not published until later with Yvonne Choquet-Bruhat (see [6, 7, 8], with the latter reference including a history of the positive mass conjecture up until the time of publication). Unfortunately, our methods were not strong enough to prove a global version, which is what would have been needed to prove the full conjecture, since as shown later, new stronger methods were needed to prove the full conjecture ([37, 38], see below).

Anyway, so I thought, let's see what Chern has to say about the positive mass conjecture. It was a long shot, since Chern was essentially a Riemannian and not a Lorentzian geometer, but it was worth a try as there wasn't much to lose and the elevator ride was about two minutes, with stops, and thus was the perfect amount of time for an elevator pitch, even though the concept of an elevator pitch did not exist in 1971 (curiously, the earliest potential origin story for the concept of an elevator pitch is for a year later in 1972 [47]). Nevertheless, not knowing this chronology then, I went ahead and gave Chern an impromptu elevator pitch, and so explained in capsule form the problem, the importance of the problem, and the work Jerry and I had done on it.

Chern was interested and intrigued. We arrived at our destination, the ninth floor of Evans Hall within what appeared to be the two minute limit, disembarked, and talked some more on the landing about the problem (thereby perhaps violating elevator pitch protocol). Later that afternoon, I brought over to Chern the relevant papers that Jerry and I had relied on to tackle the problem, talked some more, and agreed to follow up whenever some new information might appear. Later that week, during the daily math tea on the tenth floor of Evans Hall, Chern and I talked some more about the problem and he said that he would discuss it with his excellent graduate student Yau and would follow up if anything new came up. Unfortunately, Chern and I never followed up, but fortunately, Yau discussed the problem with his graduate student Rick Schoen at Stanford, and as they say, the rest is history, as Yau and Schoen went on to solve this important and central problem in general relativity in 1979 [37, 38][1]. But it all started with my holding the door open for Chern and a two minute elevator pitch.

Looking back, this may have been the greatest elevator pitch in the history of mathematics, as it certainly was the first one, preceding as it did by one year the very concept of an elevator pitch.

5. What's My Bottom Line?

Yau and I have often met at several conferences and meetings throughout the years. After our greetings, talking math, and catching up, once over tea we were talking casually about how the business aspects involved in the mathematical publishing of books and journals are changing due to the internet, and Yau, somewhat jokingly, asked "So Fischer, what's your bottom line?", which later became somewhat of an inside joke between the two of us, being somewhat of a *double entendre* (but only one meaning is discussed here). So here it is, my bottom line, as it relates to this early period of my mathematical life, but in reality relates to my whole life, mathematical or not.

An inauspicious meeting with Yau in the photocopy room of Campbell Hall, a re-meeting with Jerry Marsden in the same room, an elevator ride with Professor Chern, all events which probabilistically should never even have happened, all led to important changes and developments in my mathematical life and so also to some degree in mathematics. What if these events hadn't happened? Of course, I can't say, but I can say that other events would have happened that would have taken their place, certainly leading to different consequences, perhaps some better and perhaps some not, but I believe that the track of everyone's life is somewhat stable against such perturbations and that in the long run, each of us develops in some recognizable form from where we began, independently of small perturbations of our spacetime trajectory, so that the end surely depends somewhat on the beginning. So my bottom line for all young mathematicians, or for that matter for anyone else of any age, is the following,

> *Doors are always opening. Opportunity is everywhere, in photocopy rooms, in hallways, in elevators. The future isn't out there, somewhere. It's here, and now. All you have to do to find it is to show up, to be here, to play your part in the action. Everything else will follow.*

[1] "I learnt the problem of positive mass conjecture from a big lecture given by Robert Geroch (physicist from Chicago) during a big AMS conference in geometry in the summer of 1973 at Stanford".—S.-T. Yau

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