Open Problems

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Note. This column is edited by Pengfei Guan (McGill University), Yizhao Hou (Caltech), Jun Li (Stanford University), Kefeng Liu (UCLA), Zhouping Xin (Chinese University of Hong Kong), Hao Xu (Zhejiang University; Managing editor), Shing-Tung Yau (Harvard University; Chairman of the Committee), Zhiwei Yun (MIT), and Wei Zhang (MIT). The readers are welcome to propose the solutions. The authors may send their solutions to Hao Xu (mathxuhao@gmail.com) and post the solutions in MathSciDoc (http://archive.ymsc.tsinghua.edu.cn/). The correct solutions will be announced and some souvenirs will be awarded.—*The Editors*

Problem 2018001 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

The existence of integrable complex structures is an important problem. It is well known that the existence of almost complex structure on an even-dimensional manifold can be reduced to homotopic theory. However, how to deform the almost complex structure to an integrable one is far more difficult. The power of Riemann-Roch formula for complex manifolds allows Kodaira to classify complex surfaces. The only unknown complex surface that are not Kähler are those which are called class VII_0. There were examples given by Kodaira, Inoue and Bombieri. They are the only examples of the surfaces admit no closed curve. This was proved by Bogomolov [1] and Li-Yau-Zheng [2]. It remains an open problem to classify those that have curves. Such surfaces have first Betti number equal to one.

In higher dimension, there are many more non-Kähler complex manifolds. For example, the twistor spaces of anti-self-dual 4-manifolds are complex threefolds which are not Kähler. Riemann-Roch in higher dimension is not as powerful as in 2-dimension. I was motivated by this fact to conjecture that every almost complex manifold with dimension greater than two admits an integrable complex structure, although it may not be homotopic to the original almost complex structure.

[1] F.A. Bogomolov, *Surfaces of class VII_0 and affine geometry*, Math. USSR-Izv. **21:1** (1983), 31–73.

[2] J. Li, S.-T. Yau and F. Zheng, *A simple proof of Bogomolov's theorem on class VII_0 surfaces with* $b_2 = 0$, Illinois J. Math. **34** (1990), 217–220.

Problem 2018002 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

This problem is about the construction of geometric structures on a manifold. Classically, a 2-dimensional surface admits many structures such as conformal structures and projective structures. They are good as the moduli space is finite dimensional and it means that the structures are reasonably canonical. The moduli space is not completely understood, even with modern technology. We like to construct geometric structure on topological space whose moduli space is finite dimensional. This becomes very tough when the dimension is greater than two.

The geometrization conjecture of Thurston solves the problem for 3-manifold beautifully. The approach of Hamilton-Perelman based on Hamilton's Ricci flow approach is still complicated to be understood thoroughly.

On the other hand, the construction of geometric structures over 4-dimensional manifolds are far more complicated. Algebraic surfaces have given a large set

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of beautiful examples of 4-manifolds. They have to play an important role in any approach to understand 4-manifolds. How do we tell which 4-dimensional manifolds admit an algebraic structure, besides some topological constraints given by Hodge theory and Riemann-Roch formula. The Miyaoka-Yau inequality gives a nontrivial topological constraint that went beyond Riemann-Roch. Donaldson invariants or the Seiberg-Witten invariants give different constraints on smooth structures of 4-dimensional manifolds.

Even for a much weaker question on the existence of symplectic structure on a 4-manifold, the answer is not satisfactory, despite there are constructions due to Gompf et al. In the theory of mirror symmetry, the category of symplectic structures is supposed to be dual to the category of complex structures. It is not clear whether this should tell us some relationship between moduli space of symplectic manifolds and the moduli space of complex surfaces. On the other hand, a very important structure called self-dual structure can exist on many 4-manifolds, due to a theorem of Cliff Taubes [1]. The problem is that we do not know whether there can be infinite number of components of self-dual structures on a given smooth manifold. It should be noted that due to the theorem of David Gieseker [2] and later by Viehweg [2] that if we fix the Hilbert polynomial, the number of components of moduli space of projective structure on a 4-manifold is bounded. Is there any way to find and prove a similar statement for the moduli space of self-dual structures on a 4-manifold?

Anti-self-dual 4-manifolds give rise to twistor space which has an integrable complex structure. So we should ask whether there are infinite number of components of moduli space of integrable complex structures over such complex 3-dimensional manifold or not. Perhaps we can classify these non-Kähler manifolds according to their algebraic dimension. If two anti-self-dual manifolds have the same twistor spaces up to birational type, what can we say about the relation between the two original anti-self-dual spaces? Can we construct interesting geometric invariants on the twistor space from the anti-self-dual structure? For higher dimensional hyperkähler manifolds, we can also construct twistor space. Similar questions can be asked. Conversely, can we construct interesting topological invariants on 4-manifolds based on the existence of anti-self-dual structures or the complex structures on twistor space.

If my conjecture that every almost complex manifold with dimension greater than two admits an integrable complex structure is correct, it opens up a nice way to construct geometric structures over even-dimensional manifolds with almost complex structure (the existence of which can be reduced to homotopy problem). But we have little experience on non-Kähler complex manifolds. Some structure on top of integrable complex structure is needed. In recent years, the concept of balanced metric is studied. It would be great to find conditions for a complex manifold to admit balanced structure. In the subject of algebraic manifolds, objects such as holomorphic vector bundles are important. Can we construct such objects by solving some differential equations over a complex vector bundle assuming some topological conditions? The major integrability condition for a complex vector bundle over a complex manifold to admit integrable complex structure comes from Riemann-Roch formula. Can we formulate these conditions clearly and can one find other obstructions?

[1] C.H. Taubes, *Self-dual Yang-Mills connections on non-self-dual 4-manifolds*, J. Differential Geom. **17** (1982), 139–170.
[2] D. Gieseker, *Global moduli for surfaces of general type*, Invent. math. **43** (1977) 233–282.
[3] E. Viehweg, *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. 30. Springer-Verlag, Berlin, 1995.

Problem 2018003 (Symplectic Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Cliff Taubes [1] established one of the most fundamental work in symplectic geometry by proving the existence of pseudo-holomorphic curves for almost complex structure compatible with the symplectic form, based on information from the Seiberg-Witten invariant, which is a topological invariant. However, this only works in 4-manifolds. What happens in higher dimensional symplectic manifolds? For example, can there be more than one symplectic structure on complex projective space CP^n with $n > 2$?

In the theory of mirror symmetry, symplectic geometry is mirror to complex structures. What is the correct concept of period integral in symplectic geometry as analogue of periods in algebraic geometry. And do they satisfy some kind of Picard-Fuchs equation? For a stable holomorphic bundle over an algebraic manifold, it is again stable when restricted to a generic hyperplane section. Since stable holomorphic bundle is mirror to special Lagrangian submanifold in symplectic geometry, presumably there is some special property if we intersect special Lagrangian submanifold with certain symplectic submanifolds.

Is there good concept of moduli space of complexified symplectic structures? If a natural moduli space exists, what kind of structure would it have? Would it admit symplectic structure or complex structure? The concept of period of integral is important in algebraic geometry. Is there such concept in symplectic geometry that can be used to reflect symplectic structure? Is there analogue of Torelli theorem? For even-dimensional manifolds with dimension greater than 6, is there any obstruction for the existence of symplectic structure besides the obvious cohomology conditions?

[1] C. Taubes, *SW⇒Gr: from the Seiberg-Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc. **9** (1996), 845–918.

Problem 2018004 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Suppose a compact Riemannian manifold M can be stratified by a finite union of submanifolds: Besides a finite number of them are compact with positive scalar curvature, the rest of them can be foliated by complete submanifolds (possibly varying dimensional leaves) with positive scalar curvature. Prove that the ambient manifold M admits a metric with positive scalar curvature. When the foliation is obtained by orbits of $SU(2)$-action, this was proved by Lawson and Yau [1]. The case when all leaves are same dimension, this appeared in the recent work of Weiping Zhang [2], who took my suggestion to generalize the above theorem of Lawson-Yau in 1994 when he visited Harvard. An important question is a noncompact Lie group G such as $SL(2)$-action where we like to construct a left invariant metric on G and transplant the metric to the orbits of the action. In order to do so, there may be certain assumption on the orbit structure to build a global metric with positive scalar curvature.

[1] B.H. Lawson and S.-T. Yau, *Scalar curvature, non-abelian group actions, and the degree of symmetry of exotic spheres*, Comment. Math. Helv. **49** (1974), 232–244.

[2] W. Zhang, *Positive scalar curvature on foliations*, Ann. Math. **185** (2017), 1035–1068.

Problem 2018005 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Ray and Singer [1] defined the Ray-Singer torsion for Riemannian manifods using zeta functions. Ray-Singer conjectured that for 3-dimensional manifolds, the analytic invariant is the same as the Reidemeister torion which is related to triangulation of the 3-dimensional manifolds. The later torsion was used by Milnor to distinguish the topological type of lens spaces which are homotopy equivalent to each other. The conjecture of Ray-Singer was proved by Cheeger [2] and Müller [3]. Later, Ray and Singer [4] defined their torion for Kähler manifolds. They are called holomorphic torsion. It appeared in genus 1 curve counting invariant from BCOV theory. Can one define an analogue of Reidemeister torsion in holomorphic category, perhaps using Čech theory?

[1] D.B. Ray and I.M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Adv. Math. **7** (1971), 145–210.

[2] J. Cheeger, *Analytic torsion and the heat equation*, Ann. of Math. (2) **109** (1979), 259–322.

[3] W. Müller, *Analytic torsion and R-torsion of Riemannian manifolds*, Adv. Math. **28** (1978), 233–305.

[4] D.B. Ray and I.M. Singer, *Analytic torsion for complex manifolds*, Ann. of Math. (2) **98** (1973), 154–177.