
Geometry of Symplectic Log Calabi–Yau Pairs

by Tian-Jun Li* and Cheuk Yu Mak†

Contents

1	Introduction	42
2	Topology of Cycle of Spheres in a Rational Surface	44
2.1	The Sequence $S(D)$ and the Boundary Torus Bundle	45
2.2	Toric Minimal Pairs	45
3	Algebraic Geometry of Looijenga Pairs	46
3.1	Torelli and Deformation	46
3.2	Cusp Singularities	46
4	Deformation Classes of Symplectic Log CY Pairs	46
4.1	Operations and Minimal Pairs	46
4.2	Classification by Homology Equivalence	47
4.3	Anti-Canonical Sequences	47
5	Contact Aspects	48
5.1	Trichotomy	48
5.2	Symplectic Fillings	49
	References	49

1. Introduction

Let Y be a smooth rational surface and let $D \subset Y$ be an effective reduced anticanonical divisor. Such pairs (Y, D) , called anti-canonical pairs, have a rich geometry. They were first investigated systematically by Looijenga, and by Friedman etc in the 80s. Note that $Y - D$ comes with a canonical (up to scaling) nowhere-vanishing 2-form Ω with simple poles along D . When the intersection matrix of D is negative definite, D can

be contracted and Y becomes a singular analogue of a K3 surface (a normal complex analytic surface with trivial dualizing sheaf). Motivated by mirror symmetry, Gross, Hacking and Keel introduced new ideas in a series of papers on Looijenga pairs, beginning with [8]. In particular, they proved Torelli type results in [9] conjectured by Friedman. In this regard, it was shown in [27] that the symplectic cohomology of $X - D$ is canonically isomorphic to the vector space of global sections of the structure sheaf of its mirror.

We will survey the smooth topology, algebraic geometry, symplectic geometry and contact geometry of anti-canonical pairs in Sections 2, 3, 4, 5 respectively.

Let X be a smooth, oriented 4 dimensional manifold. A topological divisor of X refers to a connected configuration of finitely many closed embedded, oriented, labeled smooth surfaces $D = C_1 \cup \cdots \cup C_k$ in X such that each intersection between two surfaces is transversal and positive, no three C_i intersect at a common point, and D has empty intersection with ∂X . A topological divisor D is often described by a plumbing graph with vertices corresponding to the surfaces C_i and edges corresponding to intersection points. Associated to D there are plumbed neighborhoods N_D as well as the boundary plumbed 3-manifold Y_D , which are all well-defined up to orientation-preserving diffeomorphisms.

Given a topological divisor $D = C_1 \cup \cdots \cup C_k$ in X , we use $[C_i]$ to denote the homology class of C_i in $H_2(X)$ and $H_2(N_D)$, $r(D) = k$ to denote the length of D , and $S(D) = (s_1, \cdots, s_{r(D)})$ to denote the sequence of self-intersection numbers. $H_2(N_D)$ is freely generated by C_i . The intersection matrix of D is the k by k square matrix $Q_D = (s_{ij} = [C_i] \cdot [C_j])$, where \cdot is used for any of the pairings $H_2(X) \times H_2(X), H^2(X) \times H_2(X), H^2(X) \times H^2(X, \partial X)$.

* School of Mathematics, University of Minnesota, Minneapolis, MN, US
E-mail: tjli@math.umn.edu
† DPMMS, University of Cambridge, Cambridge, UK
E-mail: cym22@dpmms.cam.ac.uk

Via the Lefschetz duality for N_D , the intersection matrix Q_D can be identified with the natural homomorphism $Q_D : H_2(N_D) \rightarrow H_2(N_D, Y_D)$. We use homology and cohomology with \mathbb{Z} coefficient unless otherwise specified.

For a symplectic 4-manifold (X, ω) a symplectic divisor is a topological divisor D with each C_i symplectic and having the orientation positive with respect to ω . Let K_ω be the symplectic canonical class of (X, ω) .

Definition 1.1. A symplectic log Calabi-Yau pair (X, D, ω) is a closed symplectic 4-manifold (X, ω) together with a nonempty symplectic divisor $D = \cup C_i$ representing the Poincare dual of $-K_\omega$. A symplectic log Calabi-Yau pair is called a symplectic Looijenga pair if each C_i is a sphere, called an elliptic log Calabi-Yau pair if D is a torus.

Here are some quick observations, which have well known analogues in the holomorphic category.

Lemma 1.2. For a symplectic log Calabi-Yau pair (X, D, ω) ,

- $c_1(X - D, \omega) = 0$, and $(X - D, \omega)$ is minimal in the sense it has no symplectic sphere with self-intersection -1 .
- $D = \cup C_i$ is either a torus or a cycle of spheres.
- (X, ω) is a rational or elliptic ruled symplectic 4-manifold. In particular, $\kappa(X, \omega) = -\infty$. D is a cycle of spheres only when (X, ω) is rational.
- $b^+(Q_D) = 0$ or 1.

Proof. The vanishing of $c_1(X - D)$ follows directly from the definition and $X - D$ being minimal follows directly from the adjunction formula. The 2nd bullet is also proved by the adjunction formula. Let g_i be the genus of C_i . Then

$$-[C_i] \cdot [C_i] - \sum_{j \neq i} [C_j] \cdot [C_i] = K_\omega \cdot [C_i] = -[C_i] \cdot [C_i] + 2g_i - 2.$$

So $2g_i - 2 = -\sum_{j \neq i} [C_j] \cdot [C_i] \leq 0$, namely, $g_i \leq 1$ for each i . If $g_i = 1$ for some i , then $\sum_{j \neq i} [C_j] \cdot [C_i] = 0$ which implies that C_i is the only component. The remaining case is that $g_i = 0$ for each i . In this case, $\sum_{j \neq i} [C_j] \cdot [C_i] = 2$ for each i and clearly D is a cycle of spheres.

Since D is a nonempty symplectic divisor representing $-K_\omega$ we have $K_\omega \cdot [\omega] < 0$. It follows from [17], [22] that (X, ω) is rational or ruled and admits a genus 0 Lefschetz fibration over a Riemann surface Σ . Let F be the fibre class. Since $K_\omega \cdot F = -2$ and D represents $-K_\omega$ the projection of D to Σ has nonzero degree. Since $D = \cup C_i$ is either a torus or a cycle of spheres, the genus of Σ is at most 1.

The last bullet follows from the fact that $b^+(X) = 1$. \square

Therefore elliptic pairs and Looijenga pairs are exactly the symplectic log Calabi-Yau pairs with

length 1 and at least 2 respectively. We remark that symplectic log Calabi-Yau pairs have vanishing relative symplectic Kodaira dimension (cf. [16]). The following is the main result in [12].

Theorem 1.3 (Symplectic deformation). *Two symplectic log Calabi-Yau pairs are symplectic deformation equivalent if they are homologically equivalent. In particular, each symplectic deformation class contains a Kähler pair.*

Moreover, two symplectic log Calabi-Yau pairs are strictly symplectic deformation equivalent if they are strictly homologically equivalent.

Let us explain the various equivalence notions in the theorem (see [29] for a thorough discussion of equivalence notions for symplectic manifolds). Let (X^0, D^0, ω^0) and (X^1, D^1, ω^1) be two symplectic pairs with $r(D^0) = r(D^1) = k$. They are said to be homologically equivalent if there is an orientation preserving diffeomorphism $\Phi : X^0 \rightarrow X^1$ such that $\Phi_*[C_j^0] = [C_j^1]$ for all $j = 1, \dots, k$. The homological equivalence is said to be strict if, in addition, $\Phi^*[\omega^1] = [\omega^0]$. When $X^0 = X^1$, they are said to be symplectic homotopic if (D^0, ω^0) and (D^1, ω^1) are connected by a family of symplectic divisors (D', ω') , and they are further said to be symplectic isotopic if ω' can be chosen to be a constant family. (X^0, D^0, ω^0) and (X^1, D^1, ω^1) are said to be symplectic deformation equivalent if they are homotopic, up to an orientation preserving diffeomorphism. They are said to be strictly symplectic deformation equivalent if they are symplectic isotopic, up to an orientation preserving diffeomorphism.

A sequence (s_i) of integers is said to be anti-canonical if it is realized as $S(D)$ for a symplectic log Calabi-Yau pair (X, D, ω) . Combined with Theorem 3.1 in [3], we obtain

Corollary 1.4. *Given a anti-canonical sequence (s_i) , there are only finitely many symplectic deformation types of symplectic log Calabi-Yau pairs (X, D, ω) with $S(D) = (s_i)$.*

There is an algorithm to write down the anti-canonical sequences, starting from the list of minimal pairs and reverse the minimal reduction process in [12]. It is interesting to compare anti-canonical sequences with spherical circular sequences. A spherical circular sequence is the sequence of a cycle of symplectic spheres in a rational surface with minimal complement. An anti-canonical sequence (s_i) is said to be rigid if, for any cycle of symplectic spheres $D \subset (X, \omega)$ with $S(D) = (s_i)$ and $(X - D, \omega)$ minimal, (X, D, ω) is a symplectic log Calabi-Yau pair.

Theorem 1.5 (Anti-canonical sequences, [14]). *Each spherical circular sequence with $b^+ = 1$ is anti-canonical, and each anti-canonical sequence with $b^+ = 1$ is rigid.*

From the contact point of view, symplectic log Calabi-Yau pairs are separated into 3 groups, as stated in the following theorem. Here, $Kod(Y, \xi)$ is the contact Kodaira dimension introduced in [13].

Theorem 1.6 (Contact trichotomy, [14]). *Let (X, D, ω) be a symplectic log Calabi-Yau pair, Q_D the intersection matrix of D and (s_i) the self intersection sequence.*

- (i) *If Q_D is negative definite, then D admits convex neighborhoods inducing the same contact 3-manifold (Y_D, ξ_D) , which only depends on $S(D)$ and has $Kod \leq 0$.*
- (ii) *If $b^+(Q_D) = 1$, up to local symplectic deformations, D admits concave neighborhoods inducing the same contact 3-manifold (Y_D, ξ_D) , which only depends on $S(D)$ and has $Kod = -\infty$.*
- (iii) *If $b^+(Q_D) = 0$ but Q_D is not negative definite, then it does not admit a regular neighborhood with contact boundary.*

Golla and Lisca considered a large family \mathcal{F} of torus bundles and showed that these torus bundles are equipped with contact structures arising from Looijenga D with $b^+(Q_D) = 1$ (Theorem 2.5 in [6]). They also showed, for a subfamily of these torus bundles, such a contact structure is the unique universally tight contact structure with vanishing Giroux torsion (Theorem 1.2 in [6]). This led them to formulate the following conjecture.

Conjecture 1.7 ([6]). *For a concave cycle D of symplectic spheres, the contact structure ξ_D on Y_D is universally tight.*

Moreover, they investigated Stein (and symplectic) fillings and classified in many cases up to diffeomorphism (Theorems 3.1, 3.2, 3.5 in [6]). On the other hand, Ohta and Ono classified symplectic fillings of simple elliptic singularities up to symplectic deformation (Theorems 1, 1', 2 in [25]). Using these results and Corollary 1.4, we establish the following finiteness result.

Corollary 1.8 (Symplectic fillings, [14]). *Suppose (X, D, ω) is a symplectic log Calabi-Yau pair with $b^+(Q_D) = 1$. Then*

- *There are finitely many (at least 1) Stein fillings of (Y_D, ξ_D) up to symplectic deformation, all having $b^+ = 0$. Moreover, for a Looijenga pair, all Stein fillings have $c_1 = 0$.*
- *This is also true for minimal symplectic fillings.*

We end the survey discussing the geography of Stein fillings for negative definite Q_D .

The first author is grateful for the opportunity to speak at the 'Perspectives of Mathematics in the 21st Century: Conference in Celebration of the 90th Anniversary of Mathematics Department

of Tsinghua University'. The authors are also grateful to Kaoru Ono for his interest and useful discussions. The authors were supported by NSF grants DMS 1065927 and 1207037, and are supported by NSF grant 1611680.

2. Topology of Cycle of Spheres in a Rational Surface

In this section we review some homological facts about topological divisors, especially cycles of spheres, and we refer to [21], [6] and [14] for details. We first introduce a pair of basic operations for topological divisors.

Definition 2.1. *Toric blow-up is the operation adding a sphere component with self-intersection -1 between an adjacent pair of components C_i and C_{i+1} and reducing the self-intersection of C_i and C_{i+1} by -1 . Toric blow-down is the reverse operation.*

Notice that there is a natural labeling for these operations.

Two pairs (X, D^0) and (X, D^1) are said to be toric equivalent if they are connected by toric blow-ups and toric blow-downs. D is said to be toric minimal if no component is an exceptional sphere. Here, an exceptional sphere is a sphere with self-intersection -1 .

They can be performed in the holomorphic and symplectic categories. In the holomorphic category they are often referred as corner blow-up/down.

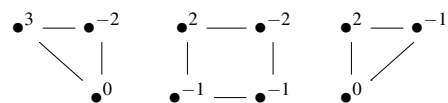
Lemma 2.2. *The following are preserved under a toric blow-up/down:*

- *D being a cycle of spheres,*
- *the non-degeneracy of the intersection matrix Q_D ,*
- *the oriented diffeomorphism type of the plumbed 3-manifold Y_D .*

The 1st bullet is obvious, while the 2nd bullet is by a direct computation. The 3rd bullet is part of Proposition 2.1 in [21].

Here is an example to illustrate how a sphere with $s = 0$ can be used to "balance" the self-intersection of the two sides by performing a toric blow-up and a toric blow-down.

Example 2.3 (Toric move). *The following three cycles of spheres are toric equivalent:*



From now on D is either a smooth torus or a cycle of smooth spheres. When D is a torus with self-intersection s , the boundary 3-manifold is the circle bundle with Euler number s .

2.1 The Sequence $S(D)$ and the Boundary Torus Bundle

When D is a cycle of spheres the labeling is taken to be cyclic. The orientation of D is a cyclic labeling up to permutation. We will assume now that D is a cycle of spheres with the self-intersection sequence $S(D) = (s_i)$. Let $s(D) = \sum_{i=1}^{r(D)} (s_i + 2)$ denote the self-intersection number of D .

Lemma 2.4 (cf. Theorem 2.5 and Theorem 3.1 in [6]). *Let D be a cycle of spheres in X and $V = X - N_D$.*

- $H_2(N_D) = \mathbb{Z}^{r(D)} = H^2(N_D), H_1(N_D) = H^1(N_D) = \mathbb{Z}, H_3(N_D) = H^3(N_D) = 0$.
- $H_1(Y_D) \rightarrow H_1(N_D)$ is a surjection. If Q_D is non-degenerate, then $b_1(Y_D) = 1$ and the map $H_1(Y_D) \rightarrow H_1(N_D)$ has a finite kernel, $H_2(Y_D) = H^1(Y_D) = \mathbb{Z}$ and the map $H_2(Y_D) \rightarrow H_2(N_D)$ is trivial.
- Suppose Q_D is non-degenerate and $b_1(X) = 0$, then $b_1(V) = b_3(V) = 0$, $b_2(V) = b_2(X) - r(D) - 1$ and the map $\mathbb{Z} = H_2(Y_D) \rightarrow H_2(V)$ is injective.

Here are obvious restrictions on homologous components of D from the cycle condition.

Lemma 2.5. *For a cycle of spheres D ,*

- At most three components are homologous in X . There are three homologous components only if $r(D) = 3$.
- There are a pair of homologous components only if $r(D) \leq 4$.
- If $[C_i] = [C_{i+1}]$ for some i then $r(D) = 3, s_i = s_{i+1} = 1$, or $r(D) = 2, s_i = s_{i+1} = 2$.

When $b^+(X) = 1$ there are various restrictions on components with non-negative self-intersection. Let $r^{\geq 0}(D)$ denote the number of components with self-intersection ≥ 0 .

Lemma 2.6. *Suppose D is a cycle of spheres in X with $b^+(X) = 1$.*

- If C_i and C_j are not adjacent and $s_i \geq 0, s_j \geq 0$, then $[C_i] = [C_j]$ and $s_i = s_j = 0$.
- $r^{\geq 0}(D) \leq 4$.
- $r^{\geq 0}(D) = 4$ only if $r(D) = 4, s_i = 0$ for each i and $[C_1] = [C_3], [C_2] = [C_4]$.
- Suppose $r(D) \geq 3$. If $s_i \geq 0, s_{i+1} \geq 0, s_i s_{i+1} \geq 1$ for some i , then $[C_i] = [C_{i+1}]$ and $s_i = s_{i+1} = 1$. This is only possible when $r(D) = 3$.

These constraints follow easily from the $b^+(X) = 1$ condition. The following lemma, derived from Lemmas 2.5 and 2.6, is very useful for Theorems 1.5, 1.6 and 1.8.

Lemma 2.7 ([14]). *Suppose D is a cycle of spheres in X with $b^+(X) = 1$. Up to cyclic permutation and orientation of D , we have*

- If $r(D) \geq 5$, then $r^{\geq 0}(D) \leq 2$. When $r^{\geq 0}(D) = 2$, $s_1 \geq 0, s_2 = 0$.
- If $r(D) = 4$ and $r^{\geq 0}(D) \geq 3$, then $S(D) = (k \geq 0, 0, l < 0, 0), [C_2] = [C_4], l + k \leq 0$.
- If $r(D) = 4$ and $r^{\geq 0}(D) = 2$, then either $S(D) = (0, l_1 < 0, 0, l_2 < 0), [C_1] = [C_3]$ or $(s_i) = (k \geq 0, 0, l_1 < 0, l_2 < 0), l_1 + l_2 + k \leq 0$.
- If $r(D) = 3$ and $r^{\geq 0}(D) = 3$, then the only possibilities of $S(D)$ are (i) $(1, 1, 1), [C_1] = [C_2] = [C_3]$, (ii) $(1, 1, 0), [C_1] = [C_2]$, (iii) $(2 \geq k \geq 0, 0, 0)$.
- If $r(D) = 3$ and $r^{\geq 0}(D) = 2$, then the only possibilities of $S(D)$ are (i) $(1, 1, p < 0), [C_1] = [C_2]$, (ii) $(k \geq 0, 0, p < 0), p + k \leq 2$.
- If $r(D) = 2$ and $r^{\geq 0}(D) = 2$, then the only possibilities of $S(D)$ are $(4, 1), (4, 0), (3, 1), (3, 0), (2, 2), (2, 1), (2, 0), (1, 1), (1, 0), (0, 0)$.
- If $r(D) = 2$ and $r^{\geq 0}(D) = 1$, then $S(D) = (k \geq 0, p < 0)$.
- If $r(D) = 2$ and $r^{\geq 0}(D) = 0$, then $S(D)$ is one of $(-1, -1), (-1, -2), (-1, -3)$.

To describe the plumbed 3-manifold Y_D , we introduce the matrix in $SL_2(\mathbb{Z})$ for a sequence of integers $(-t_1, \dots, -t_k)$,

$$A(-t_1, \dots, -t_k) = \begin{pmatrix} -t_k & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -t_{k-1} & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} -t_1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Lemma 2.8 (Theorem 6.1 in [21], Theorem 2.5 in [6]). *For a cycle of spheres D with self-intersection sequence $S(D) = (s_1, \dots, s_k)$, the plumbed 3-manifold Y_D is the oriented torus bundle T_A over S^1 with monodromy $A = A(-s_1, \dots, -s_k)$. The intersection matrix Q_D is non-degenerate if the trace of $A(-s_1, \dots, -s_k) \neq 2$.*

2.2 Toric Minimal Pairs

Lemma 2.9. *Any cycle of sphere is toric equivalent to a toric minimal one or one with sequence $(-1, p)$. If $S(D) = (-1, p)$, then Q_D is degenerate only if $p = -4$.*

Suppose D is a toric minimal cycle of spheres with sequence $S(D) = (s_i)$. Then

- $b^+(Q_D) \geq 1$ if and only if $s_i \geq 0$ for some i .
- Q_D is negative definite if $s_i \leq -2$ for all i and less than -2 for some i . Q_D is negative semi-definite but not negative definite if $s_i = -2$ for each i .
- Q_D is non-degenerate if either $s_1 \geq 0$ and $s_i \leq -2$ for $i \geq 2$, or $s_1 = s_2 = 0$ and $s_i \leq -2$ for $i \geq 3$,

The first statement is by definitions (notice that we do not allow nodal components). The second statement is obvious. Bullets 1, 2 are well-known (cf. Lemma 8.1 in [21]). To prove the 3rd bullet, by Lemma 2.8, we just need to that the trace of the monodromy matrix is not equal to 2, which is a direct calculation using Lemma 5.2 in [21].

Each toric minimal, negative definite cycle D with $s(D) \leq -2$ has a dual cycle \check{D} , with the property that the

plumbed manifolds Y_D and $Y_{\check{D}}$ are orientation reversing diffeomorphic (Theorem 7.1 in [21]). To describe the dual cycle we use the 2 by k matrix $\begin{pmatrix} a_1 & \dots & a_k \\ b_1 & \dots & b_k \end{pmatrix}$ to represent the sequence $(a_1, -2, \dots, -2, a_2, \dots, a_k)$, where $a_i \leq -3$ and there are b_i many -2 between a_i and a_{i+1} . For a negative definite toric minimal cycle D with $s(D) \leq -2$, we have either two a_i terms or $a_i \leq -4$ for some i . The dual cycle \check{D} is represented by the 2 by k matrix $\begin{pmatrix} \check{a}_i & = & -b_i - 3 \\ \check{b}_i & = & -a_{i+1} - 3 \end{pmatrix}$. It is easy to check that \check{D} is also toric minimal, negative definite and $s(\check{D}) \leq -2$. A remark is that we can also view the elliptic pairs (s) and $(-s)$ as dual pairs in the sense that boundary 3-manifolds are orientation reversing diffeomorphic.

3. Algebraic Geometry of Looijenga Pairs

In this section we very briefly review some basic results of Looijenga pairs (Y, D) , which have or might have symplectic analogues. Please consult the survey article [3] and [8].

3.1 Torelli and Deformation

There are several versions of the Torelli theorem. The following is Theorem 8.5 in [3].

Theorem 3.1 (A global Torelli). *Given Looijenga pairs and an isomorphism of lattices μ compatible with D , there is a isomorphism f of Looijenga pairs such that $\mu = f^*$ if and only if μ preserves the nef cone.*

Two anticanonical pairs are said to be (holomorphically) deformation equivalent if they are both isomorphic to fibers of a family of anticanonical pairs over a connected base. The following two statements are given in Theorem 3.1 and Theorem 5.14 in [3] respectively.

Theorem 3.2. *There are only finitely many deformation types of Looijenga pairs with the same self-intersection sequence. Two Looijenga pairs are deformation equivalent if they are homology equivalent.*

3.2 Cusp Singularities

A cusp singularity is the germ of an isolated, normal surface singularity such that the exceptional divisor of the minimal resolution is a cycle of smooth rational curves D meeting transversely. For normal surface singularities, there is a notion of Kodaira dimension κ^δ , and Gorenstein surface singularities with $\kappa^\delta = 0$ are simple elliptic singularities and cusp singularities (cf. [26] and the references therein).

Cusp singularities come in dual pairs, and their minimal resolutions are given as dual cycles. Every

pair of dual cycles embed in a Hirzebruch–Ionue surface as the only curves. A cusp singularity is called rational if its minimal resolution is realized as the anti-canonical divisor of a rational surface. By the Mumford–Grauert criterion, any toric minimal, negative definite Looijenga pair (Y, D) arises as the minimal resolution of a rational cusp singularity. Looijenga proved that a cusp is rational if its dual cusp is smoothable and he conjectured the converse is also true. The Looijenga conjecture was proved in [8] via mirror symmetry and later by integral-affine geometry in [2].

4. Deformation Classes of Symplectic Log CY Pairs

In this section we give a brief outline of the proof of Theorem 1.3 and Theorem 1.5.

4.1 Operations and Minimal Pairs

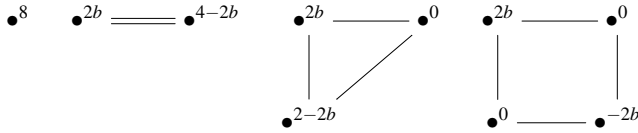
It involves the operations of non-toric blow-up/down and the notion of minimal models. A *non-toric blow-up* of D is the proper transform of a symplectic blow-up centered at a smooth point of D . A non-toric blow-down is the reverse operation which symplectically blows down an exceptional sphere not contained in D . These operations preserve the log Calabi–Yau condition and there are analogues in the holomorphic category, sometimes referred as interior blow-up/blow-down.

A symplectic log Calabi–Yau pair (X, D, ω) is called minimal if (X, ω) is minimal, or (X, D, ω) is a symplectic Looijenga pair with $X = \mathbb{C}P^2 \# \mathbb{C}P^2$. For any symplectic log Calabi–Yau pair (X, D, ω) , we apply first a maximal sequence of non-toric blow-downs using [19] and then a maximal sequence of toric blow-downs. The resulting toric minimal pair, which is actually minimal due to [28], is called a minimal model of (X, D, ω) .

We enumerate the minimal symplectic log Calabi–Yau pairs (modulo cyclic symmetry), all of them having length less than 5.

- Case (A): The base genus of X is 1. D is a torus.
- Case (B): $X = \mathbb{C}P^2$, $c_1 = 3h$.
 - (B1) D is a torus,
 - (B2) D consists of a h -sphere and a $2h$ -sphere, or
 - (B3) D consists of three h -spheres.
- Case (C): $X = \mathbb{S}^2 \times \mathbb{S}^2$, $c_1 = 2f_1 + 2f_2$, where f_1 and f_2 are the homology classes of the two factors.
 - (C1) D is a torus.
 - (C2) $r(D) = 2$ and $[C_1] = bf_1 + f_2, [C_2] = (2-b)f_1 + f_2$.
 - (C3) $r(D) = 3$ and $[C_1] = bf_1 + f_2, [C_2] = f_2, [C_3] = (1-b)f_1 + f_2$.
 - (C4) $r(D) = 4$ and $[C_1] = bf_1 + f_2, [C_2] = f_1, [C_3] = -bf_1 + f_2, [C_4] = f_1$.

The graphs in (C1), (C2), (C3) and (C4) are given respectively by



- Case (D): $X = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, $c_1 = f + 2s$, where f and s are the fiber class and section class with $f \cdot f = 0$,

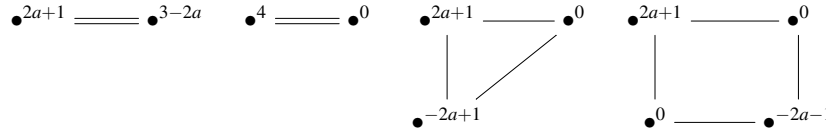
$$f \cdot s = 1 \text{ and } s \cdot s = 1.$$

(D2) $r(D) = 2$, and either $([C_1], [C_2]) = (af + s, (1 - a)f + s)$ or $([C_1], [C_2]) = (2s, f)$.

(D3) $r(D) = 3$ and $[C_1] = af + s, [C_2] = f, [C_3] = -af + s$.

(D4) $r(D) = 4$ and $[C_1] = af + s, [C_2] = f, [C_3] = -(a + 1)f + s, [C_4] = f$.

The graphs in (D2), (D3) and (D4) are given respectively by



4.2 Classification by Homology Equivalence

There are two steps to prove Theorem 1.3. One step is to show that each (strict) homology type of minimal pairs contains a unique (strict) deformation class via a combination of pseudo-holomorphic curve techniques and Thurston type symplectic construction in the setting of a pair of a symplectic 4-manifold with a smooth symplectic surface.

We also introduce marked divisors and establish the invariance of their (strict) deformation class under toric and non-toric blow-up/down operations (cf. also [25]). This invariance property reduces Theorem 1.3 to the minimal case. The statement that each symplectic deformation class contains a Kähler pair is not stated in [12] but it follows from the proof outlined above since each minimal pair clearly deforms to a Kähler pair (cf. Section 3 in [12] and Theorem 2.4 in [3]) and blow-up/down can be performed in the Kähler category.

We remark that Theorem 1.3 should also apply to the cases of irreducible nodal spheres and cuspidal spheres using [1] and [24] respectively.

Proof of Corollary 1.4. By Theorem 1.3, every symplectic deformation class contains a Kähler pair. The finiteness of Looijenga pairs follows directly from Theorem 3.2. For elliptic symplectic log Calabi-Yau pairs, where the sequences are of length 1, the finiteness is more straightforward—it follows from the finiteness of symplectic deformation types in the case of minimal pairs for each (s) , where $s = 0, 8, 9$ (cf. Section 3 in [12]), and the fact that there is only one way to (non-toric) blow up, up to deformation. \square

4.3 Anti-Canonical Sequences

Due to the classification of minimal symplectic log Calabi-Yau pairs, it is a combinatorial prob-

lem to determine the anti-canonical sequences. There are also various conditions on spherical circular sequences with $b^+ = 1$ in Lemma 2.9, Lemma 2.7, Lemma 2.6. The first statement of Theorem 1.5 that every spherical circular sequence with $b^+ = 1$ is anti-canonical is deduced from these lemmas, the list of minimal pairs, the observation that whether a spherical circular sequence is anti-canonical only depends on its toric equivalence class, and

Proposition 4.1. *Suppose $D \subset (X, \omega)$ is a cycle of spheres in a rational surface (X, ω) with minimal complement. Then $s(D) \leq 9$, and $S(D) \neq (5 + l, -l)$ with $l \geq 2$.*

D represents $c_1(X, \omega)$ if

- $s_i \geq -1$ for any i , or
- $S(D) = (1, -p_1 + 1, -p_2, \dots, -p_{l-1}, -p_l + 1)$ with $p_i \geq 2$, $l \geq 2$.

This proposition is proved using Theorem 6.10 in [25], Proposition 3.14 in [16], Theorem 3.1 in [6], and a direct verification to exclude $(5 + l, -l)$ with $l \geq 0$.

For the second statement of Theorem 1.5 that any anti-canonical sequence with $b^+ = 1$ is rigid, it follows from the following propositions and the observation that whether an anti-canonical sequence is rigid only depends on its toric equivalence class.

Proposition 4.2. *Suppose (s_i) is an anti-canonical sequence and it belongs to one in the following list.*

- $(1, -p_1 + 1, -p_2, \dots, -p_{l-1}, -p_l + 1)$ with $p_i \geq 2$, $l \geq 2$ so $r(D) \geq 3$.
- $(0, 0, 0, n)$ with $n \leq 0$.
- $(1, 1, p)$, $p \leq 1$.
- $(1, p)$ with $p \geq 4$.
- $(0, n)$ with $n \leq 4$.
- $s_i \geq -1$ for each i .
- $(-1, -2)$ and $(-1, -3)$.

Then (s_i) is rigid.

Proposition 4.3. *Suppose (X, D, ω) is a symplectic Looijenga pair with $b^+(Q_D) = 1$. Then $S(D)$ is toric equivalent to one in Proposition 4.2.*

Proposition 4.2, except for the last bullet, is proved using Proposition 7.1 in [25], Theorems 3.1, 3.2, 3.5 in [6] and similar arguments. The cases $(s_i) = (-1, -2)$ and $(-1, -3)$ are more delicate, requiring a blowup trick. Proposition 4.3 is proved by Lemmas 2.9, 2.7, 2.6, the toric move in Example 2.3 and induction on the length of D .

5. Contact Aspects

Let (X, D, ω) be a symplectic log Calabi-Yau pair. A neighborhood N' of D is called a concave (resp. convex) neighborhood if N' is a concave (resp. convex) symplectic manifold. D is called concave (resp. convex) if for any neighborhood N' of D , there is a concave (resp. convex) plumbing neighborhood $N_D \subset N'$. A necessary condition for D to be either convex or concave is ω being exact on the boundary of any plumbing neighborhood. Here is a local criterion.

Lemma 5.1. *$\omega|_{Y_D}$ is exact if and only if there is a solution for z to the equation $Q_D z = a$, where $a = ([\omega] \cdot [C_1], \dots, [\omega] \cdot [C_k])$ is the area vector. In particular, this holds if Q_D is non-degenerate. Moreover, this condition only depends on the toric equivalence class.*

The first statement is observed in [11]. Moreover, tori blow-up/down is a local operation that does not change the diffeomorphism type of Y_D and the exactness of $\omega|_{Y_D}$. One can also check that the solvability for $Q_D z = a$ is stable under toric blow-up/down by simple linear algebra. When X is a closed manifold, we also have the following criterion.

Lemma 5.2. *Suppose X is a closed manifold with intersection matrix Q_X . Let $I_1 = \iota_*(H_2(D); \mathbb{R}) \subset H_2(X; \mathbb{R})$ and $I_2 \subset H_2(X; \mathbb{R})$ be Q_X -orthogonal to I_1 in $H_2(X; \mathbb{R})$. If the span of $I_1 \cup I_2$ is $H_2(X; \mathbb{R})$, then $\omega|_{Y_D}$ is exact. The existence of I_2 is preserved under toric blow-up and toric blow-down.*

We also recall two criteria for symplectic divisors to be contact and the definition of contact Kodaira dimension.

Theorem 5.3 ([5], [20]). *A negative definite symplectic divisor is convex.*

Theorem 5.4 ([11]). *Let $D \subset (W, \omega_0)$ be a symplectic divisor. If Q_D is not negative definite and ω_0 restricted to the boundary of D is exact, then ω_0 can be locally deformed through a family of symplectic forms ω_t on W keeping D symplectic and such that (D, ω_1) is a concave divisor. Moreover, the contact structure ξ_D on Y_D is canonically associated to D in this case and in the negative definite case.*

Definition 5.5 ([13], [15]). *Let (W, ω) be a concave symplectic 4-manifold with contact boundary (Y, ξ) . (W, ω) is called a Calabi-Yau cap of (Y, ξ) if $c_1(W)$ is a torsion class, and it is called a uniruled cap of (Y, ξ) if there is a contact primitive β on the boundary such that $c_1(W) \cdot [(\omega, \beta)] > 0$.*

The contact Kodaira dimension of a contact 3-manifold (Y, ξ) is defined in terms of uniruled caps and Calabi-Yau caps. Precisely, $\text{Kod}(Y, \xi) = -\infty$ if (Y, ξ) has a uniruled cap, $\text{Kod}(Y, \xi) = 0$ if it has a Calabi-Yau cap but no uniruled caps, $\text{Kod}(Y, \xi) = 1$ if it has no Calabi-Yau caps or uniruled caps.

5.1 Trichotomy

Theorem 1.6 is based on the following observation in [14] (cf. also Theorem 2.5 in [6]).

Proposition 5.6. *For a symplectic log Calabi-Yau pair (X, D, ω) , ω is exact on Y_D if and only if Q_D is negative definite or $b^+(Q_D) = 1$.*

This result is proved by the local criterion Lemma 5.1, Lemma 2.9, Lemma 2.7, Lemma 2.6, the toric move in Example 2.3, and by applying the I_2 -criterion Lemma 5.2 to the following list of log Calabi-Yau pairs (X, D, ω) with $r(D) \leq 4$ and $b^+(Q_D) = 1$.

1. (B2) in the list of minimal models; $I_2 = \emptyset$; $S(D) = (1, 4)$.
2. (C2) with $b = 1$; $I_2 = \emptyset$; $S(D) = (2, 2)$.
3. (B3); $I_2 = \emptyset$; $S(D) = (1, 1, 1)$.
4. Non-toric blow-ups of (B3) on C_3 and its proper transforms; $I_2 = \{e_j - e_{j+1}, 1 \leq j \leq \alpha - 1\}$; $S(D) = (1, 1, 1 - \alpha)$.
5. Non-toric blow-ups of (C3) on C_3 and its proper transforms; $I_2 = \{e_j - e_{j+1}, 1 \leq j \leq \alpha - 1\}$; $S(D) = (0, 0, 2 - \alpha)$.
6. (C4) with $b = 0$; $I_2 = \emptyset$; $S(D) = (0, 0, 0, 0)$.
7. Non-toric blow-ups of (C4) with $b = 0$ on C_4 and its proper transforms; $I_2 = \{e_j - e_{j+1}, 1 \leq j \leq \alpha - 1\}$; $S(D) = (0, 0, 0, -\alpha)$.

For Case (iii) of Theorem 1.6, it follows from Proposition 5.6 that ω is not exact on Y_D . For Case (i) of Theorem 1.6, Q_D is negative definite and hence there is a convex plumbing neighborhood N_D with contact boundary (Y_D, ξ_D) by Theorem 5.3. Notice that $P = X - N_D$ is a symplectic cap of Y_D with vanishing c_1 , namely, it is a Calabi-Yau cap. It follows that $\text{Kod}(Y_D, \xi_D) \leq 0$. For Case (ii) of Theorem 1.6, it follows from Theorem 5.4 and Proposition 5.6 that, up to a local symplectic deformation, there is a concave plumbing neighborhood N_D with contact boundary (Y_D, ξ_D) . Moreover, since D is symplectic and represents $c_1(X)$, for any contact primitive α of $\omega|_{Y_D}$, we have $c_1(N_D) \cdot [(\omega, \alpha)] = c_1(X)|_{N_D} \cdot [(\omega, \alpha)] = D \cdot [(\omega, \alpha)] = D \cdot [\omega] > 0$. Thus N_D is a uniruled cap.

Remark 5.7. Applying Theorem 1.6, Theorem 1.3 and Proposition 4.1 in [9], it is not hard to prove the following statement: For a symplectic log Calabi-Yau pair (X, D, ω) with $b^+(Q_D) = 1$, there exists a Kähler log Calabi-Yau pair $(\bar{X}, \bar{D}, \bar{\omega})$ in its symplectic deformation class such that \bar{D} is the support of an ample line bundle. Then $(\bar{X} - \bar{D}, \bar{\omega})$ provides a Stein filling with $b^+ = 0$ and $c_1 = 0$.

5.2 Symplectic Fillings

In the context of torus bundles, Golla-Lisca investigated symplectic fillings in the case $b^+(Q_D) = 1$. Here is a summary of their results.

Theorem 5.8 (Theorems 1.1, 3.1, 3.5 in [6]). *For a large family \mathcal{F} of torus bundles T_A arising from D with $b^+(Q_D) = 1$, all Stein fillings of $(T_A = Y_D, \xi_D)$ have $c_1 = 0$, $b_1 = 0$ and the same b_2 . Moreover, up to diffeomorphism, there are only finitely many Stein fillings, and there is a unique Stein filling if $|\text{tr}A| < 2$. Here A is the monodromy matrix of Y_D . These results also hold for minimal symplectic fillings for this family, except possibly 3 torus bundles with $|\text{tr}A| < 2$.*

According to Corollary 1.8, the finiteness property holds more generally.

Proof of Corollary 1.8. By Theorem 1.6 (Y_D, ξ_D) is fillable and all the symplectic fillings have $b^+ = 0$. For Looijenga pairs, the Stein fillability follows from Remark 5.7. For an elliptic pair with self-intersection $s > 0$, there is an obvious Stein filling diffeomorphic to the neighborhood of a torus with self-intersection $-s$. The finiteness of symplectic fillings for elliptic pairs is proved in [25] (see Theorem 5.9).

Now observe that if D is concave and (Stein) rigid then any (Stein) symplectic filling of (Y_D, ξ_D) is the complement of a symplectic log CY pair with the same self-intersection sequence. Now we invoke the second statement of Theorem 1.5 and Corollary 1.4 to conclude the finiteness of Stein symplectic fillings for all Looijenga pairs and the finiteness of symplectic fillings except for the toric equivalence classes of $(-1, -2), (-1, -3)$. Clearly, the fillings have vanishing c_1 . \square

Together with Theorems 1.3 and 1.8 in [15], Theorem 1.6 has the following consequence: when Q_D is negative definite, the Betti numbers of exact fillings of (Y_D, ξ_D) are bounded. For elliptic pairs, we have the following:

Theorem 5.9 (Theorem 2 in [25]). *Any simple elliptic singularity has finite number of symplectic fillings, arising either from a smoothing or the minimal resolution.*

For Looijenga pairs, when D is negative definite and toric minimal, ξ_D coincides with the contact

structure arising from the corresponding cusp singularity and hence is Stein fillable with a Stein filling diffeomorphic to N_D . Notice that $b_1(N_D) = 1$ by Lemma 2.4. We provide some explicit Betti number bounds for Stein fillings below when D is negative definite.

Proposition 5.10 ([14]). *Suppose that D is toric minimal and negative definite and $V = X - N_D$. If U is a Stein filling of Y_D , then $X_U = U \cup V$ has either $b^+ = 1$ or 3, and $b^+(X) = 1 + b^+(U) + b_2^0(U), b_2^0(U) + b_1(U) = 1$.*

When $b^+(X_U) = 1$, X_U is rational or an integral homology Enriques surface, and U is negative definite with $b_1(U) = 1$. In this case $e(U) = b^-(U)$, where e is the Euler number.

When $b^+(X_U) = 3$, X_U is an integral homology K3, $(b_2^+(U), b_2^0(U), b_1(U)) = (1, 1, 0)$ or $(2, 0, 1)$. In either case, $c_1(U) = 0$ and $2 \leq e(U) \leq 21$.

Finally, we discuss the potential implication of Proposition 5.10 for Stein fillings of cusp singularities. By the now confirmed Looijenga conjecture which states that a cusp singularity is smoothable if and only if has a rational dual, a smoothing of a cusp singularity provides a Stein filling with $b^+ = 1$. In light of this, Proposition 5.10 provides some evidence to the following symplectic/contact analogue of the Looijenga conjecture.

Speculation 5.11. *If a cusp singularity does not have a rational dual, then it admits only negative definite Stein fillings.*

References

- [1] J.-F. Barraud. Nodal symplectic spheres in $\mathbb{C}P^2$ with positive self-intersection. *Internat. Math. Res. Notices* 495–508, 1999.
- [2] P. Engel. A proof of Looijenga’s conjecture via integral-affine geometry. arXiv:1409.7676.
- [3] R. Friedman. On the geometry of anticanonical pairs. arXiv:1502.02560.
- [4] R. Friedman and R. Miranda. Smoothing cusp singularities of small length. *Math. Ann.* 263:185–212, 1983.
- [5] D.T. Gay and A.I. Stipsicz. Symplectic surgeries and normal surface singularities. *Algebr. Geom. Topol.* 9(4):2203–2223, 2009.
- [6] M. Golla and P. Lisca. On Stein fillings of contact torus bundles. *Bull. Lond. Math. Soc.* 48:19–37, 2016.
- [7] R.E. Gompf. A new construction of symplectic manifolds. *Ann. of Math.* 143(3):527–595, 1995.
- [8] M. Gross, P. Hacking and S. Keel. Mirror symmetry for log Calabi-Yau surfaces I. arXiv:1106.4977, 2011.
- [9] M. Gross, P. Hacking and S. Keel. Moduli of surfaces with an anti-canonical Cy. arXiv:1211.6367, 2012.
- [10] F. Lalonde and D. McDuff. J-curves and the classification of ruled symplectic 4-manifolds. *Contact and symplectic geometry Publ. Newton Inst.* 8, (Cambridge, 1994), 3–42, Cambridge Univ. Press, Cambridge, 1996.
- [11] T.-J. Li and C.Y. Mak. Symplectic Divisorial Capping in Dimension 4. arXiv:1407.0564.
- [12] T.-J. Li and C.Y. Mak. Symplectic log Calabi-Yau surface-deformation class. *Adv. Theor. Math. Phys.* 20(2):351–379, 2016.

- [13] T.-J. Li and C.Y. Mak. The Kodaira dimension of contact 3-manifolds and geography of symplectic fillings. arXiv:1610.06870.
- [14] T.-J. Li and C.Y. Mak. Symplectic log Calabi-Yau surface-contact aspects. In preparation.
- [15] T.-J. Li, C.Y. Mak and K. Yasui. Calabi-Yau caps, uniruled caps and symplectic fillings. *Proc. Lond. Math. Soc.*, (3) 114(1):159–187, 2017.
- [16] T.-J. Li and W. Zhang. Additive and relative Kodaira dimension. *Geometry and Analysis*, No. 2, 18:103–135, Adv. Lect. Math. (ALM), Int. Press., 2011.
- [17] A. Liu. Some new applications of general wall crossing formula, Gompf’s conjecture and its applications. *Math. Res. Lett.* 3(5):569–585, 1996.
- [18] E. Looijenga. Rational surfaces with an anti-canonical divisor. *Ann. of Math.* 114(2):267–322, 1981.
- [19] D. McDuff and E. Opshtein. Nongeneric J-holomorphic curves and singular inflation. *Algebr. Geom. Topol.* 15:231–286, 2015.
- [20] M. McLean. The growth rate of symplectic homology and affine varieties. *Geometric And Functional Analysis.* 22(2):369–442, 2012.
- [21] W.D. Neumann. A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves. *TAMS* 268(2):299–344, 1981.
- [22] H. Ohta and K. Ono. Notes on symplectic 4-manifolds with $b^+ = 1$, II. *Int. J. Math.* 7:755–770, 1996.
- [23] H. Ohta and K. Ono. Simple singularities and symplectic fillings. *J. Diff. Geom.* 69(1):1–42, 2005.
- [24] H. Ohta and K. Ono. Symplectic 4-manifolds containing singular rational curves with (2,3)-cusp. *Seminars & Congres* 10:233–241, 2005.
- [25] H. Ohta and K. Ono. Symplectic fillings of the link of simple elliptic singularities. *J. Reine Angew. Math.* 565:183–205, 2003.
- [26] H. Ohta and K. Ono. An inequality for symplectic fillings of the link of a hypersurface K3 singularity. *Algebraic topology: old and new*, 93–100, Banach Center Publ., 85, Polish Acad. Sci. Inst. Math., Warsaw, 2009.
- [27] J. Pascaleff. On the symplectic cohomology of log Calabi-Yau surfaces. arXiv:1304.5298, 2013.
- [28] M. Pinsonnault. Maximal compact tori in the Hamiltonian group of 4-dimensional symplectic manifolds. *J. Mod. Dyn.* 2(3):431–455, 2008.
- [29] D. Salamon. Uniqueness of symplectic structures. *Acta Math. Vietnam.* 38(1):123–144, 2013.
- [30] C.H. Taubes. The Seiberg-Witten and Gromov invariants. *Math. Res. Lett.* 2(2):221–238, 1995.
- [31] M. Usher. Kodaira dimension and symplectic sums. *Comment. Math. Helv.* 1(84):57–85, 2009.