From the Borel–Serre Compactification to Curve Complex of Surfaces

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Abstract. In this paper, we describe the interaction and similarity between locally symmetric spaces and moduli spaces of Riemann surfaces, through the example of how the Borel–Serre compactification of locally symmetric spaces led to the curve complex of surfaces, which is a fundamental object in low dimensional topology and geometric group theory.

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1. Introduction

The modular group $\mathrm{SL}(2,\mathbb{Z})$ is a basic group, and its action on the upper-half plane $\mathbf{H}^2$ has several important generalizations. For example, one generalization consists of arithmetic groups Γ and their action on symmetric spaces X of noncompact type, and another generalization consists of mapping class groups Mod_g of compact surfaces and their actions on Teichmüller spaces $\mathcal{T}_g$. In the former case, quotient spaces are locally symmetric spaces, and in the latter case, they are the moduli spaces $\mathcal{M}_g = \mathrm{Mod}_g\backslash\mathcal{T}_g$ of compact Riemann surfaces, which were defined by Riemann in 1857.

Like $\mathrm{SL}(2,\mathbb{Z})$, many arithmetic groups Γ are not cocompact, i.e., the locally symmetric spaces $\Gamma\backslash X$ are not compact. One natural problem is to compactify $\Gamma\backslash X$ which are suitable for various applications. In this paper, we discuss the celebrated Borel–Serre compactification [BS] of $\Gamma\backslash X$ which is a basic tool to understand the topology of Γ and $\Gamma\backslash X$, and explain how its generalization to $\mathcal{M}_g$ led to the introduction of the notion of curve complex of compact Riemann surfaces by Harvey [Har].

2. Borel–Serre Compactification of Locally Symmetric Spaces

Borel and Serre wrote joint 10 papers, including a write-up of Grothendieck's work on the Grothendieck–Riemann–Roch theorem: *Le théorème de Riemann-Roch*. Bull. Soc. Math. France 86 (1958) 97–136.

Among these papers, the one with by far the largest citation number, 205, is the paper [BS]: *Corners and arithmetic groups*.

To satisfy the possible curiosity of the reader, the citation numbers of the other 9 papers are

$$46+4+0+68+68+42+21+4+3=256,$$

where the paper with the 0 citation number is an announcement of the paper [BS].[1]

One natural question is: *What is contained in this paper?* Another question is: *Why is it so special and influential?*

The purpose of this paper is to answer these questions.

2.1 Statements

The main results of [BS] include the following:

1. The construction of the Borel–Serre compactification $\overline{\Gamma\backslash X}^{BS}$ of locally symmetric spaces $\Gamma\backslash X$.
2. A description of the topology of $\partial\overline{\Gamma\backslash X}^{BS}$ in terms of the Tits building of the linear algebraic group corresponding to the space $\Gamma\backslash X$.
3. Application of the compactification $\overline{\Gamma\backslash X}^{BS}$ to the duality property and determination of the cohomology dimension of arithmetic groups Γ.

We will also discuss the following generalizations:

1. The Borel–Serre compactification of moduli space $\mathcal{M}_g$ of Riemann surfaces.
2. The curve complex defined by Harvey as an analogue of Tits buildings of linear semisimple algebraic groups.
3. Duality property and determination of cohomological dimension of mapping class group Mod_g by Harer.

Specifically, the result of Borel and Serre [BS] can be stated as:

Theorem 2.1. *Let Γ be any arithmetic subgroup of a semisimple Lie group G. Assume the $\mathbb{Q}$-rank r of Γ is positive, and hence $\Gamma\backslash X$ is noncompact. Then Γ is a* **virtual duality group** *(or a generalized Poincaré duality group) of dimension $\dim X - r$, but is not a* **virtual Poincaré duality group**.

Corollary 2.2. *The* **virtual** *cohomological dimension of Γ, $\mathrm{vcd}(\Gamma)$, is equal to $\dim X - r$.*

The Borel–Serre compactification $\overline{\Gamma\backslash X}^{BS}$ was used to prove the duality property of and to compute the virtual cohomological dimension of arithmetic groups Γ. This was the first proof and is still the only proof up to now.

[1] These citation numbers were copied from MathSciNet in the middle of April, 2017.

2.2 Motivations and Definitions

In the following we will describe motivations of constructing the Borel–Serre compactification $\overline{\Gamma\backslash X}^{BS}$ and explain all the terminologies in the statements in the previous subsection.

One of the most basic and important infinite discrete groups is $\mathbb{Z}$, which is a discrete subgroup of the Lie group $\mathbb{R}$. It admits several generalizations. The obvious ones are $\mathbb{Z}^n \subset \mathbb{R}^n$, and more generally lattices $\Lambda \subset \mathbb{R}^n$. These are abelian groups.

Another class of generalizations consists of nonabelian groups, for example, $\mathrm{SL}(2,\mathbb{Z})$, which a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$, and $\mathrm{SL}(n,\mathbb{Z}) \subset \mathrm{SL}(n,\mathbb{R})$, $n \geq 2$, which form a natural family containing $\mathrm{SL}(2,\mathbb{Z})$.

The group $\mathrm{SL}(2,\mathbb{Z})$ is an arithmetic subgroup of $\mathrm{SL}(2,\mathbb{R})$. More generally, any subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$ commensurable with $\mathrm{SL}(2,\mathbb{Z})$ is also called an **arithmetic subgroup**. We recall that two subgroups Γ_1, Γ_2 of a common group are called *commensurable* if the intersection $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2.

Similarly, $\mathrm{SL}(n,\mathbb{Z})$ and its subgroups of finite index are also arithmetic subgroups of $\mathrm{SL}(n,\mathbb{R})$. Important examples include congruence subgroups.

The reason for calling them arithmetic subgroups is that arithmetic deals with $\mathbb{Z} \subset \mathbb{R}$. The general definition of arithmetic groups is as follows.

Definition 2.3. *Let $\mathbf{G} \subset \mathrm{GL}(n,\mathbb{C})$ be a linear algebraic group defined over $\mathbb{Q}$. Then $G = \mathbf{G}(\mathbb{R})$ is a Lie group with finitely many connected components. Let $\mathbf{G}(\mathbb{Z}) = \mathbf{G}(\mathbb{Q}) \cap \mathrm{GL}(n,\mathbb{Z})$. Then any subgroup Γ of $\mathbf{G}(\mathbb{Q})$ which is commensurable with $\mathbf{G}(\mathbb{Z})$ is called an arithmetic subgroup of $\mathbf{G}$.*

Some people also call any subgroup of G which is commensurable with $\mathbf{G}(\mathbb{Z})$ an arithmetic subgroup. Any arithmetic subgroup Γ is a discrete subgroup of the Lie G, since $\mathbb{Z}$ is a discrete subgroup of $\mathbb{R}$. In some sense, arithmetic subgroups are essentially the only way to produce cofinite volume subgroups, i.e., lattices, of semisimple Lie groups G, except for $G = \mathrm{SL}(2,\mathbb{R})$. For example, the Poincaré polygon theorem allows to produce many discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$, but there is no analogue of it for other symmetric spaces.

This is the reason for all these questions and results on rigidity of lattices of semisimple Lie groups by Selberg, who had a motto that whatever cannot be constructed should not exist, and by Piateski–Shiparo, Mostow, Margulis.

Arithmetic subgroups of Lie groups enjoy many desirable properties, hence providing important examples of infinite discrete groups in geometric group theory.

These properties of arithmetic subgroups Γ come from, or proofs for them depend on, their actions on symmetric spaces, which also give rise to arithmetic locally symmetric spaces $\Gamma\backslash X$. Locally symmetric spaces are interesting and special spaces, and they play a basic role in the theory of automorphic forms and automorphic representations. Many moduli spaces can be identified with locally symmetric spaces.

We will concentrate on the case when G is a semisimple Lie group. Let Γ be an arithmetic subgroup of Γ. Then it enjoys the following **finiteness properties**:

1. Γ is finitely generated and finitely presented.
2. Γ has finitely many conjugacy classes of finite subgroups.
3. Γ has finite cohomological dimension.
4. Γ admits finite index subgroups Γ' which are torsion-free.

2.3 Group Cohomology and Cohomological Properties

For a topological space M, its basic topological invariants include cohomological groups $H^i(M)$ and homological group $H_i(M)$. They can have coefficients in $\mathbb{Z}$, $\mathbb{R}$, or other rings, fields, and local systems.

For a group Γ, it also has cohomological and homological groups. It can be defined as follows. There is a CW-complex $B\Gamma$ such that

$$\pi_1(B\Gamma) = \Gamma, \quad \text{for } i \geq 2, \quad \pi_2(B\Gamma) = 1.$$

The space $B\Gamma$ is called the **classifying space** of Γ and is unique up to homotopy equivalence. Using the classifying space $B\Gamma$, we can define cohomology and homology groups of the group Γ as follows:

$$H^i(\Gamma) = H^i(B\Gamma), \quad H_i(\Gamma) = H_i(B\Gamma).$$

More generally, if A is a $\mathbb{Z}\Gamma$-module, then we can define cohomology and homology groups with coefficient A,

$$H^i(\Gamma, A), \quad H_i(\Gamma, A).$$

Definition 2.4. *The cohomology dimension of* Γ *is*

$$\mathrm{cd}(\Gamma) = \sup\{i \mid H^i(\Gamma, A) \neq 0, \text{ for some } \Gamma-\text{module } A\}.$$

Proposition 2.5. *If* Γ *contains nontrivial elements of finite order, i.e., torsion elements, then* $\mathrm{cd}(\Gamma) = +\infty$.

Many natural groups such as $\mathrm{SL}(n, \mathbb{Z})$, $\mathrm{Sp}(2g, \mathbb{Z})$ and the mapping class groups Mod_g are not torsion-free.

If Γ admits torsion-free subgroups Γ' of finite index, we can define **virtual cohomology dimension** by

$$\mathrm{vcd}(\Gamma) = \mathrm{cd}(\Gamma').$$

It is known that this is well-defined, i.e., independent of the choice of Γ'. As mentioned before, arithmetic subgroups always admit torsion-free finite subgroups.

For a general group Γ, there is a Milnor construction of the classifying space $B\Gamma$ which is always infinite dimensional. On the other hand, for many purposes, such as computing explicitly the cohomology and homology groups of Γ, it is desirable to have small and explicit models of $B\Gamma$. One important point about arithmetic groups Γ is that they admit explicit finite dimensional models of $B\Gamma$.

Recall that G is a semisimple Lie group. Let $K \subset G$ be a maximal compact subgroup. Then $X = G/K$ with an invariant Riemannian metric is a symmetric space of noncompact type, hence it is diffeomorphic to $\mathbb{R}^n$, $n = \dim X$. Since Γ is a discrete subgroup of G, Γ acts properly and isometrically on X.

If Γ is torsion-free, then Γ acts fixed point freely on X, and X is the universal covering space of $\Gamma\backslash X$, and hence $\Gamma\backslash X$ is a *finite dimensional model of* $B\Gamma$.

Note that it follows from the definition $H^i(\Gamma) = H^i(B\Gamma)$ that

$$\mathrm{cd}(\Gamma) \leq \dim B\Gamma,$$

for every model of $B\Gamma$. Consequently, we obtain

Proposition 2.6. *For any torsion-free discrete subgroup* Γ *of* G*,* $\Gamma\backslash X$ *is a finite dimensional model of* $B\Gamma$*, and* $\mathrm{cd}(\Gamma) \leq \dim X$.

On the other hand, if Γ is not torsion-free, then $\mathrm{cd}(\Gamma) = +\infty$, and consequently $B\Gamma$ does not admit any finite dimensional model. Therefore, $\Gamma\backslash X$ is not a model of $B\Gamma$. Since $\Gamma\backslash X$ is a natural space associated with Γ, one natural question is

Question 2.7. *Assume that* Γ *is not torsion-free, what is this space* $\Gamma\backslash X$ *as far as* Γ *is concerned?*

Given any $B\Gamma$, its universal covering space $E\Gamma = \widetilde{B\Gamma}$ has the properties:

1. Γ acts properly and fixed point freely on $E\Gamma$.
2. $E\Gamma$ is contractible.

The space $E\Gamma$ is the **universal space** for fixed-point-free and proper actions of Γ. Consequently, when $\Gamma \subset G$ is torsion-free, the symmetric space X is a finite dimensional model of $E\Gamma$.

On the other hand, when Γ contains nontrivial elements of finite order, it does not act fixed point freely on X. The Cartan fixed point theorem says that any finite isometry group acting on a complete Hadamard Riemann manifold, i.e., a simply connected and nonpositively curved manifold, has a fixed point, which can be taken as the center of gravity of any finite orbit of the group.

But the action of a non-torsion free group Γ on the symmetric space X satisfies the following conditions:

1. Γ acts properly on X.

2. For any fixed subgroup $F \subset \Gamma$, the fixed point set X^F is nonempty and contractible, since it is a totally geodesic submanifold.

Consequently, X is a finite dimensional model of $\underline{E}\Gamma$, the *universal space for proper actions of* Γ, which is characterized by the two conditions above.

2.4 Locally Symmetric Spaces and Cohomology of Arithmetic Groups

As mentioned before, arithmetic subgroups Γ lead to the locally symmetric space $\Gamma\backslash X$. Locally symmetric spaces are rather special and enjoy several finiteness and rigidity properties:

1. With respect to the measure induced from the invariant Riemannian metric on X, $\Gamma\backslash X$ has finite volume.
2. The locally symmetric space $\Gamma\backslash X$ is compact if and only if the $\mathbb{Q}$-rank of $\mathbf{G}$ is equal to 0. For example, for $\mathbf{G} = \mathrm{SL}(n)$, $\Gamma = \mathrm{SL}(n,\mathbb{Z})$, the $\mathbb{Q}$-rank is equal to $n-1 > 0$. This is consistent with that fact that $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$ is noncompact.
3. $\Gamma\backslash X$ is the interior of a compact manifold with corners (or with boundary if the $\mathbb{Q}$-rank is equal to 1), and hence has finite topology.
4. $\Gamma\backslash X$ has important rigidity properties: the Mostow strong rigidity and super-rigidity of Margulis.

The above description shows that locally symmetric spaces $\Gamma\backslash X$ are very special Riemannian manifolds (orbifolds) in geometric analysis. The spectral theory of $\Gamma\backslash X$ with respect to the Laplacian operator of the invariant Riemannian metric is closely related to the spectral theory of automorphic forms. Hence they are also basic spaces in number theory. One point of this paper is to show that they are special spaces in topology, in particular as classifying spaces.

Assume that Γ is a *torsion-free* arithmetic subgroup. As discussed earlier, $\Gamma\backslash X$ is a finite dimensional model of $B\Gamma$ and it can be used to study cohomological properties of Γ. For such a purpose, we often need compact models of $B\Gamma$, or more precisely finite CW-complexes, in order to prove finiteness properties of Γ, in particular finite cohomological properties.

More precisely, we have the following consequences of having a good compact model of $B\Gamma$:

1. Γ is finitely generated and finitely presented. It also enjoys other cohomological finiteness properties.
2. Suppose $B\Gamma$ is a compact oriented manifold without boundary. Then the cohomology groups of $B\Gamma$ satisfy the Poincaré duality property:

$$H^i(B\Gamma,\mathbb{Z}) \times H^{n-i}(B\Gamma,\mathbb{Z}) \to H^n(B\Gamma,\mathbb{Z}) \cong \mathbb{Z}.$$

or equivalently,

$$H_i(B\Gamma,\mathbb{Z}) \cong H^{n-i}(B\Gamma,\mathbb{Z}).$$

Consequently, the cohomology groups of Γ also satisfy the Poincaré duality property.

We note that a group Γ is called a *Poincaré duality group* of dimension n if for every $i = 0,\cdots,n$,

$$H_i(\Gamma,A) \cong H^{n-i}(\Gamma,A),$$

where A is any $\mathbb{Z}\Gamma$-module.

More generally, we have

Definition 2.8. *A group Γ is called a* **duality group** *(or a* **generalized Poincaré duality group***) of dimension n if there exists a $\mathbb{Z}\Gamma$-module D such that for every $i = 0,\cdots,n$, and every Γ-module A,*

$$H_i(\Gamma,A) \cong H^{n-i}(\Gamma,A\otimes D).$$

The module D is called the **dualizing module** *of Γ.*

One basic result in the theory of cohomology of groups (see [Br, Theorem 10.1, p. 220]) is the following result.

Proposition 2.9. *If Γ is a duality group of dimension n, then $\mathrm{cd}(\Gamma) = n$.*

This is an effective way to compute $\mathrm{cd}(\Gamma)$, though groups with $\mathrm{cd}(\Gamma) < +\infty$ are often not duality groups.

The above discussions, especially those on properties resulting from compact models of $B\Gamma$, show that there is an issue when $\Gamma\backslash X$ is noncompact, which is the case with many basic examples of arithmetic subgroups such as $\Gamma \subset \mathrm{SL}(n,\mathbb{Z}), \mathrm{Sp}(2g,\mathbb{Z})$. This suggests the problem of constructing a compactification of $\Gamma\backslash X$ which gives a compact model of $B\Gamma$.

Now back to a torsion-free arithmetic group $\Gamma \subset G$. If $\Gamma\backslash X$ is compact, then Γ is a Poincaré duality group of dimension $\dim X$.

A natural question is what if $\Gamma\backslash X$ is noncompact? The above setup may suggest that in this case, Γ is a duality group, but not a Poincaré duality group. It turns out that the answer is positive and the proof depends crucially on a suitable compactification of $\Gamma\backslash X$. One of the main results of [BS] can be stated as

Theorem 2.10. *Let $\Gamma \subset G$ be a torsion-free arithmetic subgroup of a semisimple Lie group. Assume the $\mathbb{Q}$-rank r of $\mathbf{G}$ is positive, and hence $\Gamma\backslash X$ is noncompact. Then Γ is a duality group of dimension $\dim X - r$, but is not a Poincaré duality group.*

Corollary 2.11. *Under the same assumption on Γ as in Theorem 2.10, the cohomological dimension of Γ is equal to $\dim X - r$.*

Example 2.12. When $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ is a torsion-free subgroup of finite index, the $\mathbb{Q}$-rank r is equal to 1. Since $X = \mathbb{H}^2$, $\dim X - r = 2 - 1 = 1$. Therefore, $\mathrm{cd}(\Gamma) = 1$.

This conclusion can also be seen from the fact that Γ is a free group in this case. Note that the fundamental group of any noncompact surface is a free group.

When Γ is not torsion-free, it admits finite index torsion-free subgroups Γ'. In this case, we say Γ is a *virtual duality group* if Γ' is a duality group. Then the more general result of Borel–Serre [BS] can be stated as:

Theorem 2.13. *Let $\Gamma \subset G$ be any arithmetic subgroup of a semisimple Lie group G. Assume the $\mathbb{Q}$-rank r of* **G** *is positive. Then Γ is a* **virtual duality group** *of dimension $\dim X - r$, but is not a* **virtual** *Poincaré duality group. Consequently, the* **virtual** *cohomological dimension of Γ is equal to $\dim X - r$.*

2.5 Borel–Serre Compactification of Locally Symmetric Spaces

To prove the above results on cohomological properties of arithmetic groups, Borel and Serre [BS] defined a compactification $\overline{\Gamma\backslash X}^{BS}$ of $\Gamma\backslash X$, which is a compact model of $B\Gamma$ with the following properties:

1. $\overline{\Gamma\backslash X}^{BS}$ is a compact manifold with corners and hence is a finite CW-complex by triangulation.
2. The inclusion $\Gamma\backslash X \hookrightarrow \overline{\Gamma\backslash X}^{BS}$ is an homotopy equivalence, and hence $\overline{\Gamma\backslash X}^{BS}$ is a compact model of $B\Gamma$.
3. The topology of the boundary $\partial\overline{\Gamma\backslash X}^{BS}$ can be described explicitly and is homotopy equivalent to a bouquet of spheres of dimension $r-1$, where r is the $\mathbb{Q}$-rank of $\Gamma\backslash X$.

The third condition will be explained more precisely later, and it leads to the duality property of Γ and an explicit determination of its dualizing module.

Borel and Serre constructed the compactification $\overline{\Gamma\backslash X}^{BS}$ in [BS] in the following steps:

1. Construct a partial compactification (or bordification) $\overline{X}^{BS}$ which is a real analytic manifold with corners whose boundary faces $e_{\mathbf{P}}$ are parametrized by proper $\mathbb{Q}$-parabolic subgroups **P** of **G**, and whose interior is equal to X.
2. The action of Γ on X extends to a **proper action** on $\overline{X}^{BS}$.
3. The quotient $\Gamma\backslash\overline{X}^{BS}$ gives the compactification $\overline{\Gamma\backslash X}^{BS}$.

When Γ is torsion-free, Γ acts fixed point freely on $\overline{X}^{BS}$, and $\overline{X}^{BS}$ is a **cocompact** model of $E\Gamma$.

Example 2.14. Before we explain the general construction, we consider a special example when $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ is a torsion-free subgroup of finite index, and $X = \mathbb{H}^2$.

In this case, $\Gamma\backslash\mathbb{H}^2$ is noncompact and has finite area with respect to the hyperbolic metric. Consequently, $\Gamma\backslash\mathbb{H}^2$ is the union of a compact core and finitely many cusp neighborhoods.

The most obvious compactification of $\Gamma\backslash\mathbb{H}^2$ is obtained by adding one point for each cusp neighborhood to get a compact Riemann surface $\overline{\Gamma\backslash\mathbb{H}^2}^*$.

But there is a problem: the inclusion

$$\Gamma\backslash\mathbb{H}^2 \hookrightarrow \overline{\Gamma\backslash\mathbb{H}^2}^*$$

is not a homotopy equivalence.

Note that each cusp neighborhood of $\Gamma\backslash\mathbb{H}^2$ is topologically an *open cylinder*. To obtain a compactification of $\Gamma\backslash\mathbb{H}^2$ without changing its topology, we should add a *boundary circle* at infinity.

By adding a boundary circle to every cusp of $\Gamma\backslash\mathbb{H}^2$, we obtain the Borel–Serre compactification $\overline{\Gamma\backslash\mathbb{H}^2}^{BS}$. It satisfies all the desired properties.

The above construction of $\overline{\Gamma\backslash\mathbb{H}^2}^{BS}$ is explicit but could not be generalized directly to other locally symmetric spaces $\Gamma\backslash X$, since it is not group-theoretical enough.

Now we follow a procedure which can be generalized. The upper triangular subgroup

$$P_\infty = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \neq 0, b \in \mathbb{R} \right\}$$

is a $\mathbb{Q}$-*parabolic subgroup* of $\mathrm{SL}(2,\mathbb{R})$. It is the stabilizer of $i\infty$, a boundary point in $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$, with respected to the action of $\mathrm{SL}(2,\mathbb{R})$ on the compactification

$$\overline{\mathbb{H}^2} = \mathbb{H}^2 \cup \partial\mathbb{H}^2 = \mathbb{H}^2 \cup \mathbb{R} \cup \{\infty\}.$$

Other proper parabolic subgroups defined over $\mathbb{Q}$ are stabilizers of the rational boundary points in $\mathbb{Q} \subset \partial\mathbb{H}^2$. They are conjugates of P_∞ by elements of $\mathrm{SL}(2,\mathbb{Q})$, since $\mathrm{SL}(2,\mathbb{Q})$ acts transitively on the rational boundary points $\mathbb{Q} \cup \{\infty\}$.

The parabolic subgroup P_∞ acts transitively on $\mathbb{H}^2$, and the a, b parameters in elements of P_∞ give the x, y-coordinates of $\mathbb{H}^2$:

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot i = a^2 i + ab.$$

The unipotent subgroup of P_∞ is

$$N_{P_\infty} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{R} \right\},$$

and its orbits give the horocircles of $i\infty$ (or horizontal lines), which are parallel translates of x-coordinates in $\mathbb{C}$.

The orbit of the diagonal subgroup

$$A_{P_\infty} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a > 0 \right\}$$

through the point i gives the y-coordinate.

General Borel–Serre Compactification

For a general symmetric space $X = G/K$, every $\mathbb{Q}$-parabolic subgroup P of G admits a **Langlands decomposition**:

$$P = N_P A_P M_P,$$

where N_P is the unipotent radical of P, A_P is the split component, and M_P is a reductive subgroup. The Langlands decomposition of P and the transitive action of P on X gives a **horospherical decomposition** of X,

$$X = P \cdot x_0 \cong N_P \times A_P \times X_P,$$

where $x_0 = K \in X = G/K$, and $X_P = M_P/M_P \cap K = M_P \cdot x_0$.

We note that the space X_P is a symmetric space of nonpositive curvature, hence is contractible, and N_P is also contractible. This generalizes the xy-coordinate decomposition of $\mathbf{H}^2$ and is hence called the horospherical decomposition of X.

The **boundary face** or **component** e_P of the parabolic subgroup P is

$$e_P = N_P \times X_P.$$

It is added at the infinity of X in the direction corresponding to P (or rather in the direction of the positive chamber corresponding to P), or at the infinity of A_P. Then we obtain the *Borel–Serre partial compactification*

$$\overline{X}^{BS} = X \cup \coprod_P e_P = X \cup \coprod_P N_P \times X_P.$$

Proposition 2.15. *The Borel–Serre partial compactification $\overline{X}^{BS}$ is a real analytic manifold with corners. The codimension of the corner is equal to the $\mathbb{Q}$-rank of $\mathbf{G}$, and Γ acts properly and real analytically on $\overline{X}^{BS}$. The quotient $\overline{\Gamma\backslash X}^{BS}$ is a compact real analytic manifold with corners, with the interior equal to $\Gamma\backslash X$.*

Since the inclusion $X \hookrightarrow \overline{X}^{BS}$ is a homotopy equivalence, the same is true for the inclusion of the quotients

$$\Gamma\backslash X \hookrightarrow \overline{\Gamma\backslash X}^{BS}.$$

Example 2.16. Now we explain the Borel–Serre partial compactification $\overline{\mathbf{H}^2}^{BS}$ in terms of the above group theoretical formulation. The group P_∞ admits the Langlands decomposition

$$P_\infty = N_{P_\infty} A_{P_\infty} M_{P_\infty} \cong N_{P_\infty} \times A_{P_\infty},$$

where

$$M_{P_\infty} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a = 1, -1 \right\}.$$

The x, y-coordinates of $\mathbb{H}^2$ can be described by the above decomposition

$$\mathbb{H}^2 = P_\infty \cdot i \cong N_{P_\infty} \times A_{P_\infty}.$$

Using this decomposition, we can add a copy of $N_{P_\infty} \cong \mathbb{R}$ at the infinity of A_{P_∞}. This is the boundary face e_{P_∞} associated with the parabolic subgroup P_∞.

The Borel–Serre partial compactification $\overline{\mathbb{H}^2}^{BS}$ is obtained by adding one copy of $\mathbb{R}$ to every rational boundary point of $\mathbb{H}^2$. We can also imagine to blow up every *rational* boundary point of $\mathbb{H}^2$, or of its compactification $\overline{\mathbb{H}^2}$, into a copy of $\mathbb{R}$. It is a real analytic manifold with boundary, and the quotient $\Gamma\backslash\overline{\mathbb{H}^2}^{BS}$ is also a compact real analytic manifold with boundary, where each boundary component is a circle. It is the Borel–Serre compactification of $\Gamma\backslash\mathbb{H}^2$.

Topology of the Boundary $\partial\overline{X}^{BS}$

For each parabolic subgroup P, the boundary component e_P is contractible since both N_P and X_P are contractible. Therefore, the topology of the boundary $\partial\overline{X}^{BS}$ is controlled by the inclusion relations between these boundary components.

It can be shown that when P is bigger, its boundary component e_P is also bigger in the sense: if P' is a subgroup of P, then $e_{P'}$ is in the closure of e_P. This is most clear in the case of $\mathbf{G} = \mathrm{SL}(2) \times \mathrm{SL}(2)$.

The above description of the boundary $\partial\overline{X}^{BS}$ shows that its topology is related to the Tits building of $\mathbf{G}$.

Definition 2.17. *Given a linear semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$, its Tits building $\Delta(\mathbf{G})$ of $\mathbf{G}$ is an infinite simplicial complex whose simplices are parametrized by proper $\mathbb{Q}$-parabolic subgroups of $\mathbf{G}$ satisfying the following conditions:*

1. *Denote the simplex for a $\mathbb{Q}$-parabolic subgroup P by σ_P. Then for two parabolic subgroups P_1, P_2, σ_{P_1} is contained in the closure of σ_{P_2} as a face if and only if P_1 properly contains P_2.*
2. *Maximal proper $\mathbb{Q}$-parabolic subgroups of $\mathbf{G}$ corresponds to simplices of dimension 0.*
3. *For every simplex σ_P, its vertices correspond to the maximal proper $\mathbb{Q}$-parabolic subgroups which contain P.*

The **topology of Tits building** $\Delta(\mathbf{G})$ can be described by the Solomon–Tits Theorem as follows.

Proposition 2.18. *Let r be the $\mathbb{Q}$-rank of $\mathbf{G}$. Then $\Delta(\mathbf{G})$ is homotopy equivalent to an infinite bouquet of spheres of dimension $r-1$.*

When $r=1$, each sphere of dimension 0 consists of two points $1,-1$ in $\mathbb{R}^1$. Every proper $\mathbb{Q}$-parabolic subgroup is both maximal and minimal. So $\Delta(\mathbf{G})$ is just a countable union of points. This happens when $\mathbf{G}=\mathrm{SL}(2)$.

Note that the inclusion relation for the boundary components e_P is the opposite of the inclusion relation for the simplices σ_P of the Tits building $\Delta(\mathbf{G})$. Both e_P and σ_P are cells. Therefore, we have

Proposition 2.19. *The Borel–Serre boundary $\partial\overline{X}^{BS}$ and the Tits building $\Delta(\mathbf{G})$ are cell-complexes dual to each other, and hence they are homotopy equivalent. Consequently, $\partial\overline{X}^{BS}$ is homotopy equivalent to a bouquet of infinitely many spheres of dimension $r-1$.*

Once we have the above properties of the Borel–Serre compactification $\overline{X}^{BS}$, it follows from the general argument using the Poincaré–Lefschetz duality for manifolds with boundary (or with corners) that a torsion-free arithmetic group Γ is a duality group of dimension $\dim X - r$, the dualizing module D is equal to $H_{r-1}(\partial\overline{X}^{BS})=\mathbb{Z}^\infty$.

We outline the arguments. For more details, see [BS, Theorem 11.4.1, Theorem 11.4.2], [Iv, Theorem 6.1, Corollary 6.1], [Br, pp. 209–211], and [IJ, §3]. To compute $\mathrm{cd}(\Gamma)$, we use the equality

$$\mathrm{cd}(\Gamma)=\max\{n \mid H_c^n(\overline{X}^{BS},\mathbb{Z})\neq 0\},$$

where $H_c^n(\overline{X}^{BS},\mathbb{Z})$ denotes the cohomology with compact support. Denote the dimension of X by d. Then the Poincaré–Lefschetz duality theorem implies

$$H_c^n(\overline{X}^{BS},\mathbb{Z})=H_{d-n}(\overline{X}^{BS},\partial\overline{X}^{BS},\mathbb{Z}).$$

Since $\overline{X}^{BS}$ is contractible, the long exact sequence gives

$$H_{d-n}(\overline{X}^{BS},\partial\overline{X}^{BS},\mathbb{Z})\cong H_{d-n-1}(\partial\overline{X}^{BS},\mathbb{Z}).$$

Since $\partial\overline{X}^{BS}$ is homotopy equivalent to a bouquet of infinitely many spheres of dimension $r-1$, we conclude that for $n>0$,

$$H_c^n(\overline{X}^{BS},\mathbb{Z})\neq 0$$

if and only if $n=d-r$, and

$$H_c^{d-r}(\overline{X}^{BS},\mathbb{Z})\cong\mathbb{Z}^\infty.$$

This shows that

$$\mathrm{cd}(\Gamma)=\dim X - r.$$

Furthermore, by [Br, Proposition 7.5, p. 209], it implies Γ is a duality group with the dualizing module equal to $H^{d-r}(\Gamma,\mathbb{Z}[\Gamma])$, which, by [BS, Theorem 11.4.2], is equal to

$$H_c^{d-r}(\overline{X}^{BS},\mathbb{Z})\cong\mathbb{Z}^\infty.$$

Therefore Γ is a duality group but not a Poincaré duality group.

The Case When Γ Is Not Torsion-Free

We mentioned before that when Γ is torsion-free, $\overline{X}^{BS}$ is a model of $E\Gamma$ with a *compact quotient*, since $\overline{\Gamma\backslash X}^{BS}$ is a model of $B\Gamma$.

We also mentioned that when Γ is not torsion-free, X is a model of $\underline{E}\Gamma$. When $\Gamma\backslash X$ is not compact, X is not a *cofinite* (or *cocompact*) model of $\underline{E}\Gamma$. One question is

Question 2.20. *Is $\overline{X}^{BS}$ is a* **cofinite** *model of $\underline{E}\Gamma$ when Γ is not torsion-free?*

The answer is positive and was given in [J1, Theorem 3.2]. This is important in the **integral Novikov conjectures** in geometric topology [J1, Theorem 3.1].

3. Analogy Between Arithmetic Groups and Mapping Class Groups

In the above discussion of the Borel–Serre compactification of locally symmetric spaces, we used the example of the group $\mathrm{SL}(2,\mathbb{Z})$ acting on the upper halfplane $\mathbf{H}^2$. We want to interpret it as a moduli space in order to obtain another generalization.

3.1 Moduli Space and Teichmuller Space

For every point $\tau\in\mathbf{H}^2$, there is a lattice $\mathbb{Z}+\tau\mathbb{Z}$ in $\mathbb{C}$, which gives a compact Riemann surface of genus 1,

$$\mathbb{Z}+\tau\mathbb{Z}\backslash\mathbb{C}.$$

It follows from the uniformization theorem for Riemann surfaces that every compact Riemann surface is of this form, and the locally symmetric space $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$ can be identified with the moduli space $\mathcal{M}_1$ of compact Riemann surfaces of genus 1.

From this perspective, it is clear that one generalization of $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$ consists of the moduli spaces $\mathcal{M}_g$ of compact Riemann surfaces of genus $g\geq 1$.

The space $\mathcal{M}_g$ was introduced by Riemann in 1857. Many people studied it after Riemann, including Klein, Poincaré, Hurwitz, Severi, Fricke. But Teichmüller was the first one who made substantial progress towards understanding its topology and complex analytic structure.

Briefly, after defining $\mathcal{M}_g$, Riemann counted the number of effective complex parameters needed to describe points of $\mathcal{M}_g$ and came up with the answer $3g-3$ when $g\geq 2$. The so-called **Riemann's moduli problem** is to make $\mathcal{M}_g$ into a complex space such that its complex dimension is equal to $3g-3$. For more detailed discussions on its complicated history, see the papers [J2] [J3].

Teichmüller realized that nontrivial automorphisms of Riemann surfaces will produce singularities of $\mathcal{M}_g$, and he introduced the notion of marked

Riemann surfaces to kill automorphisms and defined the moduli space of marked Riemann surfaces, which is called *Teichmüller space* $\mathcal{T}_g$. He defined a natural metric, hence a topology, on $\mathcal{T}_g$, and showed that $\mathcal{T}_g$ is homeomorphic to $\mathbb{R}^{6g-6}$. He also announced a natural complex structure on $\mathcal{T}_g$ so that it becomes a complex manifold. In fact, he was the first person to define the notion of fine moduli spaces, and $\mathcal{T}_g$ is a fine moduli space of marked Riemann surfaces.

To go back to the moduli space $\mathcal{M}_g$ of unmarked Riemann surfaces, we need the mapping class group Mod_g of genus g, which is defined as

$$\mathrm{Mod}_g = \mathrm{Diff}^+(S_g)/\mathrm{Diff}^0(S_g),$$

where S_g is a compact oriented surface of genus g.

The group Mod_g acts on $\mathcal{T}_g$ by changing the markings of Riemann surfaces, and we have the following identification:

$$\mathcal{M}_g = \mathrm{Mod}_g \backslash \mathcal{T}_g.$$

When $g=1$, $\mathcal{T}_1 = \mathbb{H}^2$, and $\mathrm{Mod}_1 = \mathrm{SL}(2,\mathbb{Z})$.

It can be shown that Mod_g acts properly and holomorphically on $\mathcal{T}_g$. Consequently, $\mathcal{M}_g$ is a complex orbifold of dimension $3g-3$. This solves the moduli problem of Riemann mentioned earlier.

The above discussion suggests a strong analogy between the following three pairs:

1. $\mathcal{T}_g \Longleftrightarrow X = G/K$.
2. $\mathrm{Mod}_g \Longleftrightarrow \Gamma$.
3. $\mathcal{M}_g \Longleftrightarrow \Gamma\backslash X$.

There are several reasons which make such an analogy interesting. The most obvious one is that they have a common root in the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbf{H}^2$. On the other hand, there are also some subtle differences. For example, when $g \geq 2$, $\mathcal{T}_g$ with the natural Teichmüller metric is never a homogeneous space, even though it is contractible. This follows from the famous theorem of Royden on the automorphism group of $\mathcal{T}_g$.

The above analogy has motivated many questions about Mod_g, $\mathcal{T}_g$ and $\mathcal{M}_g$. Though formulations of questions look similar, methods to answer them are often very different. Maybe this makes the analogy more interesting: the beauty and attractiveness lies in the half way. See [J5] for other aspects of the analogy between these two classes of objects.

There are many questions one can ask. Here are several related to the earlier discussions of this paper.

Question 3.1.
1. *Can we compute its cohomological dimension of* Mod_g*?*
2. *Can we prove that* Mod_g *is a virtual duality group? (Note that* Mod_g *is not torsion-free.)*
3. *Is there a Borel–Serre compactification of* $\mathcal{M}_g$ *which arises from a partial Borel–Serre compactification of* $\mathcal{T}_g$*?*
4. *Is there an analogue of the Tits building which describes the boundary of the partial Borel–Serre compactification of* $\mathcal{T}_g$*?*

It is satisfying and also nontrivial that the answers to all these all turn out to be yes. We will explain them in the rest of this section.

3.2 Compactifications of Moduli Spaces and Curve Complexes

We start with compactifications. First we note that $\mathcal{M}_g$ is noncompact, since Riemann surfaces can degenerate. One differential geometric way to view this is that when $g \geq 2$, every compact Riemann surface of genus g admits a canonical hyperbolic metric. With respect to the hyperbolic metric, each homotopy class of simple nontrivial closed curve contains a unique simple closed geodesic. We can obtain a degenerating family of hyperbolic surfaces by pinching simple closed geodesics.

If two simple closed geodesics are disjoint, we can pinch them separately and independently. Otherwise, we cannot due to the collar theorem for hyperbolic surfaces. The resulting pinched surfaces are complete, noncompact hyperbolic surfaces of finite area. By adding such hyperbolic surfaces of finite area with Euler characteristic equal to $2-2g$, which correspond to stable Riemann surfaces of Euler characteristic $2-2g$, to the boundary of $\mathcal{M}_g$, we obtain the Deligne–Mumford compactification $\overline{\mathcal{M}_g}^{DM}$ of $\mathcal{M}_g$. In fact, Deligne and Mumford showed that $\overline{\mathcal{M}_g}^{DM}$ is an irreducible projective variety.

But there is one problem with the Deligne–Mumford compactification $\overline{\mathcal{M}_g}^{DM}$ for the purpose of this paper. The inclusion

$$\mathcal{M}_g \hookrightarrow \overline{\mathcal{M}_g}^{DM}$$

is not a homotopy equivalence.

Therefore, it is not an analogue of the Borel–Serre compactification $\overline{\Gamma\backslash X}^{BS}$. In [Hav], Harvey constructed an analogue of the Borel–Serre compactification for $\mathcal{M}_g$ by imitating the procedures for $\overline{\Gamma\backslash X}^{BS}$.

1. Construct the Borel–Serre partial compactification $\overline{\mathcal{T}_g}^{BS}$ by adding at infinity products of suitable Teichmuller spaces and Euclidean spaces, which are analogues of boundary components of parabolic subgroups. (Note the similarity with the two factors X_P and N_P in the boundary component e_P.) Briefly, the Fenchel–Nielsein coordinates of the Teichmüller space $\mathcal{T}_g$ are similar to the horospherical coordinates of symmetric spaces.
2. These boundary components of $\overline{\mathcal{T}_g}^{BS}$ are parametrized by pinching collections of disjoint simple closed geodesics. One can build

an infinite simplicial complex whose simplices correspond to collections of disjoint simple closed curves on the base compact oriented surface S_g of genus g.

This is the so-called **curve complex** of the surface S_g, denoted by $\mathcal{C}(S_g)$, and corresponds to the Tits building $\Delta(\mathbf{G})$ used in the Borel–Serre partial compactification $\overline{X}^{BS}$.

3. The action of the mapping class group Mod_g on $\mathcal{T}_g$ extends to a proper action on $\overline{\mathcal{T}_g}^{BS}$, and the quotient

$$\mathrm{Mod}_g \backslash \overline{\mathcal{T}_g}^{BS}$$

is the Borel–Serre compactification of $\mathcal{M}_g$.

The curve complex $\mathcal{C}(S_g)$ is the crucial notion and we explain more about it. Let S_g be the base compact oriented surface of genus g which is used to define markings on Riemann surfaces. Then the curve complex $\mathcal{C}(S_g)$ is an infinite simplicial complex such that

1. The vertices of $\mathcal{C}(S_g)$ correspond to homotopy classes of simple closed nontrivial curves in S_g.
2. Simplexes of $\mathcal{C}(S_g)$ correspond to distinct homotopy classes of **disjoint** simple closed nontrivial curves in S_g.

Since the boundary components added to $\overline{\mathcal{T}_g}^{BS}$ are parametrized by the simplices of $\mathcal{C}(S_g)$ and are contractible, we have the following result:

Corollary 3.2. *The Borel–Serre boundary $\partial\overline{\mathcal{T}_g}^{BS}$ is homotopy equivalent to the curve complex $\mathcal{C}(S_g)$.*

Harer [Har] showed an analogue of the Solomon–Tits Theorem.

Proposition 3.3. *For $g \geq 2$, the curve complex $\mathcal{C}(S_g)$ is of the homotopy type of a bouquet of spheres of dimension $2g-2$.*

Using this, he proved the following result.

Proposition 3.4. *The mapping class group Mod_g is a duality group of cohomological dimension $\mathrm{cd}(\mathrm{Mod}_g) = 4g-5$.*

Harer also proved other fundamental results on cohomological properties of Mod_g. In proving all these results, the curve complex $\mathcal{C}(S_g)$ and related complexes play an important role. See [Har] for more details.

It seems that one natural question was not asked nor answered in his work.

Question 3.5. *Is Mod_g not a virtual Poincare duality group?*

This is related to a question about the topology of curve complex $\mathcal{C}(S_g)$. Harer's result says that $\mathcal{C}(S_g)$ is homotopy equivalent to a bouquet of spheres. But he did not say **how many spheres** there are in the bouquet. There could be *none*. If this is the case, Mod_g is a virtual Poincare duality group.

In [IJ], we showed that

Proposition 3.6. *The curve complex $\mathcal{C}(S_g)$ is homotopy equivalent to a bouquet of* **infinitely many** *spheres, and hence Mod_g is not a virtual Poincare duality group.*

This is consistent with the result for noncocompact arithmetic subgroups in Theorem 2.1.

3.3 Application of Curve Complexes of Surfaces

When Harvey [Hav] introduced the curve complex $\mathcal{C}(S_g)$, it was more like a formal analogy of Tits buildings $\Delta(\mathbf{G})$ in order to parametrize the boundary components of the Borel–Serre partial compactification $\overline{\mathcal{T}_g}^{BS}$.

Later, it turned out that $\mathcal{C}(S_g)$ has many applications in low dimensional topology. For example, Masur and Minsky proved that $\mathcal{C}(S_g)$ is a hyperbolic space, and this was used by Minsky et al. to prove the ending lamination conjecture of Thurston. See the papers [Mi] and [BCM] for more details. It was also used to prove that the asymptotic dimension of Mod_g is finite in [BBF], which is a large scale geometric invariant of finitely generated groups endowed with word metric and implies that the Novikov conjectures are satisfied by Mod_g.

The curve complex was also used to understand Heegard splittings of 3-manifolds. See the paper [J4] for more references and some details about applications of curve complexes and their similarities with Tits buildings.

The curve complex $\mathcal{C}(S_g)$ is now a basic tool in geometric group theory and low dimensional topology. It is perhaps worthwhile to recall that the whole discussion started with the question to understand cohomology properties of arithmetic subgroups of semisimple Lie groups in the paper of Borel and Serre [BS].

Let us conclude with a quote from Serre who told the author of this paper about 20 years ago at University of Michigan: **"Everything is cohomology!"**

We hope that the discussion in this paper gives some support to his assertion. The reader can also take a look at the works of Serre and Grothendieck to see how cohomology theories appeared and were used.

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