Multilinear Problems

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Abstract. The purpose of this paper is to survey some developments in the study of two classes of problems in mathematical analysis related to oscillatory integrals, Radon transforms, time-frequency analysis and resolution of singularities.

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1. Introduction

Oscillatory integrals and singular integral operators have long been of interest in harmonic analysis and mathematical physics. They have emerged as powerful analytic tools in various problems, ranging from PDEs to geometry and number theory. Multilinear problems have attracted tremendous interest during the last two decades and have led to many important break throughs in harmonic analysis and many other areas. Most notably, the full resolution of the Vinogradov's main conjecture by Bourgain, Demeter, and Guth [1].

In this paper, we will survey on several multilinear analogues of oscillatory integrals and singular integral operators that have attracted the interest of many researchers during last two decades. The first class of problems being considered can be phased as the following multilinear oscillatory integral form:

Problem A [11]. Let U be some open set in $\mathbb{R}^N$ and $\phi : U \to \mathbb{R}$ be some appropriate function (weight). A pair of inputs $[S,\Pi]$ consists of a real-valued function S defined on U and a collection of surjective linear transforms $\Pi = (\pi_1,\dots,\pi_J)$, where $\pi_j : \mathbb{R}^N \to \mathbb{R}^{n_j}$, for $j = 1,\dots,J$. Associated to this pair of inputs is the following multilinear oscillatory integral form

$$M(S,\Pi) = \int_{\mathbb{R}^N} e^{i\lambda S(x)}\phi(x)\prod_{j=1}^{J} f_j(\pi_j(x))dx, \tag{1.1}$$

where λ is a real-valued parameter. The three problems to be investigated are the following.

1. (Existence of decay) for some appropriate input $[S,\Pi]$, prove an estimate of the form

   $$|M(S,\Pi)| \le C|\lambda|^{-\delta}\prod_{j=1}^{J}\|f_j\|_{p_j}, \tag{1.2}$$

   which holds for some $\delta > 0$, for all λ and for a class of functions $f_j : \mathbb{R}^{n_j} \to \mathbb{C}$, where C is some constant independent of the functions f_j and λ.
2. (Optimality of decay) Compute the optimal δ in terms of the input $[S,\Pi]$ and the multiindex $(p_1,\dots,p_J)$.
3. (Uniformity of the bound) Prove the constant C can be chosen to be uniform over a class of functions S.

The problem is formulated in such a general form that it contains many important research questions in analysis. By taking $N = 1$, $J = 0$ and $\phi(x)$ a smooth cut-off function, (1.2) corresponds to scalar oscillatory integrals; by letting $J = N$, $\pi_j(x) = x_j$, $f_j(x_j) = e^{i\xi_j x_j}$ and all $p_j = \infty$, (1.2) can be interpreted as an estimate for the Fourier transform of the surface $(x,S(x))$. Consequently, sharp and uniform estimates of (1.2) automatically imply sharp and uniform estimates for the corresponding Fourier transform. The study of this problem is also of independent interest for it concerns a new kind of stability for oscillatory integrals, which is referred to as "algebraic stability". To see this, let $f_j(\cdot) = e^{ig_j(\cdot)}\chi_j(\cdot)$, where $g_j : \mathbb{R}^{n_j} \to \mathbb{R}$ is any measurable function and χ_j is the characteristic function of $\mathbb{R}^{n_j} \cap U$. Then (1.2) implies

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$$\left|\int_{\mathbb{R}^N} e^{i(\lambda S(x)+\sum_{j=1}^J g_j(\pi_j(x)))}\phi(x)dx\right| \leq C|\lambda|^{-\delta},$$

with the constant C being uniform in all choices of $\{g_j\}_{1\leq j\leq J}$. The term $\sum_{j=1}^J g_j(\pi_j(x))$ in the exponent is called an algebraic perturbation and the estimates are stable under such a perturbation. In another word, we are estimating oscillatory integrals whose phases are lying in some quotient space.

The second class of problems deals with the boundedness of a variety of multilinear singular Radon-like transforms and their maximal function analogues, whose general form can be formulated as follows:

Problem B. Let $(x,t)\in\mathbb{R}^{n+m}$ and $K:\mathbb{R}^m\to\mathbb{R}$ be some Caderón-Zygmund kernel and $\gamma_j:\mathbb{R}^{n+m}\to\mathbb{R}$ be smooth functions, for $j=1,\dots,J$. Define the following multilinear singular Radon-like transform associated to the collection of surfaces $\{(x,t,\gamma_j(x,t)\}_{1\leq j\leq J}$

$$T(f_1,\dots,f_J)(x) = \text{p.v.}\int\prod_{j=1}^J f_j(\gamma_j(x,t))K(t)dt$$

and its maximal function analogue

$$M(f_1,\dots,f_J)(x) = \sup_{\epsilon>0}\frac{1}{|B(0,\epsilon)|}\int_{|t|<\epsilon}\left|\prod_{j=1}^J f_j(\gamma_j(x,t))\right|dt.$$

Prove estimates of the form

$$\|T(f_1,\dots,f_J)\|_r \leq C\prod_{j=1}^J \|f_j\|_{p_j}, \text{ and}$$
$$\|M(f_1,\dots,f_J)\|_r \leq C\prod_{j=1}^J \|f_j\|_{p_j}, \tag{1.3}$$

for some appropriate tuples $(p_1,\dots,p_J,r)$. In addition, characterize the maximal convex hull of these tuples in terms of the geometry or algebra or analysis of the surfaces $\{(x,t,\gamma_j(x,t))\}_{1\leq j\leq J}$. Finally, prove uniform estimates of (1.3) in the sense the constant C is uniform in a class of surfaces $\{(x,t,\gamma_j(x,t)\}_{1\leq j\leq J}$.

These two classes of multilinear problems are closely related to each other just like in the linear case: the theory of (multilinear) oscillatory integrals provides a framework for the study of (multilinear) singular Radom transforms. It is extremely challenging, if possible, to give an complete answer to either problem above even in the bilinear (i.e. setting $J=2$). What lies in the heart of them is the role of geometry (in particular, curvature) in the theory of multilinear operators.

The rest of this paper is organized as follows. In Section 2, we will present some classical results for oscillatory integrals and singular integral operators. After that, we will take the opportunity to describe some recent joint work of the author with Philip T. Gressman, Jingwei Guo and Xiaochun Li. The last section consists of some interesting open problems in this area. We hope that progress on these problems can eventually advances the understanding of curvature in the theory of multilinear operators.

2. Classical Results

2.1 Linear and Multilinear Oscillatory Integrals

The theory for oscillatory integrals in one dimension is well-established. The very well-known van der Corput lemma provides a completely satisfying answer. The lemma states that there is an absolute constant $C_k\in\mathbb{R}$ such that for any real-valued function $S\in C^k(I)$ on some interval $I\subset\mathbb{R}$, if $S^{(k)}(x)\geq 1$ on I (assuming S' is monotone when $k=1$), then

$$\left|\int_I e^{i\lambda S(x)}dx\right| \leq C_k|\lambda|^{-\frac{1}{k}}.$$

This estimate shares two important features: sharpness (the decay is optimal) and uniformity (the constant C_k is independent of I and other information of the phase S). However, the theory for higher dimensions is a lot less complete and the progress has been slow, because, among many other reasons, the singularities and the geometry of the phases involved may themselves be substantially more complicated. In dimension higher than two, estimates for oscillatory integrals may fail to simultaneously possess the features of sharpness and of stability. In his fundamental work [71], Varchenko indeed constructed a polynomial phase whose optimal decay rate for the corresponding oscillatory integral is unstable under small perturbations. Due to the additional "algebraic perturbations" from the multilinear form, the situation for the multilinear theory can be substantially more complicated. One fundamental question in this direction is to identify the class of all inputs $[S,\Pi]$ such that decay estimates of the form (1.2) can occur. Only very few cases are known and novel ideas and techniques are needed to advance the understanding.

However, much progress has been made in some particular cases of (1.2) with emphasis on only one of the above features. A very effective approach to prove sharp estimates is based on resolution of singularities, dating back to Varchenko [71]. For scalar oscillatory integrals, he obtained sharp decay rate estimates for arbitrary real analytic phases in dimension two and higher dimensions under a certain non-degenerate condition (often referred to as Varchenko's condition). Significant progress in this direction has been made by Collins, Greenleaf and Pramanik [14], Greenblatt [26, 28, 29], Ikromov and Muller [33, 34], Karpushkin [35] and many others. In the direction of oscillatory integral operators, Phong and Stein [54–57] developed a systematic method that can efficiently handle the low dimension cases, as well as some higher dimensions cases (joint with Sturm [58]). In particular, they established the operator analogue of Varchenko's results, namely, the sharp L^2 estimates for the one-dimensional oscillatory integral operator with arbitrary real analytic phases:

Theorem 1 (Phong and Stein [56]). *Let $S(x,y)$ be a real analytic function defined on some neighborhood of $0 \in \mathbb{R}^2$ and $\chi(x,y)$ be a smooth cut-off function supported in a small neighborhood of 0. Then*

$$\left\| \int_{-\infty}^{\infty} e^{i\lambda S(x,y)} \chi(x,y) f(y) dy \right\|_{L^2(\mathbb{R})} \le C|\lambda|^{-\frac{1}{2\delta}} \|f\|_{L^2(\mathbb{R})},$$

where δ is the Newton distance of the reduced Newton polyhedron of S.

For general smooth phases, nearly sharp estimates was obtained my Seeger, and the sharp estimates were eventually established by Rychkov [61] and Greenblatt [25]. In dimension higher than two, Phong, Stein and Sturm [58] and Carbery and Wright [4] obtain nearly sharp estimates (up to a logarithmic factor) for the corresponding multilinear operators for polynomial-type phases. Other related works in the two-dimensional cases include Greenleaf and Seeger [30], Yang [74, 75], Pramantik and Yang [59], Shi and Yan [63], and the author [73].

In the direction of uniformity/stability, a result from Karpushkin [35] guarantees the optimal decay rates for the scalar oscillatory integral is stable under arbitrary perturbations in dimension two (note that Varchenko's example shows that no such stability results can hold for general phase in dimension higher than two). The work from Carbery, Christ and Wright [3] is worth mentioning. They establish uniform estimates (*a la* the one-dimensional van der Corput lemma) for scalar oscillatory integrals and the sublet set analogue in all dimensions:

Theorem 2 (Carbery, Christ and Wright [3]). *Let $\alpha = (\alpha_1, \dots, \alpha_n) \neq 0$ be a multi-index, and suppose that at least one of its entries α_j is greater than or equal to two. Then there exist $\epsilon > 0$ and $C < \infty$, depending only on α and on n, such that for any smooth function $S: Q = [-1,1]^n \to \mathbb{R}$ satisfying $\partial_x^\alpha S(x) \ge 1$, for all $\lambda \in \mathbb{R}$, the following holds*

$$|\int_Q e^{i\lambda S(x)} dx| \le C|\lambda|^{-\epsilon} \quad \text{and}$$
$$\{x \in Q : |S(x)| < |\lambda|^{-1}\} \le C|\lambda|^{-\epsilon}.$$

What is equally significant to this result is their approach, which reveals certain combinatoric essence of the problem. In general, it is extremely challenging to obtain sharp and uniform estimates simultaneously. However, under certain finite-type assumptions (the phases are polynomial type), Phong, Stein and Sturm [58] and Carbery and Wright [4] obtain the uniformity and sharpness (up to a logarithmic factor) simultaneously. We quote Phong, Stein and Sturm's result:

Theorem 3 (Phong, Stein and Sturm [58]). *Let $S : [0,1]^d \to \mathbb{R}$ be a polynomial of degree n, $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$ be a multiindex, D be the subset of $[0,1]^d$ defined by $|S^{(\alpha)}| \ge 1$ and $\frac{1}{p_j} = 1 - \frac{\alpha_j}{|\alpha|}$. Then there is a constant C depends on n and α such that*

$$\left| \int_D e^{i\lambda S(x_1, \cdots, x_d)} \prod_{j=1}^{d} f_j(x_j) dx_1 \cdots dx_d \right| \le C|\log(2+|\lambda|)|^{d-\frac{1}{2}} |\lambda|^{-\frac{1}{|\alpha|}} \prod_{j=1}^{d} \|f_j\|_{p_j}. \tag{2.4}$$

Motivated by the study of multilinear oscillatory Calderon-Zygumd theory, Christ, Li, Tao and Thiele [11] studied the decay rate estimates for (1.1) in its most general form. To move further into this general setting, we introduce two concepts. The input $[S, \Pi]$ is said to be degenerate if $S(x) = \sum_{j=1}^{J} S_j(\pi_j(x))$ for some measurable functions $S_j : \mathbb{R}^{n_j} \to \mathbb{R}$. Otherwise, the input $[S, \Pi]$ is non-degenerate. We say the input $[S, \Pi]$ has the decay property if (1.2) is true for some choice of tuple $(p_1, \dots, p_J)$. It is obvious that the decay property can not hold if the input is degenerate. Christ et al. raise the following conjecture:

Conjecture 1 (Christ, Li, Tao and Thiele [11]). *Decay property is equivalent to non-degeneracy.*

Christ et al. are able to identify certain classes of inputs $[S, \Pi]$ for this conjecture. In the first major class, each input is of co-rank 1 and the number of projections is arbitrary. The second major class is when all of the input maps are of rank 1 and the number of projections is no more than $(2N-1)$. Some new class of inputs has been recently identified by Christ [9, 10] and Christ and Oliveira e Silva [13]. Some other progress related to this conjecture includes Greenblatt [27], Gressman and the author [31, 72]. In particular, the work of Christ [9] suggests an interesting connection between the multilinear theory for oscillatory integrals and additive number theory.

2.2 Multilinear Singular Radon-Type Transforms

By letting $J = 1$, $n = m$ and $\gamma_1(x,t) = (x-t)$, the two operators defined in Problem B become the classical Hilbert transform and the Hardy-Littlewood maximal function, which are paradigms of general singular integrals and maximal functions. The study of these two types operators and their variants are central themes in modern harmonic analysis.

Since the initial breakthroughs for singular integrals along curves and surfaces by Christ, Geller, Nagel, Müller, Phong, Ricci, Riviere, Seeger, Stein, Wainger, and many others [6–8, 15, 20, 42, 43, 45–54, 60, 62], extensive research in this area of harmonic analysis has been done and a great many fascinating and important results have been established, which culminate in a general theory of singular Radon transforms (see for instance Christ, Nagel, Stein, and Wainger [12]). Recently, this theory is extended to the multi-parameters setting by Stein and Street [65–69]. Important tools for proving these classical results include the Plancherel formula, the method of TT^* and the principle of almost orthogonality (like Cotlar-Stein).

Another interesting direction is the bilinear extension of the classical Hilbert transform (and maximal functions), which can be typically written as

$$H(f,g)(x) = \text{p.v.} \int_{-\infty}^{\infty} f(x-t)g(x+t)\frac{dt}{t}. \tag{2.5}$$

The Hölder-type bounds of this bilinear operator were conjectured by Caldéron [2], motivated by the study of the Cauchy integral on Lipschitz curves. In the 1990s, this conjecture was verified by Lacey and Thiele [36, 37], in a breakthrough pair of papers.

Theorem 4 (Lacey and Thiele [36, 37]). *The bilinear Hilbert transform maps from $L^p \times L^q$ to L^r given $p,q > 1$, $r > 2/3$ and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$.*

In addition to the scaling and translation symmetries that the Hilbert transform obeys, the bilinear Hilbert transform H has modulation symmetry (another famous operator that enjoys this symmetry is Carleson's operator), namely $H(f,g) = e^{-2i\xi}H(e^{i\xi}f, e^{i\xi}g)$. Consequently, any treatment (namely, a "suitable" decomposition of the operator) can efficiently handle this operator should also be invariant under modulations. Inspired by the fundamental works of Carleson [5] and Fefferman [21] for the pointwise convergence of Fourier transforms, a systematic and delicate method was developed by Lacey and Thiele [36, 37] to handle this situation, which is often referred to as the method of time-frequency analysis. Over the last two decades, this method has merged as a powerful analytic tool to handle several linear and multilinear operators with modulation symmetry. Important progress in this line includes the bilinear maximal function by Lacey [38], the multi-parameter singular integrals by Mascalu, Pipher, Tao and Thiele [44], the uniform estimates of the bilinear Hilbert transform by Thiele [70], and Grafakos and Li [24, 40] and etc. A recent theory of outer L^p spaces developed by Do and Thiele [19] has also stimulated new progress [16–18]. However, some of the most interesting open questions in this field are beyond the scope of this method and have a strong connection to some kind of non-abelian analysis. For instance, the trilinear Hilbert transform

$$\Lambda(f_1,f_2,f_3)(x) = \text{p.v.} \int_{\mathbb{R}} \prod_{j=1}^{3} f_j(x+jt)\frac{dt}{t}$$

has a hidden quadratic modulation symmetry, namely,

$$\Lambda(f_1,f_2,f_3) = q_\xi \Lambda(q_{-3\xi}f_1, q_{3\xi}f_2, q_{-\xi}f_3),$$

where $q_\xi(x) = e^{2\pi i\xi x^2}$. This quadratic (nonlinear) nature must be accounted for in any proposed method of analysis.

Motivated by the singular Radon transforms and the bilinear Hilbert transforms, Li [39] initialized the study of the bilinear Hilbert transform along curves (BHTaCs). Let $\Gamma(t) = (t, \gamma(t))$, where $\gamma(t)$ is a real-valued function, define

$$H_\Gamma f(x) = \text{p.v.} \int_{\mathbb{R}} f(x-t)g(x-\gamma(t))\frac{dt}{t}. \tag{2.6}$$

The new challenge in the investigation of this operator lies in the lack of a "natural" way to explore curvature of $\Gamma(t)$ in the bilinear setting. In the linear setting, there is a relatively "simple" bound, namely, the $L^2 \to L^2$ boundedness, and the curvature $\Gamma(t)$ can be utilized by coupling the Plancherel theorem with various tools in estimating scalar oscillatory integrals. Moreover, the lack of modulation symmetry of this operator indicates the method of time-frequency analysis may not be suitable. Li is able to overcome this difficulty by using a new method called the σ-uniformity, inspired by a seminar work of Gowers [23], establishing the $L^2 \times L^2 \to L^1$ boundedness when Γ is a monomial curve:

Theorem 5 (Li [39]). *Let $\gamma(t) = t^\alpha$, $\alpha > 0$ and $\alpha \neq 1$. Then $H_\Gamma f$ maps from $L^2 \times L^2 \to L^1$.*

The boundedness of the bilinear Hilbert transforms (along curves) is quite well-understood when the curves are completely flat (namely the Lacey-Thiele) and when the curves' curvature essentially vanishes in a finite order. The situation is still unclear in other cases, including ones in which the curves are asymptotically flat. A model case is when $\gamma(t)$ is a polynomial containing both linear and non-linear terms. Part of the difficulty in this case is that Lacey and Thiele's method is stable under any linear perturbations but not under any non-linear perturbation and vice versa for Li's method. This situation is somewhat similar to the one of Stein's polynomial Carleson conjecture: the cases were well-understood when the polynomial contains only the linear term or only the non-linear terms, but were not well-understood for some time when both linear and non-linear terms appeared.

3. Some Recent Work

In this section, we will describe some recent joint work of the author with Philip T. Gressman, Jingwei Guo and Xiaochun Li.

3.1 Sharp Trilinear Oscillatory Integrals [72]

The goal is to understand the following basic trilinear oscillatory integral form

(3.7)
$$\Lambda_S(f_1,f_2,f_3)=\iint_{\mathbb{R}^2} e^{i\lambda S(x,y)}\phi(x,y)f_1(x)f_2(y)f_3(x+y)\,dxdy.$$

Here $\phi(x,y)$ is a smooth cut-off function supported in a small neighborhood of 0 and the phase $S(x,y)$ is an arbitrary real analytic function on the support of ϕ. This trilinear form is a particular case of the multilinear oscillatory integral studied by Christ et al., in which they proved (non-sharp) decay rate estimates for polynomial phases in a more general setting. An important step in their approach is a reduction of the multilinear setting to a trilinear setting, which motivates the study of (3.7). Another motivation comes from the fundamental work of Phong and Stein [56] concerning sharp estimates degenerate oscillatory integral operators, of which (3.7) can be considered as a trilinear extension. As observed in [11], one expects no decay at all when $S(x,y)$ may be written as a sum $S_1(x)+S_2(y)+S_3(x+y)$ for measurable functions S_1,S_2, and S_3. When S is smooth, the (non-)degeneracy of S is captured by the action of the differential operator $D=\partial_x\partial_y(\partial_x-\partial_y)$, which annihilates sums of the form $S_1(x)+S_2(y)+S_3(x+y)$. Let $DS(x,y)=\sum_{j\geq j_0}P_j(x,y)$, where P_j is homogeneous polynomial of degree j and P_{j_0} is the first non-zero term. Define the relative multiplicity of S by $\operatorname{mult}(S)=(j_0+3)$. Then the sharp decay estimates can be characterized by $\operatorname{mult}(S)$.

Theorem 6 ([72]). *Let S and ϕ be as above. Then*

$$|\Lambda_S(f_1,f_2,f_3)|\leq C|\lambda|^{-\frac{1}{2\operatorname{mult}(S)}}\prod_{j=1}^{3}\|f_j\|_2. \tag{3.8}$$

The result (3.8) is sharp in the sense that if $\phi(0,0)\neq 0$, then

$$|\Lambda_S(f_1,f_2,f_3)|\geq C'|\lambda|^{-\frac{1}{2\operatorname{mult}(S)}}\prod_{j=1}^{3}\|f_j\|_2, \tag{3.9}$$

as $|\lambda|\to\infty$, for some $C'>0$ and some $\{f_j\}_{1\leq j\leq 3}$.

One interesting feature of the theorem is that the characterization of the exponent is quite different from the exponent for the bilinear form [56], which is in terms of the Newton polyhedron of the phase. This is due to the extra convolution structure of the trilinear form. The proof of this theorem builds upon the framework of Phong and Stein, relying on two important new ingredients. The first is a trilinear extension of Phong-Stein's operator van der Corput lemma. The second is an algorithm of two-dimensional resolution of singularities (influenced by many previous works). The algorithm is of independent interest for it can be employed to decompose a neighborhood of an isolated singular point into finitely many subregions, on which derivatives of the phase behaves like monomials.

3.2 Maximal Decay and Algebraic Stability [31]

P.T. Gressman and the author have obtained several new and unexpected findings for the trilinear form (3.7). The basic observation upon which Theorem 6 is based is an estimate of the form

$$|\Lambda_S(f_1,f_2,f_3)|\leq C|\lambda|^{-\frac{1}{6}}\|f_1\|_2\|f\|_2\|f_3\|_2$$

for phases S with $|DS|\geq c>0$ on the support of ϕ, with D defined above. Scaling arguments show that the exponent of λ cannot be improved, but a comparison to the sublevel set estimate (namely, the decay rate of the size of the set

$$\{(x,y)\in[0,1]^2:|S(x,y)|<\epsilon\}$$

as $\epsilon\to 0^+$ indicates that the decay rate $|\lambda|^{-\frac{1}{6}}$ is likely not the best possible if one considers L^p spaces on the right-hand side other than L^2. We were able to improve the decay rate from $|\lambda|^{-\frac{1}{6}}$ to $|\lambda|^{-\frac{1}{4}}$ by utilizing both oscillation structure (via the method of TT^*) and convolution structure (via the Hardy-Littlewood-Sobolev inequality) of the trilinear form, while the previous $|\lambda|^{-\frac{1}{6}}$-result employs only the oscillation structure. Note that the $|\lambda|^{-\frac{1}{4}}$ decay rate still falls short of the $|\lambda|^{-\frac{1}{3}}$ decay rate suggested by the optimal sublevel set estimate. Any improvement beyond $|\lambda|^{-\frac{1}{4}}$, if possible, is likely extremely challenging. However, when S is sufficiently degenerate, we were able to closing the gap between the sublevel set decay rate and the oscillatory integral decay rate.

Theorem 7 ([31]). *Let $S(x,y)$ be as above and $n=$mult$(S)\geq 9$. If the order of "vanishing" of S is less than $(\frac{n}{2}-2)$, then*

$$|\Lambda(f_1,f_2,f_3)|\leq C|\lambda|^{-\frac{2}{n}}\|f_1\|_\infty\|f_2\|_\infty\|f_3\|_\infty. \tag{3.10}$$

While this matching of decay rates is satisfying and natural, it should perhaps be regarded as somewhat surprising that this is possible since, among other things, the highest possible decay rate can only be achieved when f_1, f_2 and f_3 all belong to L^∞, which is not traditionally a regime in which one expects to find strong cancellation effects. This theorem also has some new and interesting consequences, for instance, the following sharp and stable estimates,

Theorem 8 ([31]). *Under the same assumptions of the above theorem, there is a constant C such that*

$$\left|\iint_{\mathbb{R}^2} e^{i(\lambda S(x,y)+P_1(x)+P_2(y)+P_3(x+y))}\phi(x,y)dxdy\right| \le C|\lambda|^{-\frac{2}{n}}$$

and

$$|\{(x,y)\in[0,1]^2 : |S(x,y)-P_1(x)-P_2(y)-P_3(x+y)|<\epsilon\}| \le C\epsilon^{\frac{2}{n}}$$

for all real-valued measurable functions P_1, P_2 and P_3.

Note that any quadratic function of two variables can be written as a sum of $P_1(x), P_2(y)$ and $P_3(x+y)$, and this theorem includes, in particular, the results for sharp and stable estimates for the oscillatory integrals (and the sublevel sets) under any quadratic perturbation, which can also be used to deduce estimates for the Fourier transform of the 2 dimensional manifold in $\mathbb{R}^6$ given by $(x,y,x^2,y^2,xy,S(x,y))$.

3.3 Oscillatory Integral Operators and a Weak van der Corput Lemma in 2D

Let $S(x,y)=\sum_{k,l\in\mathbb{N}} c_{k,l}x^k y^k$ be a real analytic function defined on a small neighborhood of the origin. The one-dimensional oscillatory integral operator associated to S is defined by

$$T_\lambda f(x) = \int_{-\infty}^{\infty} e^{i\lambda S(x,y)}\phi(x,y)f(y)dy. \tag{3.11}$$

The important geometric concept to characterize the mapping properties of this operator is the reduced Newton polyhedron $\mathcal{N}^*(S)$ of S, which is given by the convex hull of the union of all $[k,\infty)\times[l,\infty)$ with $c_{k,l}\neq 0$ and $k,l\geq 1$. The reduced Newton diagram $D^*(S)$ is the boundary of $\mathcal{N}^*(S)$, and the Newton distance is the number δ such that $(\delta,\delta)\in\mathcal{N}^*(S)$. In [73], the author shows that the $L^p\to L^p$ mapping properties of T_λ are fully captured by the reduced Newton polyhedron:

Theorem 9 ([73]). *Let S and ϕ be as above. Assume $\alpha>0$ and $\phi(0,0)\neq 0$. Then*

$$\|T_\lambda\|_{p\to p}\le C|\lambda|^{-\alpha} \iff \left(\frac{1}{p\alpha},\frac{1}{p'\alpha}\right)\in\mathcal{N}^*(S)$$

and this estimate is sharp iff $(\frac{1}{p\alpha},\frac{1}{p'\alpha})\in\mathcal{D}^*(S)$.

In particular, this theorem implies the fundamental work of Phong and Stein [56], which corresponds to the $p=2$ and $\alpha=\delta$ case. Another interesting feature of this theorem lies in its connection to a two-dimensional van der Corput Lemma. A well known result [64] states that under the assumption $\partial_x^k\partial_y^l S(x,y)\neq 0$ on the support of ϕ, the scalar oscillatory integral I_λ has a decay rate estimate $|\lambda|^{-\frac{1}{k+l}}$. Using a compactness argument, the above theorem can used to establish an operator analogue of this result:

Theorem 10 ([73]). *Assume the real analytic function S satisfies $\partial_x^k\partial_y^l S(x,y)\neq 0$ pointwisely on the support of ϕ (assumed to be compact) for some $k,l\geq 1$. Then there is a constant C independent of λ such that*

$$\|T_\lambda\|_{\frac{k+l}{k}\to\frac{k+l}{k}} \le C|\lambda|^{-\frac{1}{k+l}}. \tag{3.12}$$

This estimate for T_λ also preserves certain scaling properties that its scalar analogue I_λ doesn't: if (k,l) is a least pair such that $\partial_x^k\partial_y^l S(x,y)\neq 0$, then the decay rate $|\lambda|^{-\frac{1}{k+l}}$ is optimal for the norm of T_λ but not for I_λ in general (for instance $S(x,y)=x^k y^l$). The proof of this theorem is quite involved, which is based on the resolution algorithm built up in [72] and various ideas and techniques from many previous works, including Phong-Stein's operator van der Corput lemma, the theory for oscillatory BMO and Hardy spaces, Stein's complex interpolation, a lifting trick from Zygmund and many others.

3.4 Uniform Estimates for Bilinear Hilbert Transform Along Curves [41]

In joint work with X. Li, we studied the BHT and its maximal function analogue along polynomial curves without linear term. We are particularly interested in the full $L^p\times L^q\to L^r$ ranges for both operators. We show that these ranges can be characterized by three equivalent statements, algebraically by the maximal order of roots, geometrically by the maximal contact order, and analytically by the growth of sublevel sets.

Theorem 11 ([41]). *Let $P(t)$ be a polynomial contains no linear term and H_Γ and M_Γ be the corresponding BHT and bilinear maximal function along the curve $\Gamma(t)=(t,P(t))$. Then the following are equivalent:*

1. *All the roots of $P'(t)-1=0$ have order at most (k-1);*
2. *For any tangent line L of $(t,P(t))$ with slope equal to 1, the contact order between L and Γ is at most k.*
3. *There is a constant $C_P=C(P)$ s.t. the following sublevel set estimate is true for h sufficiently small*

$$|\{t : |P'(t)-1|<h\}| < C_P h^{\frac{1}{k-1}}.$$

4. *The operators H_Γ and M_Γ map from $L^p\times L^q$ to L^r for all $r>\frac{k-1}{k}$, $p,q>1$ satisfying $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$.*

The assumption that Γ is a polynomial curve is not essential. For instant, when $\Gamma(t) = (t, t^\alpha)$ ($\alpha > 0$ and $\neq 1$), same proof of the theorem [41] implies that H_Γ maps into L^r for $r > 1/2$. The significance is that it also provides some hope for the conjecture that the classical BHT also maps into L^r in the same range. Our result concerns the uniformity of both operators, in which we prove the bound for both operators can be uniform in the coefficients of the polynomial.

Theorem 12. *Let $P(t)$, H_P and M_P be as above. Then H_{Γ_P} and M_P map from $L^p \times L^q$ into L^r for all $r > \frac{d-1}{d}$, $p, q > 1$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. In addition, the bound is uniform in a sense that it depends on the degree of P but is independent of its coefficients and the range of p, q, r is best possible except for the endpoint case $r = \frac{d-1}{d}$.*

This result extends the uniform estimates for BHT of Thiele [70], Grafakos and Li [24, 40], and the bilinear maximal function of Lacey [38] to the curvature setting. The general outline of approach for these two theorems is as follows. The operator is decomposed into dyadic pieces H_j which are broken into a dichotomy of a finite number of terms that are "good" and an infinite collection of terms that are "bad" (corresponding to regions where one particular term in P vastly dominates all other terms in the polynomial). The finite collection is bounded from $L^p \times L^q \to L^r$ term by term, where the constraint of r ($r > \frac{k-1}{k}$ in the first theorem and $r > \frac{d-1}{d}$ in the second one) arises. The key tool employed here is the classical van der Corput lemma. The remaining infinite collection requires the most work, and is split into two parts. One is treated by using Li's σ-uniformity and certain trilinear oscillatory integral forms, in which an exponential decay in a certain index is obtained. The other is handled by applying the method of time-frequency analysis and a Whitney decomposition (when handling some error terms) with a slow (polynomial) growth bound. The final bound follows by interpolation.

3.5 Bilinear Hilbert Transform Along Plane Curves [32]

In this joint work with J.W. Guo, the author further studies the role of curvature in the boundedness of BHTaCs. In the linear setting, Nagel, Vance, Wainger, and Weinberg proved that such boundedness can be indeed characterized by the auxiliary function $h(t) = t\gamma'(t) - \gamma(t)$ (assuming $\gamma(0) = 0$). In the case $\gamma(t)$ is odd and convex, the corresponding Hilbert transform is bounded if and only if $h(t)$ has bounded double time. It turns out that the quotient between $\gamma(t)$ and $t\gamma'(t)$ seems to be the analogue of $h(t)$ in the bilinear setting. More precisely, for each dyadic number $\epsilon > 0$ define $q_\epsilon(t) = (\epsilon\gamma'(\epsilon))^{-1}\gamma(\epsilon t)$ for $|t| \in [1/4, 4]$. The conditions imposed on $\gamma(t)$ (assumed to be smooth in some neighborhood of the origin) are there exists a constant $C > 0$ such that the following holds for all $|t| \in [1/4, 4]$ and all ϵ small

$$\left|q_\epsilon^{(j)}(t)\right| < C \quad \text{for} \quad 0 \le j \le 5, \tag{3.13}$$

$$\left|q_\epsilon''(t)\right| > C^{-1}, \tag{3.14}$$

$$\left|(q_\epsilon''(t))^2 - q_\epsilon'(t)q_\epsilon'''(t)\right| > C^{-1}. \tag{3.15}$$

We prove that:

Theorem 13 ([32]). *If $\gamma(t)$ is a smooth function defined on $[-1,1]$ satisfying (3.13), (3.14) and (3.15), then the corresponding bilinear Hilbert transform is bounded from $L^2 \times L^2 \to L^1$.*

It is unclear at this moment whether these conditions are also necessary for the $L^2 \times L^2 \to L^1$ boundedness.

4. Some Open Problems

To date the theory of multilinear oscillatory integrals and multilinear singular Radon-like transforms is still in its early stage and very little is known. The classical L_2-theory, though works perfectly well in the linear setting, becomes increasingly inefficient as the level of the multi-linearity of the transform increases. Completely novel ideas are required to understand the role of curvature in the theory of multilinear operators and to develop novel analytic tools. In this section, we will propose some interesting open problems in this field.

4.1 Van der Corput in Higher Dimentions

The goal in this direction is to obtain uniform (stable) and optimal estimates for oscillatory integrals *a la* the van der Corput lemma. In the two dimensional setting, it can be formulated as follows,

Problem 1. *Let $k, l \ge 1$ be fixed integers and $\mathcal{U}_{k,l}$ be the set of smooth functions S with $\partial_x^k \partial_y^l S(x,y) \ge 1$ for all $(x,y) \in Q = [0,1]^2$. Let ϕ be a fixed smooth cut-off function supported on Q. Is there a constant C independent of λ such that*

$$\left\| \int_\infty^\infty e^{i\lambda S(x,y)} \phi(x,y) f(y) dy \right\|_{\frac{k+l}{k}} \le C|\lambda|^{-\frac{1}{k+l}} \|f\|_{\frac{k+l}{k}}, \tag{4.16}$$

for all $S \in \mathcal{U}_{k,l}$?

This problem amounts to the generalization of Theorem 10 to the smooth and uniform setting. There is evidence that a loss of a log factor might be unavoidable if either $k = 1$ or $l = 1$. The Taylor expansion of the phase fails to capture all its information and consequently the method based on resolution of singularities alone is certainly not sufficient to handle all the cases. The stopping time argument from Greenblatt [25] will be useful. A good starting point for the much more difficult problem of proving sharp and uniform estimates is to restrict the class of phases to the class of polynomial phases and allow the constant C to depend on the degrees of the polynomials. In another word, to remove the log terms from the estimates of Phong, Stein and Sturm [58].

4.2 Multilinear Oscillatory Forms

A second problem concerns higher-dimensional generalizations, namely, estimates of multilinear oscillatory integrals in form of

$$\left|\int_{\mathbb{R}^d} e^{i\lambda S(x_1,\dots,x_d)}\phi(x_1,\dots,x_d)\prod_{j=1}^d f_j(x_j)dx_1\dots dx_d\right| \lesssim_{\log} |\lambda|^{-\alpha}\prod_{j=1}^d \|f\|_{p_j}. \tag{4.17}$$

The scalar analogue of this multilinear form was first considered by Varchenko [71] for analytic phases satisfying Varchenko's non-degeneract condition. Estimates of the form in (4.17) under the assumption $\sum_{j=1}^d \frac{1}{p_j} = (d-1)$ were considered by Phong, Stein and Sturm [58] and Carbery and Wright [4]. It turns out that the behavior of (4.17) is substantially richer than anticipated by these earlier results. Some partial progress has been obtained by M. Gilula and P.T. Gressman and the author. We prove that (4.17) holds if and only if the point $(\frac{\alpha}{p_1},\dots,\frac{\alpha}{p_d})$ lies in the reduced Newton polyhedron of S under a Varchenko-type nondegenerate condition [22]. It will be interesting to find out all possible estimates for arbitrary degenerate phases:

Problem 2. *Given a real analytic function S, what is the maximal convex hull of the tuple $(\frac{1}{p_1},\dots,\frac{1}{p_d},\alpha)$ such that (4.17) is true?*

One of the main challenges of this problem is to develop a suitable resolution of singularities algorithm in higher dimensions. One can also ask a similar question for the trilinear form.

Problem 3. *For any given real analytic phase S, find the maximal convex hull $(\frac{1}{p_1},\frac{1}{p_2},\frac{1}{p_3},\alpha)$ such that*

$$\left|\iint e^{i\lambda S(x,y)}\phi(x,y)f_1(x)f_2(y)f_3(x+y)dxdy\right| \le C|\lambda|^{-\alpha}\prod_{j=1}^3\|f_j\|_{p_j}.$$

The main challenge of this problem is quite different from that of the previous one. The nondegenerate case is indeed the most challenging one: does the following holds

$$\left|\iint e^{i\lambda(x-y)^3}\phi(x,y)f_1(x)f_2(y)f_3(x+y)dxdy\right| \le C|\lambda|^{-1/3}\prod_{j=1}^3\|f_j\|_3?$$

Preliminary investigation of this question suggests the answer will be extremely complicated, in contrast, for example, to the simply-stated results of Phong, Stein, and Sturm [58]. It will be helpful to compute such convex hull for some class of model phases, for example, when the phase has relatively small order of vanishing.

4.3 Necessary and Sufficient Condition for the Boundedness of BHTaCs

Recall that a theorem from [47] states that for an even (odd) convex curve $\Gamma(t) = (t,\gamma(t))$ on the plane, the operator

$$T_\Gamma f(x) = \text{p.v.}\int_{-\infty}^{\infty} f(x-\Gamma(t))\frac{dt}{t}$$

is bounded from $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ if and only if there is a constant $C>0$ such that $\gamma'(Ct) \ge 2\gamma'(t)$ ($h(Ct) \ge 2h(t)$ where $h(t) = t\gamma'(t) - \gamma(t)$, respectively). One can ask a similar question for the BHTaCs.

Problem 4. *Let $\Gamma(t)$ be a convex even (or odd) plane curve. Find a necessary and sufficient condition for Γ so that the corresponding bilinear Hilbert transform is bounded from $L^2 \times L^2 \to L^1$.*

A first step might be to find a new proof of the monomial case, which bears more features from the classical tools for the Hilbert transform along curves. In Li's approach, he decomposes H_Γ along the critical points of the phase, namely the zeros of $(\xi t + \eta\gamma(t))' = 0$. Away from the critical points is the minor part, in which fast decay can be obtained using Fourier series and can be handled via classical tools from paraproducts. Near the critical points is the major part, in which one applies the method of stationary phase. Unlike the linear theory, the resulting decay is not sufficient to obtain any useful estimates and the oscillatory terms are critical in the bilinear theory. One also needs to work simultaneously in the time and phase plane, in which the problem can be reduced to certain bilinear restriction problems and certain trilinear oscillatory integrals respectively. By employing the method of TT^*, sharp $L^2 \times L^2 \times L^2$ estimates are obtained, which can only handle half of the major part (referred as the first major part). The difficulty in the other half (referred as the second major part) is overcome by the σ-uniformity trick. It will be significant to find a method completely based on TT^* to handle the second major parts, providing a possible entrance for classical tools (like Cotla-Stein's almost orthogonality principle) to the bilinear theory.

4.4 Multilinear Hilbert Transform Along Curves (MHTaCs)

While the boundedness of the multilinear Hilbert transform ($n \geq 3$) might be beyond the scope of current techniques in analysis, its curvature analogue might be somewhat more approachable.

Problem 5. *Let $\gamma_j : \mathbb{R} \to \mathbb{R}$ be smooth functions, for $1 \leq j \leq n$. Defined the MLHaC*

$$M(f_1, \dots, f_n)(x) = \text{p.v.} \int_{-\infty}^{\infty} \prod_{j=1}^{n} f_j(x - \gamma_j(t)) \frac{dt}{t}$$

Does M map $L^{p_1} \times \cdots \times L^{p_n}$ into L^r?

To the author's best knowledge, no bound is known for this operator for $n \geq 3$, even for the model case $\gamma_j(t) = t^j$. Again, the difficulty lies in the lack of systematic methods to explore the curvature in the multilinear setting. The classical L^2 theory becomes less and less efficient as the level of multi-linearity increases. All of this calls for further development for the theory of multilinear oscillatory integrals. For instance, the MHTaCs are intrinsically connected to the estimates of the form

$$\left| \int e^{i\lambda S(\xi_1, \dots, \xi_n)} \phi(\xi_1, \dots, \xi_n) g_{n+1}(\xi_1 + \cdots + \xi_n) \times \prod_{j=1}^{n} g_j(\xi_j) d\xi_1 \dots d\xi_n \right| \lesssim |\lambda|^{-\delta} \prod_{j=1}^{n+1} \|\hat{g}_j\|_{p_j}.$$

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