Topological Strings and Their Applications in Mathematics

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Abstract. In this note, we briefly review the mathematical structures of topological strings and their applications in mathematics. We focus on the discussions of integrality structures in topological strings and large N Chern-Simons/topological string duality. More precisely, prediction of the number of rational curves in quintic, LMOV conjectures of open topological strings and framed knots, derivation of Mariño-Vafa formula and interpretation of Rogers-Ramanujan identities by topological string will be described.

1. Introduction

For decades, we have witnessed the great development of string theory and its powerful impact on mathematics. There have been a lot of marvelous results revealed by string theory. It was found that the mysterious string duality deeply relate different aspects of mathematics. The topological sector of string theory leads to a simplified model of string theory, topological string theory. Mirror symmetry and large N Chern-Simons/topological string duality are two of most important dualities in topological string theory.

Two different topological twists of string theory give rise to A-model and B-model topological strings respectively, they are related by mirror symmetry. A-model is described by Gromov-Witten theory in mathematics [39], while B model is described by BCOV theory [11, 18] which can be regarded as a quantized version of variation of Hodge structures. The mirror symmetry predicts that the A-model on a Calabi-Yau 3-fold X is equivalent to the B-model on a mirror Calabi-Yau 3-fold $\hat{X}$. The first important application of the mirror symmetry is the discovery of the enumerative formula for the numbers of rational curves in a quintic Calabi-Yau 3-fold [16] which will be reviewed in Section 2.

The study of large N Chern-Simons/topological string duality was originated in physics by an idea that gauge theory should have a string theory explanation [37]. In 1992, Witten [88] constructed a topological string theory on cotangent bundle T^*M of a 3-manifold M with N D-branes to explain the $SU(N)$ Chern-Simons gauge theory on M. In 1998, Gopakumar and Vafa [36] proposed that the topological string theory on T^*S^3 with N D-branes constructed by Witten [88] is dual to the string theory on the resolved conifold $\hat{X}$ at large N. In 2000, Ooguri and Vafa [75] extended the above picture to explain the Chern-Simons theory on S^3 with a knot inside by open topological string theory on resolved conifold with a corresponding Lagrangian submanifold associated to this knot. The large N duality of Chern-Simons/topological string is very mysterious since it relates the topological invariants of three manifolds and knots to the Gromov-Witten invariant of Calabi-Yau 3-folds, this duality provides new insights both in physics and mathematics. For example, by considering a framed unknot in S^3, Mariño and Vafa [72] discovered a remarkable closed formula for Hodge integrals up to three Chern classes of Hodge bundle on moduli spaces of curves. This Mariño-Vafa formula, proved in [54, 74], has powerful applications in intersection theory of moduli space of curves. It implies Witten conjecture [87, 43], ELSV

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formula [26], λ_g-integrals [29] and several other interesting Hodge integral identities. We will review the Chern-Simons theory, large N duality and LMOV conjecture for framed knots in Section 4. The derivation of Mariño-Vafa formula will be illustrated in Section 5.

On the other hand, Ooguri and Vafa [75] reformulated the open topological string generating function in terms of new integral invariants capturing the spectrum of M2 branes ending on M5 branes embedded in the resolved conifold. Later, Labastida, Marinõ and Vafa [58] refined the analysis of [75] and conjectured a more precise integrality structure. Such integrality structures imply the infinite product structures for open topological string partition functions which will be showed in Section 3. In particular, in Section 6, by studying the open topological string partition function on the trivial toric Calabi-Yau 3-fold $\mathbb{C}^3$, we show that the corresponding infinite product formula gives a "1-deformed Rogers-Ramanujan formula" which includes the celebrated Rogers-Ramanujan identities [38] as the special cases. Thus it provides a topological string interpretation of the Rogers-Ramanujan identities.

2. Closed Topological Strings

In quantum mechanics, in order to compute the amplitude of a particle propagating in a space X, we should consider the contributions of all the maps from all the graphs to X. The basic idea of string theory is that the particle is replaced by a string, all the possible paths of a string are also replaced by all the surfaces in X. Similarly, the string amplitude should be computed by studying all the contributions of the maps from surfaces to X [4]. Superstring theory is the string theory with supersymmetry incorporated. Topological string is a topological variant of superstring theory by modifying the supersymmetry. When X is a Calabi-Yau manifold, we have two ways to obtain topological string theory, the A-model depending on the Kähler structure of X, and the B-model depending on the complex structure of X [19].

2.1 Closed Topological String Partition Function

The mathematical theory for A-model is described by Gromov-Witten theory [39]. Let $\overline{\mathcal{M}}_{g,n}(X,Q)$ be the moduli space of stable maps $(f,\Sigma_g,p_1,..,p_n)$, where $f:\Sigma_g\to X$ is a holomorphic map from the nodal curve Σ_g to the Kähler manifold X with $f_*([\Sigma_g])=Q\in H_2(X,\mathbb{Z})$. In general, $\overline{\mathcal{M}}_{g,n}(X,Q)$ carries a virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(X,Q)]^{vir}$ in the sense of [12, 62], whose virtual dimension is given by:

$$\mathrm{vdim}[\overline{\mathcal{M}}_{g,n}(X,Q)]^{vir}=\int_Q c_1(X)+(\dim X-3)(1-g)+n.$$

When X is a Calabi-Yau 3-fold, i.e. $c_1(X)=0$, then $\mathrm{vdim}[\overline{\mathcal{M}}_{g,0}(X,Q)]^{vir}=0$. The genus g, degree Q Gromov-Witten invariants of X is defined by

$$K^X_{g,Q}=\int_{[\overline{\mathcal{M}}_{g,0}(X,Q)]^{vir}}1.$$

In A-model, the genus g closed free energy $F^X_g(\omega)$ of X is the generating function of Gromov-Witten invariants $K^X_{g,Q}$

$$F^X_g(\omega)=\sum_{Q\neq 0}K^X_{g,Q}e^{-Q\cdot\omega},$$

where ω is the complexified Kähler parameter of X. Let

$$F^X(g_s,\omega)=\sum_{g\geq 0}g_s^{2g-2}F^X_g(\omega)\quad Z^X(g_s,\omega)=\exp(F^X(g_s,\omega))$$

be the total free energy and partition function, where g_s is the string coupling constant. The central question in topological string theory is how to compute this partition function $Z^X(g_s,\omega)$ or every Gromov-Witten invariants $K^X_{g,Q}$.

2.2 Gopakumar-Vafa Formula

Usually, the Gromov-Witten invariants $K^X_{g,Q}$ are rational numbers. In 1998, Gopakumar and Vafa [35] expressed the total free energy $F^X(g_s,\omega)$ in terms of the generating function of integer-valued invariants $N^X_{g,Q}$ obtained by counting BPS states in M theory:

$$\begin{aligned}(1)\quad F^X(g_s,\omega)&=\sum_{g\geq 0}g_s^{2g-2}\sum_{Q\neq 0}K^X_{g,Q}e^{-Q\cdot\omega}\\&=\sum_{g\geq 0,d\geq 1}\sum_{Q\neq 0}\frac{1}{d}N^X_{g,Q}\left(2\sin\frac{dg_s}{2}\right)^{2g-2}e^{-dQ\cdot\omega}.\end{aligned}$$

Obviously, genus 0 part of the formula (1) gives

$$\sum_{Q\neq 0}K^X_{0,Q}e^{-Q\cdot\omega}=\sum_{Q\neq 0}N^X_{0,Q}\sum_{d\geq 1}\frac{1}{d^3}e^{-d\cdot Q\omega}.$$

which is called the multiple covering formula [5].

Remark 2.1. The invariants $N^X_{g,Q}$ are called Gopakumar-Vafa invariants in literatures. A central problem in topological string is how to define the Gopakumar-Vafa invariants directly. We refer to [41, 48, 42, 71] for some approaches in this direction.

2.3 Example 1: Quintic X_5

Let $X_5\subset\mathbb{C}P^4$ be a nonsingular quintic hypersurface, which is a compact Calabi-Yau 3-fold. In this case, the genus 0 free energy and multiple covering formula gives

$$(2)\qquad F^{X_5}_0(T)=\sum_{d\geq 1}K^{X_5}_{0,d}e^{dT}=\sum_{d\geq 1}N^{X_5}_{0,d}\sum_{k\geq 1}\frac{1}{k^3}e^{kdT}.$$

Mirror symmetry transfers the difficult computation of $F_0^{X_5}(T)$ to the simple computation in B-model on mirror quintic family $\hat{X}_t$ which is dealt with by the theory of variation of Hodge structures [69]. Let

$$\partial_t^4 f(t) - 5e^t(5\partial_t+1)\cdots(5\partial_t+4)f(t) = 0$$

be the associated Picard-Fuchs equation of the family $\hat{X}_t$, and define

$$\sum_{j=0}^{\infty} I_j(t)k^j = e^{kt}\sum_{d=0}^{\infty} e^{dt}\frac{\prod_{j=1}^{5d}(5k+j)}{\prod_{j=1}^{d}(k+j)^5}. \tag{3}$$

By Frobenius method, one can see that $\{I_0(t), I_1(t), I_2(t), I_3(t)\}$ forms a basis of solutions of the equation (3). Let $J_k(t) = \frac{I_k(t)}{I_0(t)}$ and $T = J_1(t)$, in their celebrated work [16], Candelas, de la Ossa, Green and Parkes made the following prediction by using mirror symmetry

$$F_0^{X_5}(T) = \frac{5}{2}(J_1(t)\cdot J_2(t) - J_3(t)). \tag{4}$$

Therefore, one can compute all $K_{0,d}^{X_5}$ and $N_{0,d}^{X_5}$ by formula (4), for examples $N_{0,1}^{X_5} = 2875$ and $N_{0,2}^{X_5} = 609250$. They conjectured that these numbers gave the numbers of rational curves of degree d in X_5, while it is a difficult question in classical enumerative geometry. Topological string theory provides an effective way to compute these numbers. From then on, topological string theory attracted the interests of algebraic geometers. Kontsevich [44] introduced the moduli space of stable maps and the method of localization [2] to compute the genus 0 Gromov-Witten invariant $K_{0,d}^{X_5}$. Then the Candelas-de la Ossa-Green-Parkes formula (4) was later proved by Lian-Liu-Yau [52] and Givental [33]. We refer to [19] for more details.

As to the genus 1 free energy $F_1^{X_5}(T) = \sum_{d\geq 1} K_{1,d}^{X_5}e^{dT}$, Bershadsky-Cecotti-Ooguri-Vafa [11] obtained the following formula also by the computations on mirror manifold

(5)
$$F_1^{X_5}(T) = \frac{25}{12}(J_1(t)-t) - \log\left(I_0(t)^{31/3}(1-5^5e^t)^{1/12}J_1'(t)^{1/2}\right).$$

Formula (5) was proved by Zinger [92].

For higher genus free energy $F_g^{X_5}(T)$, Huang-Klemm-Quackenbush [40] have made the predictions for g up to 51 by using the BCOV theory [11] and mirror symmetry. To calculate all genus Gromov-Witten invariants of the quintic X_5 is a central problem in Gromov-Witten theory, see [20, 17, 30] for some related works.

2.4 Example 2: Toric Calabi-Yau 3-Folds

A toric Calabi-Yau 3-fold is a toric variety with trivial canonical bundle [13] which is noncompact. Because of its toric symmetry, the geometric information of a toric Calabi-Yau 3-fold is encoded in a trivalent graph named "toric diagram" [9] which is the gluing of some trivalent vertices. The topological string partition function Z^X of a toric Calabi-Yau 3-fold X can be computed by using the method of topological vertex [9, 53]. The integrality of the invariants $N_{g,Q}^X$ for toric Calabi-Yau 3-fold X in Gopakumar-Vafa formula (1) was later proved by P. Peng [76] and Konishi [45].

3. Open Topological Strings

Let us now consider the open sector of topological A-model of a Calabi-Yau 3-fold X with a submanifold $\mathcal{D}$, we assume dim $H_1(\mathcal{D},\mathbb{Z}) = 1$ for convenience of the following discussion. The open sector topological A-model can be described by holomorphic maps ϕ from open Riemann surface of genus g with l-holes $\Sigma_{g,l}$ to X, with Dirichlet condition specified by $\mathcal{D}$. These holomorphic maps are called open string instantons. More precisely, an open string instanton is a holomorphic map $\phi: \Sigma_{g,h} \to X$ such that $\partial\Sigma_{g,l} = \cup_{i=1}^l \mathcal{C}_i \to \mathcal{D} \subset X$ where the boundary $\partial\Sigma_{g,l}$ of $\Sigma_{g,l}$ consists of l connected components $\mathcal{C}_i$ mapped to Lagrangian manifold $\mathcal{D}$ of X. Therefore, an open string instanton ϕ is described by the following two different kinds of data: the first is "bulk part" which is given by $\phi_*[\Sigma_{g,l}] = Q \in H_2(X,\mathcal{L})$, and the second is "boundary part" which is given by $\phi_*[\mathcal{C}_i] = w_i\gamma$, for $i = 1,..l$, where γ is the generator of $H_1(\mathcal{D},\mathbb{Z})$ and $w_i^\alpha \in \mathbb{Z}$. Let $w = (w_1,...,w_l) \in \mathbb{Z}^l$, "open Gromov-Witten invariants" $K_{w,g,Q}^X$ are determined by the data w, Q in the genus g. See [63, 47] for mathematical aspects of defining these invariants in special cases.

We take all $w_i \geq 1$ as in [72], and use the notations of partitions and symmetric functions [67]. Denote by $\mathcal{P}$ the set of all partitions including the partition 0 of 0, and $\mathcal{P}_+$ the set of nonzero partitions. Let $\mathbf{x} = \{x_1, x_2, ...\}$ be the set of infinitely many independent variables. For $n \geq 0$, let $p_n(\mathbf{x}) = \sum_{i\geq 1} x_i^n$ be a power sum symmetric function. For a partition $\mu \in \mathcal{P}_+$, let $p_\mu(\mathbf{x}) = \prod_{i=1}^l p_{\mu_i}(\mathbf{x})$. The total free energy and partition function of open topological string on X are expressed in the following forms:

$$F_{str}^{(X,\mathcal{D})}(g_s,\omega,\mathbf{x}) = -\sum_{g\geq 0}\sum_{\mu\in\mathcal{P}_+}\frac{\sqrt{-1}^{l(\mu)}}{|Aut(\mu)|}g_s^{2g-2+l(\mu)} \times \sum_{Q\neq 0} K_{\mu,g,Q}^{(X,\mathcal{D})} e^{-Q\cdot\omega}p_\mu(\mathbf{x})$$

$$Z_{str}^{(X,\mathcal{D})}(g_s,\omega,\mathbf{x}) = \exp(F_{str}^{(X,\mathcal{D})}(g_s,\omega,\mathbf{x})).$$

The central problem in open topological string theory is how to calculate the partition function $Z_{str}^{(X,\mathcal{D})}(g_s,\omega,\mathbf{x})$ or the open Gromov-Witten invariants $K_{\mu,g,Q}^{(X,\mathcal{D})}$. In the case of compact Calabi-Yau 3-folds, such as the quintic X_5, there are only a few works devoted to the study of its open Gromov-Witten invariants, for example, a complete calculation of the disk invariants of X_5 with boundary in a real Lagrangian was given in [77].

Suppose X is a toric Calabi-Yau 3-fold, and $\mathcal{D}$ is a special Lagrangian submanifold named as Aganagic-Vafa A-brane in the sense of [6, 8]. The open string partition function $Z_{str}^{(X,\mathcal{D})}(g_s,\omega,\mathbf{x})$ can be computed by using the method of topological vertex [9, 53] and the method of topological recursion developed by Eynard and Orantin [27]. The second approach was first proposed by Mariño [68], and studied further by Bouchard, Klemm, Mariño and Pasquetti [14], the equivalence of the two methods was proved in [28, 31].

The open Gromov-Witten invariants $K_{\mu,g,Q}^X$ are rational numbers in general. Just as in the closed string case [35], the open topological strings compute the partition function of BPS domain walls in a related superstring theory [75]. It follows that $F_{str}^{(X,\mathcal{D})}(g_s,\omega,\mathbf{x})$ also carries an integral expansion. This integrality structure was further refined in [56, 57, 58].

We introduce the variables

$$q = e^{\sqrt{-1}g_s},\ a = e^{-\omega}. \tag{6}$$

Let $f_\lambda(q,a)$ be a function determined by the following formula

$$F_{str}^{(X,\mathcal{D})}(g_s,\omega,\mathbf{x}) = \sum_{d=1}^{\infty}\frac{1}{d}\sum_{\lambda\in\mathcal{P}_+} f_\lambda(q^d,a^d)s_\lambda(\mathbf{x}^d), \tag{7}$$

where $s_\lambda(\mathbf{x})$ is the Schur symmetric functions [67]. Then $f_\lambda(q,a)$ has the following structure [58]:

$$f_\lambda(q,a) = \sum_{g=0}^{\infty}\sum_{Q\neq 0}\sum_{|\mu|=|\lambda|} M_{\lambda\mu}(q)N_{\mu,g,Q}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{2g-2}a^Q, \tag{8}$$

where $N_{\mu,g,Q}$ are integers which compute the net number of BPS domain walls [75] and $M_{\lambda\mu}(q)$ is defined by

$$M_{\lambda\mu}(q) = \sum_\nu \frac{\chi_\lambda(C_\nu)\chi_\mu(C_\nu)}{\mathfrak{z}_\nu}\prod_{j=1}^{l(\nu)}(q^{-\nu_j/2}-q^{\nu_j/2}), \tag{9}$$

where $\chi_\nu(C_\mu)$ is the character of an irreducible representation of the symmetric group. For convenience, we usually introduce the new integers $n_{\mu,g,Q} = \sum_\nu \chi_\nu(C_\mu)N_{\nu,g,Q}$. These integers $N_{\mu,g,Q}$ and $n_{\mu,g,Q}$ are both called LMOV invariants. The expression (8) can be rewritten as follow:

$$\begin{aligned} f_\lambda(q,a) &= \sum_{g\geq 0}\sum_{Q\neq 0}\sum_{\mu\in\mathcal{P}}\frac{\chi_\lambda(C_\mu)}{\mathfrak{z}_\mu}n_{\mu,g,Q} \\ &\times\prod_{j=1}^{l(\mu)}(q^{-\frac{\mu_j}{2}}-q^{\frac{\mu_j}{2}})(q^{-\frac{1}{2}}-q^{\frac{1}{2}})^{2g-2}a^Q. \end{aligned}$$

Combing the formula (7), we obtain the following multiple covering formula for open string illustrated in [72]:

$$\begin{aligned} &\sum_{g\geq 0}\sum_{Q\neq 0} g_s^{2g-2+l(\mu)}K_{\mu,g,Q}^{(X,\mathcal{D})}a^Q \\ &= \sum_{g\geq 0}\sum_{Q\neq 0}\sum_{d|\mu}\frac{(-1)^{l(\mu)+g}}{\prod_{i=1}^{l(\mu)}\mu_i}d^{l(\mu)-1}n_{\mu/d,g,Q} \\ &\times\prod_{j=1}^{l(\mu)}(2\sin\frac{\mu_j g_s}{2})(2\sin\frac{dg_s}{2})^{2g-2}a^{dQ}. \end{aligned} \tag{10}$$

Hence we have the following integrality structure conjecture which is called the Labastida-Mariño-Ooguri-Vafa (LMOV) conjecture for open string.

Conjecture 3.1 (LMOV conjecture for open string)**.** *Let $F_\mu^{(X,\mathcal{D})}(g_s,\omega)$ be the generating function determined by*

$$F_{str}^{(X,\mathcal{D})}(g_s,\omega,\mathbf{x}) = \sum_\mu F_\mu^{(X,\mathcal{D})}(g_s,\omega)p_\mu(\mathbf{x}),$$

then $F_\mu^{(X,\mathcal{D})}(g_s,\omega)$ has the integral expansion as in the right-hand side of the formula (10).

Remark 3.2. In the original paper [75], Ooguri and Vafa proposed the following structure for $f_\lambda(q,a)$, i.e. there exist integers $N_{\lambda,i,j}$ such that

$$f_\lambda(q,a) = \frac{\sum_{i,j}N_{\lambda;i,j}a^{i/2}q^{j/2}}{q^{1/2}-q^{-1/2}}. \tag{11}$$

The integral expansion form (8) of $f_\lambda(q,a)$ was refined in [58]. In fact, the Ooguri-Vafa invariants $N_{\lambda;i,j}$ are certain linear sums of the LMOV invariants $N_{\mu,g,Q}$.

Remark 3.3. The integral expansions (8) and (11) lead to certain infinite product formulas for open topological string partition function, see [60] for the infinite product formula by using the expansion formula (8). Similarly, expansion formula (11) implies the following infinite product formula for open topological string partition function

$$\begin{aligned} &Z_{str}^{(X,\mathcal{D})} \\ &= \prod_{\mu\neq 0}\prod_{j_1,\ldots,j_{l(\mu)}\geq 1}\prod_{l\geq 0}\prod_{i,k}\left(1-a^{\frac{i}{2}}q^{\frac{k+1}{2}+l}x_{j_1}^{\mu_1}\cdots x_{j_{l(\mu)}}^{\mu_{l(\mu)}}\right)^{\frac{\sum_\lambda \chi_\lambda(C_\mu)N_{\lambda,i,k}}{\mathfrak{z}_\mu}}. \end{aligned} \tag{12}$$

Formula (12) is closely related to the infinite product formula appearing in the studying of motivic Donaldson-Thomas invariants [49, 50, 78]. An example for the relationship of open topological string partition function on $(\mathbb{C}^3,D_\tau)$ and Poincare polynomial of the cohomological Hall algebra of a symmetric quiver was provided in [66]. Application of the formula (12) to interpret Rogers-Ramanujan identities will be illustrated in the following Section 6.

4. Large N Duality

4.1 Quantum Invariants

In his seminal paper [86], E. Witten introduced a topological invariant of a 3-manifold M defined as the partition function in quantum Chern-Simons theory. Let G be a compact gauge group which is a Lie group, and M be an oriented three-dimensional manifold. Let $\mathcal{A}$ be a $\mathfrak{g}$-valued connection on M where $\mathfrak{g}$ is the Lie algebra of G. Chern-Simons [22] action is given by

$$S(\mathcal{A}) = \frac{k}{4\pi}\int_M Tr\left(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)$$

where k is an integer called the level.

Chern-Simons partition function is defined as the path integral in quantum field theory

$$Z^G(M;k) = \int e^{iS(A)} D\mathcal{A}$$

where the integral is over the space of all $\mathfrak{g}$-valued connections $\mathcal{A}$ on M. Although the definition is not rigorous in mathematics, Witten [86] developed some techniques to deal with such invariants.

If the 3-manifold M contains a knot $\mathcal{K}$, we define $W_R(\mathcal{K}) = Tr_R Hol_{\mathcal{K}}(\mathcal{A}) = Tr_R(P\exp\oint_{\mathcal{K}}\mathcal{A})$ to be the trace of holonomy along $\mathcal{K}$ taken in representation R. Then Witten's invariant of the pair $(M,\mathcal{K})$ is given by

$$Z^G(M,\mathcal{K};R;k) = \int e^{iS(\mathcal{A})}\prod_{j=1}^{L} W_R(\mathcal{K})D\mathcal{A}.$$

We often use the following normalization form

$$P_R^G(M,\mathcal{K};k) = \frac{Z^G(M,\mathcal{K};R;k)}{Z^G(M;k)}. \tag{13}$$

When $M = S^3$ and the Lie algebra of G is a semisimple Lie algebra, Reshetikhin and Turaev [79, 80] developed a systematic way to constructed the above invariant (13) by using the representation theory of quantum groups. Their construction led to the definition of colored HOMFLY-PT invariants [57, 65], which can be viewed as the large N limit of the quantum $U_q(sl_N)$ invariants. Usually, we use the notation $W_{\lambda^1,\dots,\lambda^L}(\mathcal{L};q,a)$ to denote the (framing-independent) colored HOMFLY-PT invariants for a (oriented) link $\mathcal{L} = \bigsqcup_{j=1}^{L}\mathcal{K}_j$, where each component $\mathcal{K}_j$ is colored by an irreducible representation V_{λ^j} of $U_q(sl_N)$. Some basic structures for $W_{\lambda^1,\dots,\lambda^L}(\mathcal{L};q,a)$ were proved in [59, 60, 90]. It is difficult to obtain an explicit formula for $W_{\lambda^1,\dots,\lambda^L}(\mathcal{L};q,a)$ in any irreducible representations λ^i. We refer to [65] for an explicit formula for torus links, and a series of works due to Morozov et al. [70] and Nawata et al. [73] for the conjectural formulas for twist knots. In particular, we have the following explicit formula for a trivial knot (unknot) U:

$$W_\lambda(U;q,a) = \prod_{x\in\lambda}\frac{a^{1/2}q^{cn(x)/2} - a^{-1/2}q^{-cn(x)/2}}{q^{h(x)/2} - q^{-h(x)/2}}.$$

For a box $x = (i,j)\in\lambda$, the hook length and content are defined to be $hl(x) = \lambda_i + \lambda_j^t - i - j + 1$ and $cn(x) = j - i$ respectively. We let $W_\lambda(q)$ be the coefficient of $a^{\frac{|\lambda|}{2}}$ in $W_\lambda(U;q,a)$, i.e.

$$W_\lambda(q) = [a^{\frac{|\lambda|}{2}}]W_\lambda(U;q,a) = \prod_{x\in\lambda}\frac{q^{cn(x)/2}}{q^{h(x)/2} - q^{-h(x)/2}}. \tag{14}$$

4.2 Large N Duality

In another fundamental work of Witten [88], $SU(N)$ Chern-Simons gauge theory on a three-manifold M was interpreted as an open topological string theory on T^*M with N topological branes wrapping M inside T^*M. Furthermore, Gopakumar and Vafa [36] conjectured that the large N limit of $SU(N)$ Chern-Simons gauge theory on S^3 is equivalent to the closed topological string theory on the resolved conifold. Furthermore, Ooguri and Vafa [75] generalized the above construction to the case of S^3 with a knot $\mathcal{K}$ inside. They introduced the Chern-Simons partition function $Z_{CS}^{(S^3,\mathcal{K})}(q,a,\mathbf{x})$ for $(S^3,\mathcal{K})$ which is a generating function of colored HOMFLY-PT invariants in all irreducible representations.

$$Z_{CS}^{(S^3,\mathcal{K})}(q,a,\mathbf{x}) = \sum_{\lambda\in\mathcal{P}} W_\lambda(\mathcal{L},q,a)s_\lambda(\mathbf{x}). \tag{15}$$

Ooguri and Vafa [75] conjectured that for any knot $\mathcal{K}$ in S^3, there exists a corresponding Lagrangian submanifold $\mathcal{D}_{\mathcal{K}}$, such that the Chern-Simons partition function $Z_{CS}^{(S^3,\mathcal{K})}(q,a,\mathbf{x})$ is equal to the open topological string partition function $Z_{str}^{(X,\mathcal{D}_{\mathcal{K}})}(g_s,\omega,\mathbf{x})$ on $(X,\mathcal{D}_{\mathcal{K}})$, under the variable changes (6). They have established this duality for the case of a trivial knot U in S^3. The link case was further discussed in [58]. The large N Chern-Simons/topological string duality predicts, for any link $\mathcal{L}$ in S^3, there exist a corresponding Lagrangian submanifold $\mathcal{D}_{\mathcal{L}}$ such that

(16)

$$Z_{CS}^{(S^3,\mathcal{L})}(q,a,\mathbf{x}) = Z_{str}^{(\hat{X},\mathcal{D}_{\mathcal{L}})}(g_s,\omega,\mathbf{x}) \quad \text{for} \quad q = e^{\sqrt{-1}g_s}, a = e^{-\omega}.$$

To establish the large N duality in mathematics, first we should find a way to construct the Lagrangian submanifold $\mathcal{D}_{\mathcal{L}}$ corresponding to the link $\mathcal{L}$ in geometry. See [58, 46, 84, 24] for such constructions for some special links. Then, we need to develop some methods to compute the open sting partition function under this geometry. For the trivial knot (i.e. unknot U) in S^3, the dual open string partition function was computed by J. Li and Y. Song [63] and S. Katz and C.-C. M. Liu [47].

4.3 Integrality of the Quantum Invariants

The large N duality (16) together with the integrality structure conjecture for open topological string, i.e. Conjecture 3.1 imply that the Chern-Simons partition function $Z_{CS}^{(S^3,\mathcal{L})}(q,a,\mathbf{x})$ carries the integrality structure which is named as the LMOV conjecture for links in [59]. Furthermore, as mentioned previously, the large N duality was generalized to the case of framed knot $\mathcal{K}_\tau$ with framing $\tau \in \mathbb{Z}$ in [72], with the Chern-Simons partition $Z_{CS}^{(S^3,\mathcal{K}_\tau)}$ for framed knot $\mathcal{K}_\tau$ given by formula (17). For convenience, we only formulate the LMOV conjecture for framed knot $\mathcal{K}_\tau$ in following, although LMOV conjecture should also holds for any framed link, see [61].

Conjecture 4.1 (LMOV conjecture for framed knots or framed LMOV conjecture). *Let*

$$F_{CS}^{(S^3,\mathcal{K}_\tau)}(q,a,\mathbf{x}) = \log Z_{CS}^{(S^3,\mathcal{K}_\tau)}(q,a,\mathbf{x})$$

be the Chern-Simons free energy for a framed knot $\mathcal{K}_\tau$ in S^3. Then there exist functions $f_\lambda(\mathcal{K}_\tau;q,a)$ such that

$$F_{CS}^{(S^3,\mathcal{K}_\tau)}(q,a,\mathbf{x}) = \sum_{d=1}^{\infty}\frac{1}{d}\sum_{\lambda\in\mathcal{P}_+} f_\lambda(\mathcal{K}_\tau;q^d,a^d)s_\lambda(\mathbf{x}^d).$$

Let $\hat{f}_\mu(\mathcal{K}_\tau;q,a) = \sum_\lambda f_\lambda(\mathcal{K}_\tau;q,a)M_{\lambda\mu}(q)^{-1}$, where $M_{\lambda\mu}(q)$ is given by formula (9). Let $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, then for any $\mu \in \mathcal{P}_+$, there are integers $N_{\mu,g,Q}(\tau)$ such that

$$\hat{f}_\mu(\mathcal{K}_\tau;q,a) = \sum_{g\geq 0}\sum_{Q} N_{\mu,g,Q}(\tau) z^{2g-2}a^Q \in z^{-2}\mathbb{Z}[z^2,a^{\pm\frac{1}{2}}].$$

K. Liu and P. Peng [59] first studied the mathematical structures of LMOV conjecture for general links (as to the Chern-Simons partition (15)), which is equivalent to the framed LMOV conjecture for any links in framing zero. They provided a proof for this case by using cut-and-join analysis and cabling technique [65]. Motivated by the work [72], K. Liu and P. Peng [61] formulated the framed LMOV conjecture for any links. In [21], together with Q. Chen, K. Liu and P. Peng, we developed the ideas in [61] to study the mathematical structures hidden in framed LMOV conjecture and formulate congruence skein relations of colored HOMFLY-PT invariants.

5. Mariño-Vafa Formula

We have mentioned that Mariño and Vafa [72] generalized the large N duality to the case of knot with arbitrary framing. They studied carefully and established the large N duality between a framed unknot in S^3 and the open string theory on resolved conifold with AV-brane with help of the localization computations in [47]. By comparing the coefficient of the highest degree of Kähler parameter in this duality, they derived a remarkable Hodge integral identity which now is called the Mariño-Vafa formula. Two different mathematical proofs for Mariño-Vafa formula were given in [54] and [74] respectively.

Now, we describe the derivation of Mariño-Vafa formula more precisely. For a framed knot $\mathcal{K}_\tau$ with framing $\tau \in \mathbb{Z}$, we define the framed colored HOMFLYPT invariants $\mathcal{K}_\tau$ as follow,

$$\mathcal{H}_\lambda(\mathcal{K}_\tau,q,a) = (-1)^{|\lambda|\tau} q^{\frac{\kappa_\lambda \tau}{2}} W_\lambda(\mathcal{K},q,a),$$

where $\kappa_\lambda = \sum_{i=1}^{l(\lambda)}\lambda_i(\lambda_i - 2i + 1)$. The Chern-Simon partition function for $(S^3,\mathcal{K}_\tau)$ is given by

$$Z_{CS}^{(S^3,\mathcal{K}_\tau)}(q,a;\mathbf{x}) = \sum_{\lambda\in\mathcal{P}} \mathcal{H}_\lambda(\mathcal{K}_\tau,q,a)s_\lambda(\mathbf{x}). \tag{17}$$

In particular, for the framed unknot U_τ, we define the coefficient of $a^{\frac{|\lambda|}{2}}$ in $\mathcal{H}_\lambda(U_\tau,q,a)$ as follow

$$\begin{aligned}\mathcal{H}_\lambda(q;\tau) &:= [a^{\frac{|\lambda|}{2}}]\mathcal{H}_\lambda(U_\tau,q,a)\\ &= (-1)^{|\lambda|\tau} q^{\frac{\kappa_\lambda\tau}{2}} W_\lambda(q),\end{aligned} \tag{18}$$

where $W_\lambda(q)$ is given by formula (14).

Let $\hat{X} := \mathcal{O}(-1)\oplus\mathcal{O}(-1) \to \mathbb{P}^1$ be the resolved conifold, and D_τ be the corresponding AV-brane which is the large N duality of the framed unknot U_τ in S^3. The open string partition function for $(\hat{X},D_\tau)$ has the following structure

$$\begin{aligned}&Z_{str}^{(\hat{X},D_\tau)}(g_s,a;\mathbf{x})\\ &= \exp\left(-\sum_{g\geq 0,\mu}\frac{\sqrt{-1}^{l(\mu)}}{|Aut(\mu)|} g_s^{2g-2+l(\mu)} F_{\mu,g}^{\tau}(a)p_\mu(\mathbf{x})\right)\end{aligned}$$

where $F_{\mu,g}^\tau(a) = \sum_{Q\in\mathbb{Z}/2} K_{\mu,g,Q}^\tau a^Q$ and $K_{\mu,g,Q}^\tau$ is the open Gromov-Witten invariants defined by S. Katz and C.-C. Liu [47]:

$$K_{\mu,g,Q}^\tau = \int_{[\mathcal{M}_{g,l(\mu)}(D^2,S^1|2Q,\mu_1,\ldots,\mu_l)]} e(\mathcal{V}).$$

In particular, when $Q = \frac{|\mu|}{2}$, the computations in [47] show

$$\begin{aligned}&K_{\mu,g,\frac{|\mu|}{2}}^\tau = (-1)^{|\mu|\tau}(\tau(\tau+1))^{l(\mu)-1}\\ &\prod_{i=1}^{l(\mu)}\frac{\prod_{j=1}^{\mu_i-1}(\mu_i\tau+j)}{(\mu_i-1)!}\int_{\overline{\mathcal{M}}_{g,l(\mu)}}\frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_j\psi_j)}\end{aligned}$$

where $\Lambda_g^\vee(\tau) = \tau^g - \lambda_1\tau^{g-1} + \cdots + (-1)^g\lambda_g$. Large N duality in this case predicts

$$Z_{CS}^{(S^3,U_\tau)}(q,a;\mathbf{x}) = Z_{str}^{(\hat{X},D_\tau)}(g_s,a;\mathbf{x})$$

where $q=e^{ig_s}$. Considering the coefficients of $a^{\frac{|\mu|}{2}}$ in the following equality:

$$[p_\mu(\mathbf{x})]\log Z_{CS}^{(S^3,U_\tau)}(q,a;\mathbf{x})=[p_\mu(\mathbf{x})]\log Z_{str}^{(\hat{X},D_\tau)}(g_s,a;\mathbf{x}),$$

we obtain the Mariño-Vafa formula:

$$\sum_{g\geq 0}\frac{\sqrt{-1}^{l(\mu)}}{|Aut(\mu)|}g_s^{2g-2+l(\mu)}K^\tau_{\mu,g,\frac{|\mu|}{2}}$$
$$=\sum_{n\geq 1}\frac{(-1)^n}{n}\sum_{\bigcup_{j=1}^n\mu^j=\mu}\prod_{j=1}^n\sum_{\nu^j=\mu^j}\sum_{|\nu^j|=|\mu^j|}\frac{\chi_{\nu^j}(C_{\mu_j})}{z_\mu}q^{\frac{\tau\kappa_{\nu^j}}{2}}\mathcal{H}_{\nu^j}(q).$$

We construct the following generating function which in fact gives the open topological string free energy of $(\mathbb{C}^3,D_\tau)$:

$$F_{str}^{(\mathbb{C}^3,D_\tau)}(g_s;\mathbf{x})=\sum_{\mu\in\mathcal{P}^+}\sum_{g\geq 0}\frac{\sqrt{-1}^{l(\mu)}}{|Aut(\mu)|}g_s^{2g-2+l(\mu)}K^\tau_{\mu,g,\frac{|\mu|}{2}}p_\mu(\mathbf{x}).$$

We also consider the generating function:

$$F^\tau(q,\mathbf{x})=\sum_{\mu\in\mathcal{P}}\sum_{n\geq 1}\frac{(-1)^n}{n}$$
$$\times\sum_{\bigcup_{j=1}^n\mu^j=\mu}\prod_{j=1}^n\sum_{\nu^j=\mu^j}\sum_{|\nu^j|=|\mu^j|}\frac{\chi_{\nu^j}(C_{\mu_j})}{z_\mu}\mathcal{H}_{\nu^j}(q;\tau)p_\mu(\mathbf{x}).$$

Then the proof of Mariño-Vafa formula is equivalent to show that

$$F_{str}^{(\mathbb{C}^3,D_\tau)}(g_s;\mathbf{x})=F^\tau(q;\mathbf{x}). \tag{19}$$

Identity (19) was proved in [54] by showing that both sides of it satisfy the following cut-and-join equation:

$$\frac{\partial\mathcal{C}}{\partial\tau}=-\frac{g_s}{2}\sum_{i,j\geq 1}\left((i+j)p_ip_j\frac{\partial\mathcal{C}}{\partial p_{i+j}}\right.$$
$$\left.+ijp_{i+j}\frac{\partial^2\mathcal{C}}{\partial p_i\partial p_j}+ijp_{i+j}\frac{\partial\mathcal{C}}{\partial p_i}\frac{\partial\mathcal{C}}{\partial p_j}\right)$$

and when $\tau=0$, identity (19) holds.

Mariño-Vafa formula has powerful applications in intersection theory of moduli space of curves [3]. It implies the famous Witten conjecture [87, 43], ELSV formula [26], and various Hodge integral identities, see [55, 51, 23, 89] for discussing the applications of Mariño-Vafa formula.

6. Deformed Rogers-Ramanujan Formula

The following two Rogers-Ramanujan identities

$$\sum_{n\geq 0}\frac{q^{n^2}}{(1-q)\cdots(1-q^n)}=\prod_{n\geq 0}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})} \tag{20}$$

$$\sum_{n\geq 0}\frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)}=\prod_{n\geq 0}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})} \tag{21}$$

were first discovered by Rogers [81], and then rediscovered by Ramanujan [38], Schur [82] and Baxter [10]. Now, there have been many different proofs and interpretations for them [1, 32, 64, 15, 83]. We refer to [34] for most modern understanding of the Rogers-Ramanujan identities.

We will show that the Rogers-Ramanujan identities can be interpreted by open topological string theory on $(\mathbb{C}^3,D_\tau)$ in framing $\tau=1$. Furthermore, we hope to find the topological string interpretations for more general Rogers-Ramanujan type identities [34] in near future. We refer to [91] for more details about the results in this section.

By Mariño-Vafa formula (19), we have

$$Z_{str}^{(\mathbb{C}^3,D_\tau)}(g_s,\mathbf{x})=\exp\left(F_{str}^{(\mathbb{C}^3,D_\tau)}(g_s,\mathbf{x})\right)=\sum_{\lambda\in\mathcal{P}}\mathcal{H}_\lambda(q;\tau)s_\lambda(\mathbf{x}).$$

In particular, we let

$$\begin{aligned}Z_\tau(q,x)&:=Z_{str}^{(\mathbb{C}^3,D_\tau)}(g_s,\mathbf{x}=(x,0,0,..))\\&=\sum_{n\geq 0}\mathcal{H}_n(q;\tau)x^n\\&=\sum_{n\geq 0}\frac{(-1)^{n(\tau-1)}q^{\frac{n(n-1)}{2}\tau+\frac{n^2}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)}x^n,\end{aligned}$$

where in the last "=" formulas (18) and (14) are used.

In this case, the infinite product formula (12) is reduced to the following formula:

$$Z_\tau(q,x)=\prod_{m\geq 1}\prod_{k\in\mathbb{Z}}\prod_{l\geq 0}\left(1-q^{\frac{k+1}{2}+l}x^m\right)^{N_{m,k}(\tau)} \tag{22}$$

for any $\tau\in\mathbb{Z}$.

When $\tau\leq -1$, it turns out [66] that $Z_{-\tau}(q,x)$ is equivalent to the Poincare polynomial of the cohomological Hall algebra [50] of one vertex quiver with $-\tau$ loops, for which the infinite product formula and the integrality of $N_{m,k}(\tau)$ were proved in [78, 25].

Furthermore, the special case of formula (22) in $\tau=1$ implies the following conjecture which can be viewed as the 1-parameter deformation of the two Rogers-Ramanujan identities (20) and (21):

Conjecture 6.1. *Fix $m\geq 1$, there exist finite many positive integers $N_{m,k}$ such that*

$$\begin{aligned}&\sum_{n\geq 0}\frac{q^{n^2}}{(1-q)\cdots(1-q^n)}(q^{-\frac{1}{2}}x)^n\\&\quad=\prod_{m\geq 1}\prod_{k\in\mathbb{Z}}\prod_{l\geq 0}\left(1-q^{\frac{k+1}{2}+l}x^m\right)^{(-1)^mN_{m,k}}\end{aligned} \tag{23}$$

In particular, when $x=q^{\frac{1}{2}}$ and $x=q^{\frac{3}{2}}$, these integers $N_{m,k}$ together with the formula (23) give the two Rogers-Ramanujan identities (20) and (21).

Let us give some numerical checks for Conjecture 6.1. We introduce the polynomial

$$f_m(q) = \sum_{k\in\mathbb{Z}} N_{m,k} q^k,$$

By using Maple 13, we have computed the polynomial $f_m(q)$ for $1 \le m \le 20$. Here are a list of them for $m \le 6$:

$$\begin{aligned}
f_1(q) &= 1 \\
f_2(q) &= q, \\
f_3(q) &= q^4, \\
f_4(q) &= q^5 + q^9, \\
f_5(q) &= q^6 + q^8 + q^{10} + q^{12} + q^{16}, \\
f_6(q) &= q^7 + 2q^9 + q^{11} + 3q^{13} + q^{15} + 2q^{17} + q^{19} + q^{21} + q^{25}
\end{aligned}$$

If we let $x = q^{\frac{1}{2}}$, identity (23) becomes

(24)

$$\begin{aligned}
\sum_{n\ge 0} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)} &= \prod_{m\ge 1}\prod_{k\ge 0}\prod_{l\ge 0}\left(1 - q^{\frac{m+k+1}{2}+l}\right)^{(-1)^m N_{m,k}} \\
&= \prod_i \prod_{l\ge 0} (1-q^{i+l})^{n_i}
\end{aligned}$$

where $n_i = \sum_{m+k+1=2i} (-1)^m N_{m,k}$, our computations imply that:

$$n_i = \begin{cases} -1, & i = 5k+1 \text{ or } 5k+4, \text{ for } k \ge 0 \\ 1, & i = 5k+2 \text{ or } 5k+5, \text{ for } k \ge 0 \\ 0, & \text{otherwise,} \end{cases}$$

It turns out formula (24) gives the first Rogers-Ramanujan identity (20).

Similarly, letting $x = q^{\frac{3}{2}}$, identity (23) becomes

(26)

$$\begin{aligned}
\sum_{n\ge 0} \frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)} &= \prod_{m\ge 1}\prod_{k\in\mathbb{Z}}\prod_{l\ge 0}\left(1 - q^{\frac{3m+k+1}{2}+l}\right)^{(-1)^m N_{m,k}} \\
&= \prod_i \prod_{l\ge 0} (1-q^{i+l})^{r_i}
\end{aligned}$$

where $r_i = \sum_{3m+k+1=2i} (-1)^m N_{m,k}$, we find that:

$$r_i = \begin{cases} -1, & i = 5k+2, \text{ for } k \ge 0 \\ 1, & i = 5k+4, \text{ for } k \ge 0 \\ 0, & \text{otherwise,} \end{cases}$$

Hence formula (26) gives the second Rogers-Ramanujan identity (21).

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