Open Problems

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Note. The readers are welcome to propose the solutions. The authors should send their solutions to liu@math.ucla.edu and post their solutions in arXiv. The correct solutions will be announced and some souvenirs will be presented to the solvers.—*The Editors*

Problem 2017012 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Compute the Green's functions for a flat torus. The heat kernel for the torus can be computed by looking into the dual lattice that defines the torus (see §9.5 of [1]). Green's function can be obtained by integrating the heat kernel in time, after subtracting the constant term. But this is complicated to be expressed in a nice form. For two dimensional tori, Green's function can be expressed in terms of automorphic forms (see Chapter 2 of Serge Lang's book [2]). D'Hoker and Phong [3] studied the conformally invariant Green's function on curves of higher genus, which admits an expression in terms of theta functions (more precisely, in terms of the "prime form"). It is attractive since it depends only on the complex structure and not the metric, and it is used frequently in string theory.

In my survey article for IMU [4], I suggested to study zeros and critical points of eigenfunctions, Green's function and heat kernel. The best testing problem is on flat tori where the problem is already deep and difficult. C.-L. Wang and C.-S. Lin [5] made fundamental progress in the two dimensional case. C.-S. Lin was able to relate it to arithmetic of elliptic curves. The same problem for studying critical points of Green's function for 3-dimensional flat tori should be very interesting.

We can also ask for more general situation: Compute the Green's function for locally symmetric spaces and their critical points; What are their arithmetic properties if the discrete group is arithmetic?

[1] M. Berger, *A panoramic view of Riemannian geometry*. Springer-Verlag, Berlin, 2003. xxiv+824 pp.

[2] S. Lang, *Introduction to Arakelov theory*. Springer-Verlag, New York, 1988. x+187 pp.

[3] E. D'Hoker and D.H. Phong, *Conformal scalar fields and chiral splitting on super Riemann surfaces*, Comm. Math. Phys. **125** (1989), 469–513.

[4] S.-T. Yau, *Nonlinear analysis in geometry*. Enseign. Math. (2) **33** (1987), 109–158.

[5] C.-L. Wang and C.-S. Lin, *Elliptic functions, Green functions and the mean field equations on tori*. Ann. of Math. (2) **172** (2010), 911–954.

Problem 2017013 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

About forty years ago, I made a conjecture [1, Problem 100] that the first eigenvalue of an embedded minimal hypersurface in the $n+1$-sphere S^{n+1} is equal to n. Choi and Wang [2] proved that it is $\geq n/2$. So far no significant improvement on the work of Choi-Wang has been found. It will be interesting to find an upper bound of k (depending only on n) such that n is the k-th eigenvalue. This will show that the hyperplane passing through the origin can at most cut the hypersurface into $k+1$ pieces.

We can turn the question around. Given a compact Riemannian manifold M, when can we tell that it is a minimal hypersurface in codimension one whose complement has two distinct components in a compact space N whose Ricci curvature has a positive

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lower bound. In this case, according to the argument of [2], one can give a positive lower bound of its first eigenvalue. Note that we can allow N to have singularities as long as some integral formula holds on N.

[1] S.-T. Yau, *Problem section.* in Seminar on Differential Geometry, pp. 669–706, Ann. of Math. Stud., 102, Princeton Univ. Press, Princeton, N.J., 1982.

[2] H. I. Choi and A.-N. Wang, *A first eigenvalue estimate for minimal hypersurfaces.* J. Differential Geom. **18** (1983), 559–562.

Problem 2017014 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Classify the topology of manifolds that support a metric without minimal 2-spheres. By the work of Sack-Uhlenbeck [1], such manifolds cannot have any homotopy group with dimension greater one. But it is still possible that such manifold may still have the property of the existence of minimal 2-spheres if the topology is exotic. Meeks-Simon-Yau theorem [2] says that is the case if the three manifold admits exotic cells. It will be interesting to look for other exotic $K(p,1)$. A very interesting one is the 4-dimensional manifold constructed by Davis [3] which is covered by a contractible manifold which is not homeomorphic to Euclidean space. Does there exist a metric on such manifold which admits no minimal two-spheres? Does similar question apply to other exotic manifolds of this type?

[1] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres.* Ann. of Math. (2) **113** (1981), 1–24.

[2] W. Meeks, L. Simon and S.-T. Yau, *Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature.* Ann. of Math. (2) **116** (1982), 621–659.

[3] M. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space.* Ann. of Math. (2) **117** (1983), 293–324.

Problem 2017015 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Let M be a $2m$-dimensional Calabi-Yau manifold with a nonvanishing holomorphic $2m$-form W. There are three parallel $2m$-forms given by $\mathrm{Re}W$, $\mathrm{Im}W$ and m-fold product ω^m of the Kähler form. Take any nontrivial linear combination of them which is still a parallel form. One can define calibrated submanifold of this form. Find nontrivial examples of such submanifolds in a compact Calabi-Yau manifold.

Problem 2017016 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

This is about eigenfunctions for the Laplacian. Let us look at the eigenfunctions f with eigenvalue equal to $-\lambda^2$. we look at its zero set. Besides the question I asked many years ago about the asymptotic behavior of the one-codimensional Hausdorff measure of the nodal set, we can ask the flux of the eigenfunctions along the nodal set, namely the integral of the gradient of the eigenfunction along the nodal set. When we ask this global question for a closed manifold, this can be expressed by the L^1 norm of the eigenfunction. This was pointed out to me by Yaiza Canzani that there is a formula of Dong-Sogge-Zelditch [1,2] that allows one to do this expression. (The paper of Dong was in fact part of his Harvard thesis.) But local problem is more tricky. Namely we fix a ball in the manifold, and ask the asymptotic behavior of the same quantity within this ball.

[1] R. Dong, *Nodal sets of eigenfunctions on Riemann surfaces.* J. Differential Geom. **36** (1992), 493–506.

[2] C. D. Sogge and S. Zelditch, *Lower bounds on the Hausdorff measure of nodal sets.* Math. Res. Lett. **18** (2011), 25–37.

Problem 2017017 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Kollár [1] made the following conjecture: For two algebraic manifolds that are symplectic deformation equivalent, if one of them is rationally connected, then the other one is also rationally connected.

For 3-dimensional manifolds, this was proved by Voisin [2] under extra assumptions which was removed by Zhiyu Tian [3], who also obtained partial results in 4-dimension [4].

[1] J. Kollár, *Low degree polynomial equations: arithmetic, geometry and topology.* In European Congress of Mathematics, Vol. I (Budapest, 1996), volume 168 of Progr. Math., pp. 255–288. Birkhäuser, Basel, 1998.

[2] C. Voisin, *Rationally connected 3-folds and symplectic geometry.* Astérisque **322** (2008), 1–21.

[3] Z. Tian, *Symplectic geometry of rationally connected threefolds.* Duke Math. J. **161** (2012), 803–843.

[4] Z. Tian, *Symplectic geometry and rationally connected 4-folds.* J. Reine Angew. Math. **698** (2015), 221–244.

Problem 2017018 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Let M be a compact manifold. Let T_{ij} be a given symmetric tensor and f to be a given function defined on M. Find conditions on M, T_{ij} and f so that we can find a metric g_{ij} on M whose Ricci tensor is $fg_{ij}+T_{ij}$. This problem was originally proposed by me in 1979 [1].

[1] T. Kotake and T. Ochiai (eds.), *Non-linear problems in geometry*: proceedings of the Sixth International Symposium (Katata, September 3-8, 1979), Division of Mathematics, the Taniguchi Foundation, published by Tohoku University, 1979.

Problem 2017019 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

We conjecture that a compact complex manifold satisfies the $\partial\bar{\partial}$-lemma if and only if it can be deformed to a complex variety which is birational to a Kähler manifold. On the other hand, it was proved in [1] that the property of satisfying the $\partial\bar{\partial}$-lemma is not closed under holomorphic deformations. In [2], it was proved that for any (smooth) complex analytic family of compact complex manifolds, the central fibre must be Moishezon if the other fibres are Moishezon.

[1] D. Angella and H. Kasuya, *Cohomologies of deformations of solvmanifolds and closedness of some properties*, North-West. Eur. J. Math. **3** (2017), 75–105.

[2] D. Popovici, *Limits of Moishezon Manifolds under Holomorphic Deformations*, arXiv:1003.3605.

Problem 2017020 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

This problem is about the realization of Riemannian metric by some concrete constructions. Given an n-dimensional Riemannian metric on a manifold, it has been a long open question of how to embed it isometrically into an $n(n+1)/2$ dimensional Euclidean space. If the metric is real analytic, it can be done locally. However, this is unknown even for $n=2$ for smooth metric. The best known result in this case was due to C.-S. Lin [1] when the curvature is nonnegative. And there are works by various people [2] when the curvature is not too degenerate at the zero locus.

The global problem is even more difficult. The first Weyl problem works well for 2-dimensional surfaces when the curvature is positive, due to the works of Pogorelov [3] and Nirenberg [4]. The existence is fine if the curvature is nonnegative. But the optimal regularity is not completely understood, although there are works due to several people [5,6]. Pogorelov generalized the Weyl problem allowing the curvature to be negative on the sphere, if one allows the ambient space to be the hyperbolic space form.

The rigidity (or uniqueness) of the Weyl problem was known to many people if the embedding is C^2. But in a famous work of Pogorelov, he claimed to prove the rigidity of closed convex surface without smoothness assumption. It will be great to give a transparent proof of the theorem of Pogorelov.

There is no known rigidity theorem for n-dimensional submanifold embedded into an $n(n+1)/2$ space for $n>2$. This is unfortunate as it means we do not have a canonical way to realize such n-dimensional manifold in Euclidean space.

Infinitesimal rigidity problem for closed surfaces were also studied extensively. The most definite result was due to Blaschke [7] who showed that closed convex surfaces are infinitesimally rigid. A result of Minagawa and Rado [8] showed that generically, surfaces of rotation are infinitesimally rigid. Would that be a more general phenomena? In order words, would any closed surfaces be generally infinitesimally rigid?

Since it is likely that higher dimensional rigidity is virtually impossible (of course, one needs a rigorous argument here), we like to propose other ways to realize a Riemannian metric, which hopefully will be more canonical.

I like to illustrate the proposal in the following manner for a 3-dimensional metric. Given a 3-dimensional space M, we can find a hypersurface M' in the product space $M\times\mathbb{R}$ defined by a function f on M. Then we can conformally change M' to a new manifold M'' where the conformal factor is a function $g>0$ defined over M'. By choosing f and g suitably, can we then isometrically embed M'' into $\mathbb{R}^4$?

Perhaps under suitable chosen condition (such as positivity of curvature) on the metric of M, we can conclude the above procedure is canonical (unique).

For higher dimension, we can iterate the graph and conformal construction several times. The problem here is related to the work of Chen-Wang-Yau [9] on quasilocal mass in higher dimension.

[1] C.-S. Lin, *The local isometric embedding in $\mathbb{R}^3$ of 2-dimensional Riemannian manifolds with nonnegative curvature*. J. Differential Geom. **21** (1985), 213–230.

[2] Q. Han, J.X. Hong and C.-S. Lin, *Local isometric embedding of surfaces with nonpositive Gaussian curvature*. J. Differential Geom. **63** (2003), 475–520.

[3] A.V. Pogorelov, *Extrinsic Geometry of Convex Surfaces*, “Nauka”, Moscow, 1969; English trans., Math. Monographs 35, Amer. Math. Soc, Providence, RI, 1973.

[4] L. Nirenberg, *The Weyl and Minkowski problems in differential geometry in the large*. Comm. Pure Appl. Math. **6** (1953), 337–394.

[5] J.X. Hong, *Realization in $\mathbb{R}^3$ of complete Riemannian manifolds with negative curvature*, Comm. Anal. Geom., **1** (1993), 487–514.

[6] P. Guan and Y.Y. Li, *The Weyl problem with nonnegative Gauss curvature*, J. Differential Geom. **39** (1994), 331–342.

[7] W. Blaschke, *Ein Beweis für die Unverbiegbarkeit geschlossener konvexer Flächen*. Gött. Nachr. pp.607–610, 1912.

[8] T. Minagawa and T. Rado, *On the infinitesimal rigidity of surfaces of revolution.* Math. Z. **59** (1953), 151–163.
[9] P.-N. Chen, M.-T. Wang and S.-T. Yau, *Conserved quantities in general relativity: from the quasi-local level to spatial infinity.* Comm. Math. Phys. **338** (2015), 31–80.

Problem 2017021 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

In the foundational work of Kodaira [1], it was proved that if the Kähler class of the manifold is integral, then it can be embedded into projective space. Such Kähler manifold is equipped with a positive line bundle L. I proposed that one can approximate any Kähler metric in this Kähler class by projective embedding of the manifold using high power of L. I suggested this problem to Tian for his thesis and pointed out the method similar to what I did with Siu [2] previously can solve this problem. The problem is more complicated for those Kähler metrics that is not in a rational class. What is the best way to resolve this?

The universal embedding for Hodge manifolds is projective space, as shown by Kodaira. What about Kähler but non-projective manifolds? As was shown by Claire Voisin [3], there are Kähler manifolds that cannot be deformed to projective manifolds. Is there universal space for Kähler manifolds to be submanifolds? The algebraic dimension of the manifold should play a role. There is a conjecture that any Kähler manifold with algebraic dimension zero is a complex torus. If that is the case, can Kähler manifolds be embedded into a complex torus fiber space over the projective space? What class of complex torus fiber space over projective space admits a Kähler metric? How to classify them?

A much weaker version of Kähler manifolds are balanced manifolds. These are n-dimensional complex manifolds which admit a closed positive $(n-1, n-1)$-form. The form gives rise to a Hermitian metric which we call balanced metric. They were studied by [4] and an important fact is that if M admits a balanced metric, so is other complex manifold that is bimeromorphic to it. Such a metric plays an important role in string theory because it admits certain supersymmetry. It would be interesting to classify those balanced manifolds that have algebraic dimension zero.

[1] K. Kodaira, *On Kähler varieties of restricted type (an intrinsic characterization of algebraic varieties).* Ann. of Math. (2) **60** (1954), 28–48.
[2] Y.-T. Siu and S.-T. Yau, *Complete Kähler manifolds with nonpositive curvature of faster than quadratic decay.* Ann. of Math. (2) **105** (1977), 225–264.
[3] C. Voisin, *On the homotopy types of compact Kähler and complex projective manifolds.* Invent. Math. **157** (2004), 329–343.
[4] M.L. Michelsohn, *On the existence of special metrics in complex geometry.* Acta Math. **149** (1982), 261–295.