A.Ya. Khintchine's Work in Probability Theory

by Sergei Rogosin[*] and Francesco Mainardi[†]

Introduction

This paper deals with the work of the well-known Russian mathematician, Alexander Yakovlevich Khintchine[1] (1894–1959), one of the creator of the Russian (Soviet) school on Probability Theory.

An extended version of the paper including translation from Russian, French, German and Italian is presented in the book Rogosin, Mainardi (2010): *The Legacy of A.Ya. Khintchine's Work in Probability Theory*, Cambridge Scientific Publishers (2010).

The idea to write this paper arises from two considerations.

First of all, Probability Theory was developed in 1920–1930s so rapidly that now some of the ideas and results of this period are forgotten and are rediscovered. It is very interesting from different points of view to understand how this branch of mathematics became one of the most important and applicable mathematical discipline. In order to see it one may turn to the most fundamental concepts of the Probability Theory.

Second, not all results of the pioneers of this formative period are known to new generations of mathematicians.

Among the contributors to the "great jump" of the Probability in the above mentioned period special role belongs to Alexander Yakovlevich Khintchine. He had obtained outstanding results, which form a very clear and rigorous style of handling deep problems. He creates together with A.N. Kolmogorov the well-known school of Probability Theory at Moscow University.

In connection with the above mentioned motivation for our paper, it is especially crucial to describe the works by Khintchine because of the following reasons:

- Several important results by Khintchine are forgotten and later re-discovered.
- A number of results were published in inaccessible places and not in English.
- The concrete and clear style of Khintchine' work can help the readers understand much better the preceding and recent results.

The paper has the similar structure to that of the above mentioned book Rogosin, Mainardi (2010): *The Legacy of A.Ya. Khintchine's Work in Probability Theory*, Cambridge Scientific Publishers (2010). Section 5, describing the results by A.Ya. Khintchine on infinitely divisible distribution, is closely related to our paper Mainardi, Rogosin (2006): The origin of infinitely divisible distributions: from de Finetti's problem to Lévy-Khintchine formula, *Mathematical Methods for Economics and Finance*, **1** (2006) 37–55. [E-print http://arXiv.org/math/arXiv:0801.1910]

Short Biography of Alexander Yakovlevich Khintchine

A.Ya. Khintchine was born on July 19, 1894 in the village Kondrovo of the Kaluga region, about one and a half hundred km southwest of Moscow.

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[1] Another transliteration of his name in English is Alexander Iacovlevich Khinchin or Hintchine.

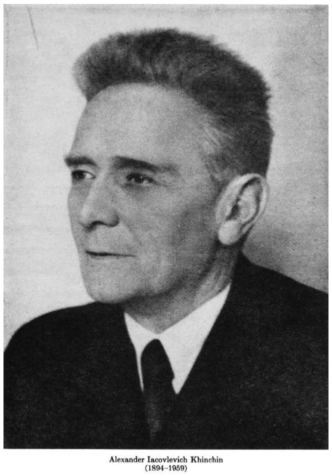

Figure 1. Alexander Iacovlevich Khinchin (1894–1959).

From 1911 to 1916 he was a student of the Physical-Mathematical faculty of the Moscow State University (MSU). All his scientific life was deeply connected with this University.

In the period of study at the University and in the first years of his research career Khintchine was under a strong influence of the ideas and personality of N.N. Luzin. It is known that A.Ya. Khintchine presented his first result at a meeting of the student mathematical club in November 1914 (see, *e.g.*, Gnedenko (1961)).

The mathematical talent of this young student was noticed at the University by his teachers. After graduation at MSU A.Ya. Khintchine was recommended for preparation to the professorship. His teaching career started in 1918 at Moscow Women's Polytechnical Institute. One year later he was invited to the Ivanovo-Voznesensk (now Ivanovo) Polytechnical Institute, and soon after he became the dean of Physical-Mathematical Faculty of newly founded Ivanovo-Voznesensk Pedagogical Institute. In 1922 the Research Institute on Mathematics and Mechanics was organized at the Moscow State University. A.Ya. Khintchine was invited to this Institute as a researcher. During a certain period he combined his research in Moscow with lecturing at Ivanovo-Voznesensk. Finally, in 1927 he got the professorship at the Moscow State University.

After his first significant publications A.Ya. Khintchine became known to the European probabilistic community. In 1928 he spent a couple of weeks at the University in Göttingen, one of the most important mathematical centers at the beginning of XX century. Here he prepared at least two papers, published in 1929 in Mathematische Zeitschrift and in Mathematische Annalen ([G 46] and [G 48], respectively).[2]

Khintchine was a member of the Soviet delegation at the International Congress of Mathematicians held in Bologna (Italy) from 3 to 10 September 1928 (see, *e.g.*, Mainardi, Rogosin (2006)). The Russian delegation was represented by 27 scientists including some prominent researchers like S. Bernstein (Karkhov), A.Ya. Khintchine (Moscow), V. Romanovsky (Tashkent) and E. Slutsky (Moscow). We note, however, that Khintchine did not present any communication so that he did not publish a paper in the Proceedings of the Congress (which appeared in 1929–1932).

Since 1927, all A.Ya. Khintchine's scientific and teaching activity was connected with the Moscow State University. He was the head of the chair of the Probability Theory, then the head of the chair of Mathematical Analysis, director of the Research Institute of Mathematics and Mechanics at MSU. He passed away on November 18, 1959, after a lengthly serious illness.

First Papers in Probability: 1924–1936

The first papers by A.Ya. Khintchine in the Probability Theory appeared in 1924. In order to understand the role of these articles it is necessary to describe the state of Probability Theory in those years. One can recall a critical review by R. von Mises who summed up the situation in the following words: "Today, probability theory is not a mathematical science" (see, *e.g.*, Cramer (1962)).

There was no satisfactory definition of the mathematical probability, and the conceptual foundations of the subject were completely obscure. Moreover, with few exceptions, mainly belonging to the French and Russian schools, authors on probability did not seem aware of the standards of rigor which in other mathematical fields, were regarded as obvious.

Already in the middle of 1920s appeared another evaluation of the Probability Theory as a branch of the Mathematical Science.

In his monograph "Fundamental Laws of Probability" ([G 35]), published in 1927, Khintchine wrote: "Up to the recent years in Europe the dominated opinion on the Probability Theory was as on science which is important and useful, but cannot depict important problems. Anyway the work of Russian mathematicians (in particular, results by P.L. Chebyshev, A.M. Lyapunov, A.A. Markov) shows us that it is not correct. The Probability Theory has an integral

[2] Throughout our paper the citation of the type [G ...] means the corresponding paper in the list of Khintchine's publications, presented by B.V. Gnedenko in his article Gnedenko (1961). We have included the revised version of this list at the end of our paper.

method deeply connected with the methods of the modern theory of functions,[3] and thus the most of the recent ideas appeared in the Mathematical Analysis have a fruitful application in the Probability Theory."

This optimistic opinion by A.Ya. Khintchine got an evident justification in the next few decades.

At the end of the 1930s, the picture has been radically changed. Mathematical probability theory received its axiomatic foundation. It became a purely mathematical discipline, with problems and methods of its own, conforming the current standards of mathematical rigorism, and entering into fruitful relationships with other branches of mathematics. At the same time, the fields of applications of the mathematical probability were steadily and rapidly growing in number and depth. It is true that nowadays there are still some "pure" mathematicians who tend to look down on the "applied" science of probability. But this attitude is expected to disappear very soon.

The tremendous development of Probability Theory, which took place in the twenty years from 1920s to 1940s was, no doubt, a joint effect of the efforts of a number of mathematicians and statisticians. However, it does not seem unlikely that future historians will ascribe its development, as far as the mathematical side of the subject is concerned, above all to the creative powers of four scientists (in alphabetic order): B. de Finetti, A.Ya. Khintchine, A.N. Kolmogorov, and P. Lévy. In fact, it may be said that the real turning point came with the publications of the following works:

- P. Lévy, *Calcul des Probabilités*, Gauthier-Villars, Paris (1925), pp. viii+350.
- B. de Finetti, Funzione caratteristica di un fenomeno aleatorio, *Memorie della R. Accademia Nazionale dei Lincei*, **4** No 5, 86–133 (1930).
- A.Ya. Khintchine, *Asymptotische Gesetze der Wahrscheinlichkeitsrechnung*, Julius Springer, Berlin, 1933.
- A.N. Kolmogorov, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Julius Springer, Berlin, 1933.
- P. Lévy, Sur les intégrales dont les éléments sont des variables aléatoires indépendentes, *Annali della R. Scuola Normale di Pisa* **3**, 337–366 (1934) and **4**, 217–218 (1935).

The first papers by Khintchine on the Probability Theory ([G 14], [G 15]) were devoted to the law of iterated logarithm for Bernoulli sequences of random variables.

Let us formulate in Khintchine's form the result in [G 14]: suppose that we have an infinite series of mutually independent trials (experiments) in each the probability of an appearance of a certain event E is equal to $p,\ 0<p<1$. Suppose that the event E is realized $m(n)$ times in first n trials and denote

$$\mu(n) = m(n) - np.$$

It is important to determine an exact upper limit for the order of $\mu(n)$ as well as to give an exact meaning of such an order.

Problem. *To find a function $\chi(n)$ satisfying the following conditions:*
for any arbitrary small $\varepsilon>0$ there exists a positive integer $n_0 = n_0(\varepsilon)$ such that with probability greater than $1-\varepsilon$ the following assertions hold:

1^0. *For all $n>n_0$*

$$\left|\frac{\mu(n)}{\chi(n)}\right| < 1+\varepsilon.$$

2^0. *There exists $n>n_0$ for which*

$$\left|\frac{\mu(n)}{\chi(n)}\right| > 1-\varepsilon.$$

Answer. *The solution of the problem is given by the formula*

$$\chi(n) = \sqrt{2p(1-p)n\log\log n}. \tag{1}$$

Any other solution is asymptotically equivalent to this one.

The papers [G 14], [G 15] initiated the work of the Moscow probability school, and Khintchine's work in the area Cramer (1962). After this work (together with the preceded work by E. Borel on the strong law of large numbers) the problem of estimation of probability of the sums of random variables occupied larger place in the investigations (see, *e.g.*, Kolmogorov's survey in Kolmogorov (1959)).

In 1925 Khitchine and Kolmogorov initiated the systematic study of the convergence of infinite series whose terms are mutually independent random variables.[4] In this paper, Khintchine proved that for countably valued random variables, convergence of means and variances guarantees almost sure (*i.e.*, with probability one) convergence of series. In order to obtain this result Khintchine used the ideas from the measure theory, namely, constructed the corresponding random variables as functions on the interval $[0,1]$ with Lebesgue measure. From the modern point of view it is not necessary to apply such a construction, but, in a sense, it was one of small stones

[3] Khintchine had in mind first of all the results from measure theory and different generalizations of the integral, which were very popular when his book appeared.

[4] See their joint paper [G 25]: Ueber Konvergenz von Reihen, deren Glieder durch den Zufall bestimmt werden, *Mat. Sb.* **32**, 668–677 (1925).

anticipated the Kolmogorov's foundation of Probability Theory.[5]

Khintchine's papers devoted to the law of the iterated logarithm and the summation of the series of random terms followed by the papers dealing with the classical problem on summation of independent random variables. He gave the particularly clear conditions of the applicability of the law of large numbers in the case of mutually independent, identically distributed summands, which is reduced to the existence of a finite mathematical expectation ([G 44]). Namely, he proved that with the probability arbitrarily close to one, $\sum_{j=1}^{n} \mathbf{x}_j/n \to \mathbf{E}(\mathbf{x}_1)$ for any sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$ of mutually independent random variables with a common distribution, having a finite expectation.

In many works, Khintchine paid a special attention to the conditions of convergence to the Gaussian Law. Among the first papers dealing with this topic, we mention the papers [G 47] and [G 48], which initiated a new direction of studies of so-called case of "large deviations."

Later he started to develop the idea of the domain of attraction of the Gaussian Law (see [G 74]). This notion, introduced by P. Lévy, has been essentially extended in the monograph by A.Ya. Khintchine ([G 92]). Let us describe the main result of the paper [G 74] again in Khintchine's form. First he gave the following definition: a distribution law $F(x)$ is said to belong to the domain of attraction of another law $G(x)$, when for the the sum S_n of n random variables, which are independent and identically distributed under the law $F(x)$, there exist numbers $\xi_n > 0$, η_n such that the distribution law of $S_n/\xi_n - \eta_n$ tends to $G(x)$ as $n \to \infty$.

Since the Gaussian law is surely the most important among the limiting laws in the above sense, it is interesting to look for a simple criterium which allows one to recognize if a given distribution law belongs to the domain of attraction of the Gaussian law.

It is known that this is the case when the integral

$$\int_{-\infty}^{+\infty} \alpha^2 \, dF(\alpha)$$

is finite, but there are laws for which this integral is infinite that belong to the domain of attraction of the Gauss law.

In [G 74] the following theorem proved.

Theorem[6] *The distribution $F(x)$ belongs to the domain of attraction of the Gauss law if and only if*

$$\lim_{x \to +\infty} \frac{x^2[1 - F(x) - F(-x)]}{\int_{-x}^{x} \alpha^2 \, dF(\alpha)} = 0. \tag{2}$$

The concept of the domain of attraction of the Gaussian Law was found to be highly connected with the conditions for convergence to the Gaussian of the normed sums of independent components. In the case of identically distributed components, Khintchine found simultaneously with P. Lévy and W. Feller, but independently of them, the necessary and sufficient conditions for convergence to the normal distribution (see [G 79]).

The Interaction with Paul Lévy

In 1928 B. de Finetti started a research regarding functions with random increments, see DeFinetti (1929a,b,c, 1930a, 1931) based on the theory of infinitely divisible characteristic functions, though he did not use such a term.[7]

The general case of de Finetti's problem, including also the case of *infinite variance*, was investigated in 1934–35 by Lévy (Levy (1934, 1935)), who published two papers in French in the Italian Journal *Annali della Reale Scuola Normale di Pisa*.

At that time Lévy (Levy (1925)) was quite interested in the so-called *stable distributions* that are known to exhibit infinite variance, except for the particular case of the Gaussian. The approach by Lévy, well described later in his classical 1937 book Levy (1954), is independent of that of Kolmogorov, as can be understood from footnotes in his 1934 paper Levy (1934). From the footnote$^{(1)}$ we learn that the results contained in his paper were presented in three communications of the Academy of Sciences (Comptes

[5] Chapter 6 of his book (Kolmogorov (1933)) Kolmogorov devoted to the results of Khintchine and himself on the applicability of the ordinary and strong law of large numbers. In the Preface to this book he wrote: "I wish to express my warm tanks to Mr. Khintchine, who has read the whole manuscript and proposed several improvements."

[6] This theorem appears as Theorem [45], pp. 192–193, in 1938 Khintchine's book, *Limit Distributions for the Sum of Independent Random Variables*. In the *Bibliographical and Historical Notes* at the end this book, p. 114, Khintchine states that theorem was proved independently and simultaneously by him, by P. Lévy: Determination générale des lois limites, *Compt. Rendus Acad. Sci. Paris* **203** (16), 698–700 (1936), and by W. Feller: Ueber den zentralen Grenzwertsatz der Wahrscheinichkeitsrechung, *Math. Z.* **40**, 521–559 (1935). According to B.V. Gnedenko and A.N. Kolmogorov (see p. 172 in the 1954 English translation of their 1949 book: *Limit Distribution for Sums of Independent Random Variables*) Levy's proof is referred to P. Lévy: Propriétés asymptotiques des sommes de variables aléatoires independentes ou enchainées, *J. Math. Pures Appl.* (ser. 9) **14**, 347–402 (1935). Like Khintchine, Gnedenko & Kolmogorov also quoted the 1935 paper by Feller in *Math. Z.* and, in addition, a further note by Feller, published in *Math. Z.* **43**, 301–312 (1937).

[7] More details on infinitely divisible distributions can be found below in Sec. 5, and also in our paper Mainardi, Rogosin (2006).

Rendus) of 26 February, 26 March and 7 May 1934. Then, in the footnote$^{(6)}$, p. 339, the Author writes:

> *[Ajouté á la correction des épreuves] Le résumé de ma note du 26 février, rédigé par M. Kolmogorov, a attiré mon attention sur deux Notes de M.B. de Finetti* (see DeFinetti (1929a, 1930a)) *et deux autres de M. Kolmogorov lui-même* (see Kolmogorov (1932a,b)), *publiées dans les Atti Accademia Naz. Lincei (VI ser). Ces dernières notamment contiennent la solution du problème traité dans le présent travail, dans le cas où le processus est homogène et où la valeur probable* $\mathbb{E}\{x^2\}$ *est finie. Le résultat fondamental du présent Mémoire apparait donc comme une extension d'un résultat de M. Kolmogorov.*

This means that P. Lévy was not aware of the results on homogeneous processes with independent increments obtained by B. de Finetti and by A.N. Kolmogorov. The final result of Lévy is known as the *Lévy canonical representation* of the infinitely divisible characteristic functions.

In a paper of 1937 Khintchine [G 81] showed that Lévy' s result can be obtained also by an extension of Kolmogorov's method: his final result is known as the *Lévy-Khintchine canonical representation* of the infinitely divisible characteristic functions. The translation from the Russian of this fundamental paper can be found in [G 81]. The theory of the infinitely distributions was then presented in German in the article [G 91] and in Russian in his 1938 book on *Limit Distributions for Sums of Independent Random Variables* [G 92].

The importance of the results by A.Ya. Khintchine has been highly evaluated by P. Lévy in his book (Levy (1970)). At p. 105 he writes:

> *"Tout cela constituait un ensemble important, d'autant plus que a théorie des lois indéfinitement divisible a été la base d'un autre chapitre important du calcul des probabilites, l'arithmétique des lois de probabilité, dont je parlerai plus loin. Sans songer à minimiser le rôle de Khintchine, qui m'a devancé sur plusieurs points et qui san doute aurait retrouvé ceux pour lesquels je l'ai devancé, sans oublier non plus les travaux antérieures de Cauchy, de Pólya, de B. de Finetti et de Kolmogorov, je crois (also pouvoir dire) que cette théorie est essentiellement mon oeuvre."*

A.Ya. Khintchine and P. Lévy were working on the foundation of the theory of stable law distributions in competition.[8] They were interested in the works of each other. Some of their results look as a replica to those obtained by the other side. One example of such papers is a small note by A.Ya. Khintchine (see [G 84]) devoted to the description of the invariant classes of distributions. The resulting theorem is a direct generalization of an analogous theorem by P. Lévy (see below). It stipulated the role of the invariant classes in the Probability Theory. Another paper is the note by A.Ya. Khintchine (see [G 85]) in which examples of stable distribution laws are constructed.

[8] Khintchine paid a lot of attention to the results by Lévy. Due to him Lévy's theory became more accessible to the international probability community.

A.Ya. Khintchine and P. Lévy published only one joint paper [G 86]. It was devoted to the proof of the following theorem on the representation formula for characteristic functions of stable distribution laws:

The characteristic function $\varphi(t)$ *(i.e. expected value of* e^{itX}*, with* X *being the random variable under consideration) of the stable distribution law is defined by*

$$\log \varphi(t) = -c\left(1 - i\beta\frac{t}{|t|}\, tang\, \frac{\pi}{2}\alpha\right)|t|^{\alpha} \qquad (3)$$

$$(c > 0,\ 0 < \alpha \le 2,\ |\beta| \le 1).$$

The proof of the theorem consists of two parts. The part concerning the case $0 < \alpha < 1$ was written by P. Lévy, and that concerning the case $1 < \alpha < 2$ was written by A.Ya. Khintchine. It means that the paper was written by correspondence and the authors did not meet to discuss the results of the paper.

Infinitely Divisible Distributions

In the middle of 1930s A.Ya. Khintchine constructed the general theory of the limit distributions for sums of independent random variables. In his fundamental paper [G 91] he obtained the following results.

- He proved the Kolmogorov conjecture, namely: the limit law for the sums of mutually independent random variables, such that every summand is neglected with respect to the sum, have to be *infinitely divisible*.[9] It happens also that the class of infinitely divisible distributions is exactly the union of those distributions which are the limits of sums of mutually independent random variables satisfying the condition that none individual summand has an influence on the value of the limit of sums. It is clear that the problem has no sense without the last condition since in this case almost all sums of such a type has no limit.
- He discovered that each partial distribution is an infinitely divisible one. Conversely, each infinitely divisible distribution is a partial distribution.
- He showed how one can determine a new random variable, having an arbitrary (but not Gaussian) stable distribution, from a sequence of independent identically distributed random variables (with distribution $H(x)$) by using a simple construction. In this manner, a model can be obtained for random variables satisfying the stable

[9] For details on the origin of the infinitely divisible distribution we refer the reader to our 2006 paper Mainardi, Rogosin (2006): The origin of infinitely divisible distributions: from de Finetti's problem to Lévy-Khintchine formula, *Mathematical Methods for Economics and Finance*, **1** (2006) 37–55. [E-print http://arXiv.org/math/arXiv:0801.1910]

distribution.[10] Possible generalizations of such constructions are discussed as well.

The above results clarify the fact that infinitely divisible distributions play an important role in the classical theory of the summation of independent random variables.

To describe the role of the results by A.Ya. Khintchine presented in [G 91] we cite here a part of the article by A.N. Kolmogorov *Limit Theorems* (Kolmogorov (1959)): "The branch connected with the so called central limit theorem on attractions of distributions of the sums of a large number independent or weakly dependent summands (scalar or vectorial) to the normal Gaussian distribution was developed in some directions:

- Sharpening the classical limit theorems on attraction by the Gaussian distribution.
- Investigation of this problem as a special case of the problem on attraction by an arbitrary infinitely divisible distribution. It was understood due to appearance of the latter class of distributions and after the proof by A.Ya. Khintchine [G 91] in 1937 the main theorem which states that the limit distribution of the sums of independent individually negligible summands can only be infinitely divisible.

This second direction was based from the very beginning on the idea of comparing of the process of forming the successive sum of independent variables as the limit process with independent increments. The latter in any case has to satisfy "unboundedly" divisible distribution laws (in the Gaussian case this idea was developed by L. Bachelier already in 1900)."

We add few words by B.V. Gnedenko in Khintchine's obituary about the importance of the class of infinitely divisible distributions for the development of the Probability Theory in the late 1930s: "The fundamental role of the infinitely divisible distributions in the Probability Theory was established by recent investigations of B. de Finetti, A.N. Kolmogorov and P. Lévy on homogeneous in time stochastic processes and of A.Ya. Khintchine and G.M. Bawly[11] on the limit laws for the sums of independent random variables."

The results by A.Ya. Khintchine of [G 91] formed one of the main starting points of the modern theory of limit distributions for the sums of independent random variables.

Closely related to the summation theory are the results by A.Ya. Khintchine on arithmetic of the distribution laws (see [G 82]). We also mention the result of Khitnchine's paper [G 83], in which a criterion for characteristic function is given, generalizing Bochner's type criterion.

Khintchine's Book on the Distribution of the Sum of Independent Random Variables

The monograph [G 92] gave a masterful presentation of the general limit theorem for the sums of independent random variables and their application to the classical problem on the convergence of the normed sums to the normal law on the stage known in 1938. The book was based on a course of lectures which A.Ya. Khintchine delivered at Moscow State University. This course attracted the interest of A.A. Bobrov, B.V. Gnedenko, D.A. Raikov towards the problems of summation of random variables.

Among the results, one finds in this book the theorem that the class of all stable distributions coincides with the class of limiting distributions for the normed sums of the form $(\mathbf{x}_1 + \mathbf{x}_2 + \ldots + \mathbf{x}_n)/A_n - B_n$, where $\mathbf{x}_k$ are independent and identically distributed random variables, and $A_n > 0$, B_n are constants.

To the best of authors' knowledge the book by A.Ya. Khintchine was never translated into another language. The results of this book were developed in the classical treatise by A.N. Kolmogorov and B.V. Gnedenko (Gnedenko,Kolmogorov (1954)) on *Limit Distributions for Sums of Independent Random Variables*[12] that appeared in Russian in 1949 and in English in 1954.

The complete presentation (translation) of original techniques and results of the book by A.Ya. Khintchine is presented in our book Rogosin, Mainardi (2010). It can be interesting to the experts in Probability Theory as well as to beginners in this branch of Mathematics since the presented results are simple, clear and precise.

[10] Another model of such a type which uses the Poisson law instead of $H(x)$ and with integrals instead of series was proposed by P. Lévy.

[11] Gregory Minkelevich Bawly (1908–1941) graduated at the Moscow State University in 1930, defended his PhD thesis under guidance of A.N. Kolmogorov in 1936. His scientific advisor greatly evaluated his results on the limit distributions for sums of independent random variables and cited him in his book with Gnedenko (Gnedenko,Kolmogorov (1954)). G.M. Bawly lost his life in Moscow in November 1941 at a bombing attack. According to Khintchine [G 92] the name of *infinitely divisible distributions* (in a printed version) is found in the 1936 article by G.M. Bawly (Bawly (1936)), that was recommended for publication in the very important starting volume of the new series of Matematičeski Sbornik. His second (and the last) paper Bawly (1937) was published in Turkey and contained a local theorem on the limit distribution of sums of independent random variables generalizing the corresponding result by von Mises.

[12] We note that the titles of both books by Khintchine and by Gnedenko & Kolmogorov are identical, although in the reference list of the book by Gnedenko & Kolmogorov (in both editions), the title of Khintchine's previous book is in some way different (*Limit Theorems for Sums of Independent Random Variables*).

Teaching Probability Theory and Analysis

Alexander Yakovlevich Khintchine had a constant and deep interest to the problems of teaching in universities as well as in secondary schools. His pedagogical ideas presented in his textbooks, monographs and special articles.[13]

In 1938–1940 he headed the physical-mathematical section of the Methodical-Educational Soviet at the Ministry of Education of the Russian Federation. When the Academy of Pedagogical Sciences of the Russian Federation was founded, he became an academician of this Academy. He was a very active as the member of the editorial board of the multi-volume "Encyclopedy of Elementary Mathematics", some volumes of which appeared in the late 1950s (see e.g. [G 151]).

In the article "Alexander Yakovlevich Khintchine" B.V. Gnedenko says (see [G 159, pp. 180–196]): "Everybody who attends either a mathematical course lectured by him or his scientific reports remembered how exceptionally careful was the formulations of the problems, how exact was the description of the status of these problems in the corresponding branch of science. The reader could forget that his ideas was alien to him at the beginning. He/she started to feel the importance and significance of the discussed ideas, and to have satisfaction from the fact that he/she could achieve a new level of knowledge and a real understanding of the complicated notions and methods. The success of A.Ya. Khintchine as a Teacher and a Creator of new Research Directions in the Probability Theory, in the Number Theory, and in the Real Analysis is based, in particular, on his way of thinking."

From this point of view, it would be natural to include into this section of our paper certain sections and chapters from his monographs in which one can see the hand of a real Master. But in this case this section would be overloaded.

We restrict ourselves to the discussion of ideas by A.Ya. Khintchine, which concern teaching in universities and in the secondary schools.

Pedagogical Credo

In the previous sections, one exhibit some features of texts written by A.Ya. Khintchine. Many of his books run into several editions. We have to mention here few words by B.V. Gnedenko from the Introduction to the 4-th edition the book *Continued Fractions* [G 73]: "Before starting to write this Introduction I have read the book once more. I feel the great pleasure of the contact with the great master again. He does not only know thoroughly the material but also can present it in such a way that a reader find himself under the influence of the author's individuality. His book is not consist simply of declarations but it contains complete proofs of all statements. Anyway (since the book is addressed mainly to the beginners), all the proofs help the reader to understand the real course of reasoning, as well as their necessity." Such sentence means that in all his books A.Ya. Khintchine is not only a scientist, but also, and mostly, a teacher.

Another example of the same type we can find by reading the small booklet "*Three Pearls of Number Theory*" which has been appeared in several Russian editions (see [G 120]). It is written in the form of a letter to a former student of Moscow University (a soldier of the second World War), who asked the author to send some mathematical pearls to him in hospital. Even in this case A.Ya. Khintchine provided the reader with three very interesting results on number theory which were proved by rather young mathematicians. The first pearl concerns with the hypothesis on arithmetic progression which was solved by the young Dutch mathematician B.L. van der Warden in 1928. The second pearl deals with the hypothesis of Landau-Shnirelman on the thickness of sums of sequences which was solved by the young American mathematician G. Mann in 1942. The third pearl is the elementary proof of Waring's problem in Logic given by the young Russian mathematician Yu.V. Linnik in 1942. Khintchine told the story of these proofs by describing in detail (but on quite elementary level) all their steps. Thus he tried to lead the reader into the process of thinking saying him: you see, these people were of the same age as you and they formulated and successfully attacked very serious mathematical questions by using deep but simple considerations.

Alexander Yakovlevich Khintchine was one of the best lecturers of Moscow State University. During several decades he delivered a course of Mathematical Analysis in universities and pedagogical institutes. His students considered these lectures as the best they have ever attended during their students' years. It is no wonder. A.Ya. Khintchine tried to carry out not only the formal mathematical machinery, not only analytical technique, but the essence and "soul" of Analysis, its basic ideas. His pedagogical credo was: "better not too much but in the best form".

Mathematics in the Secondary School

Alexander Yakovlevich Khintchine was very enthusiastic in strengthening the content of course of Mathematics in secondary schools, as well as in developing new teaching methods and methodical receipts.

[13] A number of his articles devoted to the questions of methodics and pedagogic was published as special chapters of the book A.Ya. Khintchine, "Pedagogical Articles", (*Pedagogicheskie stat'i*), Prosvestchenie, Moscow, 1963, pp. 204 (in Russian) (see [G 159]).

He had hardly worked on the problems of education at the secondary school. A collection of his papers on these problems was published in 1963 in Moscow [G 159]. In his materials prepared for publication one can find different ideas which concern the questions of mathematical education. For example, he said (see [G 159, p. 11]): "there are several types of repetition or revising of the material which pupils have to make, namely, 1) repetition before the start of academic year, 2) revising following its beginning, 3) review of lessons learned during the latter, 4) revising of the material of certain theme connected with the control of knowledge, 5) annual repetition, 6) revising the material at the preparation to exams. This is awful. Whether it is possible to teach in such a way that pupils will not forget the material in order to avoid such infinite chain of repetitions?" We see here the great concern of A.Ya. Khintchine on the important educational problem, saying that repetition is not a way out. The teacher have to find a better method making mathematical knowledge of pupil real and useful instrument in their life. The main requirements of A.Ya. Khintchine to teach mathematics in the secondary school are: a) an account of the age-specific features of the pupils can lead to the necessity of simplified presentation of ideas and notions of the science. But such simplification should not falsify and even distort the scientific content of these notions; b) replacement of the rigorous and exact definitions and proofs by fuzzy presentations having no exact sense cannot facilitate the understanding of the subject. Khintchine said that fuzzy thinking is not simpler than precise thinking.

General Ideas on Mathematical Education in Secondary Schools

A.Ya. Khintchine thought that the main goal in the methodology of mathematics as well as in the educational process as a whole is to awake the creative ideas, to develop the technique mostly suitable for it. For him the success of pedagogical process is not in the collection of good marks, but in the depth of understanding. Pupils have to use correct and rigorous logical considerations, to see the gaps in arguments. The goal of teachers is to show the ways to solve independently non-standard problems, to sort out their known proofs.

He said (see [G 159, pp. 29–30]): "Two principles I have to lay into the base of the solution of the question at what level one or another mathematical notion should be studied in the secondary school taking into account the age features of pupils and the modern scientific understanding of this notion. These principles are:

1. If age-specific features of the pupils do not allow to give to certain notion its real scientific interpretation then the conception of this notion can be simplified. It means that the school should not develop the notion to the level accepted in the modern mathematical science. It can be stopped at one of previous levels of its development. But in any case the teacher should not distort the scientific meaning of the notion to give it feature which contradict this meaning. The school should not develop any notion in the direction deviating from the way of its scientific development.
2. If one changes sharp and exact definitions, formulations and reasoning by fuzzy ones, having no exact sense, leading at the sequential application to the logical contradiction, then it cannot simplify the real understanding. Fuzzy thinking cannot be simpler than the sharp one.

We suppose also that the usual theoretical scheme of the course of mathematics in the secondary school contains a lot of archaic notion which stay out of the the main stream of mathematical development. In most cases creating specially for the school notions which are not used in the science has no methodical sense and bring only damage to real mathematical development of pupils."

Basic Mathematical Notions in the Secondary School

Khintchine considered the following mathematical notions: number, limit and function as the basic ones. Starting the description of numbers the teacher has to formulate and clarify the definition of this notion. He/she has to create among his pupils the real understanding of a number as the subject of arithmetic operations. Of course it should be done carefully, step-by-step starting with natural numbers and integers, continued by fractions (rational numbers). It is important to make the whole picture of developing the notion of a number. Special attention has to be paid to introduction of irrational numbers since it allows to do the steps creating the continuous line measure, description of continuous measuring of many physical magnitudes. Finally, on this base pupils could understand the notion of continuity, of limit and continuous function (see [G 159, pp. 44–49]). Complex numbers have to be also an element of mathematical education in the secondary school. Teacher has to show how natural is to extend real numbers in this way, how many connections has such a notion with different mathematical problems of algebra and geometry. It can be done by using different geometric illustrations.

Since the notion of limit was changing in the course of its development one has to take it into account. This notion have to be developed in the pupils' mind too. It is not good to start already with the formal modern definition. Every teacher in mathematics knows that such formalism kills many important questions dealing with this notion: why it is appeared, how it is constructed, how it is connected with another basic notions of mathematics.

Almost all methodists of Khintchine's time (as well as of our time) had supposed that the notion of function has to be a central pivot of the whole course of mathematics. Basing on it the main notion of arithmetic, algebra, geometry and trigonometry have to be created. It is really correct but can lead to certain overvaluation and even abuse. This notion has to be applied very carefully accounting the age of pupils, the nature of the considered problems etc. Their study must not consist only of drawing graphs and considering formulas. The teacher has to show the real contents of this notion, show how one can use it for the solution of mathematical problems. From the other side, there is no need to make unnecessary generalizations and extensions, such as, e.g., introduction of multi-valued functions.

The question on the most suitable form of introduction of one or another mathematical notion in the course of mathematics in the secondary schools is one of the main aims of the mathematical methodology. Before starting a discussion concerning certain concrete notion one has to understand the general principles which are characteristic for the corresponding branch of mathematics. First of all we have to answer the question what is a mathematical definition, what is its role for the secondary schools, how and in which way one has to replace the definition in the case when its complete introduction is logically impossible or methodically unreasonable.

In the paper "On mathematical notions in the secondary school" (see [G 159, pp. 85–105]) A.Ya. Khintchine tried to clarify under which conditions these notions should be posed. He answers the questions on the most suitable introduction of concrete mathematical notions. The aim of the paper was to create a real background for further discussions of these questions in order to make these discussions really productive.

Formalism in Mathematical Education in the Secondary School

Khintchine considered the formalism of the mathematical knowledge and abilities as the most heavy disadvantages of the mathematical education in the secondary schools. Those pupils who get only formal part of mathematical methods became weak in front of problems arising in the real life. They could not pose the questions in the mathematical sense and from the mathematical point of view, and could not solve the problems of such a kind.

These pupils are also in a weak positions in the universities. The courses of mathematics in the universities are even more free from the formal approaches, their study requires understanding the corresponding ideas rather than to learn by heart the formal conceptions and mathematical notions.

One more dangerous consequence of the formalism in the study of mathematics in the secondary schools is that the formal knowledge of mathematics is useless for creating of scientific world-view of the pupils.

How to fight against formalism in the concrete situation, in the framework of the concrete programs of mathematical courses? These questions are discussed by Khintchine on the base of some examples taken from programs of mathematics and from his teaching practice (see his article "Formalism in teaching of mathematics in the secondary school" in [G 159, pp. 106–127]).

Educational Effects of Mathematical Classes

The article with such title (see [G 159, pp. 128–160]) was published already after the death of A.Ya. Khintchine though it was presented by him at one of the scientific meetings of the Mathematical Office of the Research Institute on school education of the Ministry of Education of Russian Federation. Some of his ideas, presented below, seems rather modern. Let us present few of them.

The objects of Mathematics (as a Science and as a Discipline studied in the secondary schools) are quantitative relations and spatial forms of real things, but not these things themselves. In a sense it decreases an educational effect of the course of mathematics in schools.

The most known method to reach such an effect is the use of specific mathematical logical rigor for creating the general logic and so called culture of thinking. From the other side, a teacher can equip certain mathematical problems by a concrete content and it gives possibility to extend the mental outlook of pupils, and to increase their general cultural level. But it is not all possible.

First of all, we have to speak about creating the culture of thinking. The teacher has to train pupils thinking correctly. He should always use the complete argumentation. Then, the teacher has to show on the concrete examples how wrong is the way of illegal generalizations and to fight against inconsistent analogies. It is important always to follow in the proofs the complete argumentation. The creating classifications has to be also complete and consistent.

Another moment, which is important to develop, is the concrete and logical style of thinking. In any case it is very useful in any area of the future life of pupils.

On so Called "Problems on Reasoning" in the Course of Arithmetics

Even in the modern schools we may find the idea to start teaching pupils from the early age on the base of so call problems on reasoning. A.Ya. Khintchine discussed this idea basing on the material of the course of mathematics (arithmetic) for 5th year of the schools (see [G 159, pp. 161–172]).[14]

He said: "If we ask even a very good teacher of mathematics how many pupils (of e.g. 5th year of the school) can solve the problems which consist not in simple calculations but need to find a special way of solving, then we have got not very impressive statistics. Of course quite a lot of pupils can do it if before they solve a number of analogous problems. Thus the aim to develop the quick-wittedness of pupils on the base of hard and non-standard problems can not be reached even by the very good teachers. If we consider the concrete problems proposed for pupils, we can see how difficult for them is to find the best way for solving, and even more difficult for teachers is to find a methodic of creating the corresponding skills. Besides, these problems appeared to be solved by using much more simple technique few years later. We can not understand for what reason pupils have to invent an unusual and nonstandard for his/her knowledge methods if he/she will reach the solution by standard technique at the corresponding age. Besides, even solution of quite a simple problem which is obtained correctly and very fast can be made pupils more happy than heavy "creative" thinking leading to the solution which is impossible to understand completely."

Teaching Mathematical Analysis

This part is based on the article by A.I. Markushevich "A.Ya. Khintchine as a teacher of mathematical analysis" included into the book A.Ya. Khintchine, "*Pedagogical Papers*" ([G 159, pp. 173–179]).

A.Ya. Khintchine had outstanding pedagogical skills. His courses were short but free of unessential details. He always tried to explain the importance and mathematical essence of the notions and to motivate the correct formulation of the problem. Thus he prepared the audience to discovering of a new research material. This was a necessary condition for them to follow the lecturer's guidance and to go through complicated mathematical constructions. From time to time he made some departure from the main course and explain to the students how to achieve one or another pedagogical aim. He taught them to be teachers.

He was never afraid to attack very complicated pedagogical problems. The characteristic example of it is the well-known book "Eight lectures on Mathematical Analysis" ([G 116]). This book is addressed to those who adopting the machinery of the mathematical analysis tries to understand its main ideas and logic. These 8 lectures (chapters) in the book are the following: I. Continuum, II. Limits, III. Functions, IV. Series, V. Derivative, VI. Integral, VII. Series representation of functions, VIII. Differential equation. Each of this subject is discussed in the historical retrospective, but mainly the presentation is made in such a way which could form the modern point of view of the reader.[15]

Another brilliant example of realization of pedagogical credo by A.Ya. Khinchine is his "Short course of mathematical analysis" ([G 144]). "Eight lectures" are more or less free in style and contain a lot of "considerations" and explanations, but the "Short course" is mainly rigorous. It presents all necessary facts and their proofs. Another feature of the book is that step by step it develops very vague and intuitive mathematical notions brought by students from the secondary schools. One of the most characteristic examples is the introduction (on a new level) the notion of the limit. It is done in the same way as we have described before, namely, several stages from the school-type notion to the rigorous should be made by readers under the guidance of the author.

He discussed many special problems of mathematical education. Among them is, for instance, the question of organization of the independent and unassisted work of students (how modern for any high school it sounds!). A.Ya. Khintchine said: "When we are talking about independent work of students we mean first of all their consulting by teachers. In reality it means that no matter what the difficulty is (even small one which can be clarified in 10 minutes of thinking) the student appeals to a teacher and got a final answer or exact citing without any feed-back from student's side. Is it a stimulation of independent work of students?"

Special attention has to be paid from his point of view to education and re-qualification of teachers of the secondary schools. They should get really scientific knowledge in all themes studied in the secondary schools and even more. It should not be a collection of certain receipe how to show pupils a solution of one or another problem. Such collections are widely spread (up to now!).

A.Ya. Khintchine was one of the supporters of the idea to introduce elements of analysis of infinitesimals already in the course of the secondary schools. Let as recall the essence of his idea concerning this

[14] This article was never published and was found by B.V. Gnedenko in the archive by A.Ya. Khintchine.

[15] It is amazing that after more than 50 years from the first edition of the book it remains modern and can teach young people with completely new mentality and taught by another mathematical program in the secondary schools.

question. “Analysis of infinitesimals (or Calculus) is one of the most import discovering of the mankind. It has numerous applications. Calculus is important for formation of the scientific point of view of our pupils.”

Teaching Probability Theory

The contribution of A.Ya. Khintchine in the theory of functions and the number theory is great. Anyway, the most of his scientific life was connected to Probability Theory. In 1920s A.Ya. Khintchine, A.N. Kolmogorov, E.E. Slutskii and P. Lévy have discovered the tight connection between Probability Theory and the mathematical branch which studied sets and general notion of the function. Very close to the understanding of this connection came a little bit earlier E. Borel.

The stochastic processes were discovered due to fundamental works by A.N. Kolmogorov and A.Ya. Khintchine. In a sense this theory was developing the ideas of A.A. Markov to study dependent random variables (called later Markov chains).

Systematic using of the methods of the set theory and the theory of functions of real variables in probabilistic models, construction of the base of the theory of stochastic processes, extended developing of the summation theory for independent random variables, as well as introduction of the new approach to the problems of statistical physics, all these things constitute the important impact of A.Ya. Khintchine to the creation of modern Probability Theory.

Starting from the problems connected with the number theory (The Law of Iterated Logarithm) and the theory of functions (convergence of series of random summands) he extended his interest to more and more deep problems of the theory. Moreover, he attracted many young Moscow mathematicians to study these problems and thus created highly developed Moscow school of Probability. Such behavior was really characteristic feature of his pedagogical talent. He was really a great Teacher. It is remarkable that his only textbook on Probability (joint with his student B.V. Gnedenko) he entitled “Elementary Introduction to the Probability Theory” ([G 118]), but which is neither elementary nor simple.

We have to mention also, that all his books appeared due to courses of lectures which he delivered in universities. Thus the first monograph ([G 35]) on Basic Laws of Probability appeared due to his intention to understand the nature of probabilistic approach and to show it to the young people the most simple examples. Second monograph deals with his explanation of the just completed by A.N. Kolmogorov and I.G. Petrovsky theory of Markov processes ([G 65]). At last the third monograph ([G 92]) is a real course of lectures. This course attracted to the problems of summation A.A. Bobrov, B.V. Gnedenko and D.A. Raikov.

This list is really incomplete. In any of his books (even a very small booklet) it can be recognized his intention to widely describe the corresponding ideas and to involve people in solving the appeared problems. Such approach we can find in his later works on queueing theory (or how he was called it “the theory of mass service”), information theory, foundation of statistical physics. It is very interesting to read paper by A.Ya. Khintchine in which he discuss the main ideas of probability with school boys and girls (see article in “Children Encyclopedia” [G 151] and brilliant booklet [G 70]).

He supposed that elements of probability have to be taught already in the secondary schools. His main arguments were: 1) many people finished the secondary school will encounter certain statistical data; the study of certain notions of the probability theory is a good base for it; 2) by solving concrete problems of the probability theory by using formulas of combinatorics the pupils may find an interest in these formulas and try to understand on a better level the meaning of them.

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