Aspects on Mean Field Equations with Multiple Singular Sources on Flat Tori

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Abstract. This is a survey on some recent advances on the mean field equation with multiple singular sources on flat tori, mainly based on a joint project with T.-J. Kuo and C.-S. Lin. Recent theories indicate that this equation has essential relations with several different subjects, such as the complex linear ODE with the Treibich-Verdier potential, Painlevé VI equation and premodular forms. Besides reviewing the known results and announcing new results, I will also raise some open questions from these different aspects.

Introduction

Let $\tau \in \mathbb{H} = \{\tau | \operatorname{Im} \tau > 0\}$ and $E_\tau := \mathbb{C}/\Lambda_\tau$ be a flat torus in the plane with lattice $\Lambda_\tau = \mathbb{Z}\omega_1 + Z\omega_2$, where $\omega_1 = 1$, $\omega_2 = \tau$. Also $\omega_0 = 0$ and $\omega_3 = 1+\tau$. Consider the following curvature equation with four singular sources:

$$\Delta u + e^u = 8\pi \sum_{k=0}^{3} n_k \delta_{\frac{\omega_k}{2}} \text{ on } E_\tau, \tag{1.1}$$

where $\delta_{\omega_k/2}$ is the Dirac measure at $\frac{\omega_k}{2}$, and $n_k \in \mathbb{Z}_{\geq 0}$ for all k with $\sum n_k \geq 1$. By changing variable $z \mapsto z + \frac{\omega_k}{2}$, we can always assume $n_0 = \max_k n_k \geq 1$.

Not surprisingly, (1.1) is related to various research areas. In conformal geometry, a solution u of (1.1) leads to a metric $ds^2 = \frac{1}{2}e^u(dx^2+dy^2)$ with constant Gaussian curvature $+1$ acquiring *conic singularities at* $\frac{\omega_k}{2}$*'s*. Equation (1.1) also belongs to a general class of equations, the so-called *mean field equations*:

$$\Delta u + \rho \left(\frac{he^u}{\int he^u} - \frac{1}{|M|} \right) = 4\pi \sum_{j=1}^{n} \alpha_j \left(\delta_{Q_j} - \frac{1}{|M|} \right) \text{ on } M, \tag{1.2}$$

where $h(x)$ is a positive C^1 function on a compact Riemann surface M without boundary. Equation (1.2) arises not only from conformal geometry, but also from many physical problems. For example, it arises in statistical physics as the equation for the *mean field limit* of the Euler flow in Onsager's vortex model (cf. [2]), hence its name. Recently it was shown that (1.2) is related to the self-dual condensates of the Chern-Simons-Higgs model in superconductivity; see e.g. [7, 17, 19, 28, 30]. Clearly (1.2) becomes (1.1) by letting $M = E_\tau$, $h = 1$, $n = 4$, $\alpha_j = 2n_j$, $Q_j = \frac{\omega_k}{2}$ and $\rho = 8\pi \sum n_j$.

Equation (1.2) has been studied extensively. It was proved in [1, 4, 5] that outside a countable set of critical parameters ρ's, solutions u of (1.2) have uniform a priori bounds in $C^2_{loc}(M \setminus \{Q_j\}_{j=1}^n)$. Thus the topological Leray-Schauder degree d_ρ is well-defined for non-critical ρ's. Recently, an explicit degree counting formula has been proved in [6], which has the following consequence: *Suppose that* $0 < \rho \notin 8\pi\mathbb{N}$, $\alpha_j \in \mathbb{N}$ *for all* j *and the genus* $g(M)$ *of* M *is at least* 1*. Then* $d_\rho > 0$*, hence the mean field equation (1.2) has a solution.*

However, the existence of solutions of (1.1) is very challenging from the PDE point of view, because

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$\rho = 8\pi\sum n_j$ are critical parameters in the above sense and the a priori estimates fail. In fact, *the solvability of (1.1) essentially depends on the moduli τ in a sophisticated manner*. This phenomena was first discovered by Lin and Wang [25] when they studied the case $n_0 = 1$ and $n_1 = n_2 = n_3 = 0$, i.e.

$$\Delta u + e^u = 8\pi\delta_0 \text{ on } E_\tau. \tag{1.3}$$

They proved that the Green function $G(z;\tau)$ on E_τ defined by

$$-\Delta G(z;\tau) = \delta_0 - \frac{1}{|E_\tau|} \text{ on } E_\tau, \int_{E_\tau} G(z;\tau) = 0,$$

has either three or five critical points (depending on τ), and (1.3) has solutions if and only if $G(z;\tau)$ has five critical points. For example, when $\tau \in i\mathbb{R}_{>0}$ (i.e. E_τ is a rectangular torus), $G(z;\tau)$ has only three critical points $\frac{\omega_k}{2}$ ($k = 1,2,3$) and so (1.3) has *no* solution; while for $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ (i.e. E_τ is a rhombus torus), $G(z;\tau)$ has five critical points $\frac{\omega_k}{2}$ ($k = 1,2,3$) and $\pm\frac{\omega_3}{3}$, so (1.3) has solutions. Recently, (1.3) was thoroughly investigated in [14, 27], where among other things, the geometry of Ω_5 (i.e. the set of τ such that (1.3) has solutions) was studied.

For the case $n_0 = n \geq 2$ and $n_1 = n_2 = n_3 = 0$, i.e.

$$\Delta u + e^u = 8n\pi\delta_0 \text{ on } E_\tau, \tag{1.4}$$

Chai, Lin, Wang [3] and subsequently Lin, Wang [26] studied it from the viewpoint of algebraic geometry. They developed a theory to connect this PDE problem with a multiple Green function on E_τ^n, the classical Lamé equation, the associated hyperelliptic curves and premodular forms. We refer the interested readers to [3, 26] for the story of this theory.

In view of [3, 25, 26], it is natural for us to consider the general case $n_k \in \mathbb{Z}_{\geq 0}$ for all k. Our purpose of this research project is not only to study this problem from its PDE aspect, but also to explore its connection with various different subjects and to understand how these different subjects apply to each other. In the following sections, we briefly review some recent advances on (1.1) from three different aspects and raise some open questions. As we will see, quite different phenomenon happens for the general case $n_k \in \mathbb{Z}_{\geq 0}$ comparing to the case $n_1 = n_2 = n_3 = 0$.

From the Aspect of Linear ODE

This section is devoted to the connection between the mean field equation (1.1) and a complex linear ODE from the viewpoint of the integrable system. See [3] for the special case $n_1 = n_2 = n_3 = 0$, i.e. equation (1.4).

The Liouville theorem says that for any solution $u(z)$ of (1.1), there is a meromorphic function $f(z)$ in $\mathbb{C}$ such that

$$u(z) = \log\frac{8|f'(z)|^2}{(1+|f(z)|^2)^2}. \tag{2.1}$$

This $f(z)$ is called a developing map. Differentiating (2.1) leads to

$$u_{zz} - \tfrac{1}{2}u_z^2 = \{f;z\} := \left(\tfrac{f''}{f'}\right)' - \tfrac{1}{2}\left(\tfrac{f''}{f'}\right)^2. \tag{2.2}$$

Conventionally, the RHS of (2.2) is called the Schwarzian derivative of $f(z)$, denoted by $\{f;z\}$. Note that outside the singularities $E_\tau[2] := \{\frac{\omega_k}{2} \,|\, 0 \leq k \leq 3\} + \Lambda_\tau$, we have $(u_{zz} - \frac{1}{2}u_z^2)_{\bar{z}} = 0$. Furthermore, using the local behavior of $u(z)$ at $\frac{\omega_k}{2}$, we see that $u_{zz} - \frac{1}{2}u_z^2$ has at most double poles at each $\frac{\omega_k}{2}$. Therefore, $u_{zz} - \frac{1}{2}u_z^2$ is an *elliptic function* with at most *double poles* at $E_\tau[2]$.

Assume in addition that $u(z)$ is *even*, i.e. $u(z) = u(-z)$. Then $u_{zz} - \frac{1}{2}u_z^2$ is even elliptic and so has the following simple expression

$$u_{zz} - \frac{1}{2}u_z^2 = -2\left[\sum_{k=0}^{3} n_k(n_k+1)\wp(z+\tfrac{\omega_k}{2};\tau) + B\right] =: -2I(z), \tag{2.3}$$

where B is some constant and $\wp(z;\tau)$ is the famous Weierstrass $\wp$-function with periods $\omega_1 = 1$ and $\omega_2 = \tau$, because due to the evenness, $u_{zz} - \frac{1}{2}u_z^2$ has no residues at $z \in E_\tau[2]$. This fact motivates us to study the following generalized Lamé equation (GLE)

$$y''(z) = I(z)y(z) = \left[\sum_{k=0}^{3} n_k(n_k+1)\wp(z+\tfrac{\omega_k}{2};\tau) + B\right]y(z). \tag{2.4}$$

Since $\{f;z\} = -2I(z)$, a classical result says that there exist linearly independent solutions $y_1(z), y_2(z)$ of GLE (2.4) such that $f(z) = y_1(z)/y_2(z)$.

Note that GLE (2.4) becomes the classical Lamé equation when three n_k's vanish, such as $n_1 = n_2 = n_3 = 0$. GLE (2.4) is the elliptic form of Heun's equation and the potential $-\sum_{k=0}^{3} n_k(n_k+1)\wp(z+\frac{\omega_k}{2};\tau)$ is the so-called *Treibich-Verdier potential* ([34]), which is well known as an algebro-geometric finite-gap potential associated with the stationary KdV hierarchy. We refer the readers to [22, 31, 32, 33, 34] and references therein for historical reviews and subsequent developments. The precise relation between (1.1) and GLE (2.4) is stated as follows.

Theorem 2.1. *The mean field equation (1.1) has an even solution if and only if there exists $B \in \mathbb{C}$ such that the monodromy representation of GLE (2.4) is unitary (i.e. the monodromy group is conjugate to a subgroup of $SU(2)$).*

Theorem 2.1 in the case $n_1 = n_2 = n_3 = 0$ was proved in [3]. The necessary part of Theorem 2.1 is not difficult; see e.g. [16, 20]. The proof of the sufficient part is much more delicate and will be given in a coming paper.

Theorem 2.1 provides a new way to attack the PDE problem of solving (1.1). Recently, we applied Theorem 2.1 to obtain a sharp nonexistence result for (1.1). Our motivation comes from a conjecture of Lin and Wang [26], which asserts that *(1.4) has no solutions for all $n \in \mathbb{N}$ if $\tau \in i\mathbb{R}_{>0}$ (i.e. E_τ is a rectangular torus).* Geometrically, this conjecture is equivalent to assert that the rectangular torus admits no metric with constant curvature 1 and a conical singularity with angle $2\pi(1+2n)$. This conjecture seems challenging from the PDE point of view, and is known to be true only for $n=1$ ([7, 25]) and $n=2$ ([10]), but the approach does not work for $n \geq 3$. Now this conjecture is a consequence of the following sharp nonexistence result.

Theorem 2.2 ([16]). *Let $n_k \in \mathbb{Z}_{\geq 0}$ for all k with $\max_k n_k \geq 1$. If (n_0,n_1,n_2,n_3) satisfies neither*

$$\frac{n_1+n_2-n_0-n_3}{2} \geq 1, \quad n_1 \geq 1, \quad n_2 \geq 1 \tag{2.5}$$

nor

$$\frac{n_1+n_2-n_0-n_3}{2} \leq -1, \quad n_0 \geq 1, \quad n_3 \geq 1, \tag{2.6}$$

then for each $\tau \in i\mathbb{R}_{>0}$, the monodromy of GLE (2.4) can not be unitary for any $B \in \mathbb{C}$, namely equation (1.1) on E_τ has no even solutions.

It was proved in [3] that once (1.4) has a solution, then it has also an even solution. Thus Theorem 2.2 confirms the aforementioned conjecture. We want to emphasize that the statement that *the monodromy of GLE (2.4) can not be unitary* is interesting itself from the viewpoint of monodromy theory of linear ODEs.

Our condition on n_k in Theorem 2.2 is *sharp*, because Eremenko and Gabrielov [19] proved that *(1.1) has an even and symmetric solution $u(z)$ (i.e. $u(z) = u(-z) = u(\bar{z})$) on some rectangular torus E_τ if and only if (n_0,n_1,n_2,n_3) satisfies either (2.5) or (2.6).* Their approach is geometric and relies essentially on the even symmetric assumption. Theorem 2.2 improves their result because the symmetric assumption $u(z)=u(\bar{z})$ is removed. We emphasize that this improvement is not trivial at all, because our numerical computation shows that there exist $1 < b_1 < b_2 < \sqrt{3}$ such that for any $\tau = ib$ with $b \in (b_1,b_2)$,

$$\Delta u + e^u = 16\pi\delta_0 + 16\pi\delta_{\omega_3/2} \text{ on } E_\tau$$

has *no even and symmetric solutions* but does have *two even solutions*. We suspect that the even assumption $u(z)=u(-z)$ is not necessary either, namely we propose the following conjecture.

Conjecture 2.3 ([16]). *Equation (1.1) has no solution for any $\tau \in i\mathbb{R}_{>0}$ if and only if (n_0,n_1,n_2,n_3) satisfies neither (2.5) nor (2.6).*

On the other hand, Theorem 2.2 is also related to the following conjecture concerning counting solutions.

Conjecture 2.4 ([16]). *Suppose $\tau \in i\mathbb{R}_{>0}$ and $\rho \in (8\pi(n-1), 8\pi n)$, $n \in \mathbb{N}$. Then the equation*

$$\Delta u + e^u = \rho\delta_0 \quad \text{on } E_\tau$$

possesses exactly n solutions.

Conjecture 2.4 was already proved for $\rho \in (0, 8\pi)$ in [27] and for $\rho = 8\pi(n-\frac{1}{2})$ in [3]. In a coming paper, we will apply Theorem 2.2 to prove Conjecture 2.4 for $|\rho - 8\pi n| \ll 1$ and $|\rho - 8\pi(n-1)| \ll 1$.

Our proof of Theorem 2.2 is to apply the spectral theory of finite-gap potential, or equivalently the algebro-geometric solutions of stationary KdV hierarchy equations [21], and can be seen as an unexpected application of the KdV theory. More precisely, our proof is based on the study of the so-called spectral polynomial. It is well known (cf. [22, 31]) that there associates a *spectral polynomial* $Q_{(n_0,n_1,n_2,n_3)}(B;\tau)$ of B for the Treibich-Verdier potential; see Section 4 for a brief review. When $\tau \in i\mathbb{R}_{>0}$, $Q_{(n_0,n_1,n_2,n_3)}(B;\tau)$ is a polynomial of B with real coefficients. Then a natural question is whether $Q_{(n_0,n_1,n_2,n_3)}(B;\tau)$ has real and distinct zeros for $\tau \in i\mathbb{R}_{>0}$. We can prove the following surprising result on this aspect.

Theorem 2.5 ([16]). *Let $n_k \in \mathbb{Z}_{\geq 0}$ for all k with $\max_k n_k \geq 1$. Then all the zeros of $Q_{(n_0,n_1,n_2,n_3)}(\cdot;\tau)$ are real and distinct for $\tau \in i\mathbb{R}_{>0}$ if and only if $(n_0,n_1,n_2,\, n_3)$ satisfies neither (2.5) nor (2.6).*

In view of Theorem 2.5, the next key step of proving Theorem 2.2 is to show that if all the zeros of $Q_{(n_0,n_1,n_2,n_3)}(\cdot;\tau)$ are real and distinct for $\tau \in i\mathbb{R}_{>0}$, then the monodromy of GLE (2.4) can not be unitary for any $B \in \mathbb{C}$. This statement is also interesting itself from the viewpoint of monodromy theory of linear ODEs.

From the Aspect of Painlevé VI Equation

In this section, we introduce the connection between the mean field equation (1.1) and the well-known Painlevé VI equation, which seems not appear in the literature and is new. The classical Painlevé VI equation with four free parameters $(\alpha,\beta,\gamma,\delta)$ (PVI$(\alpha,\beta,\gamma,\delta)$) is written as

$$\begin{aligned}\frac{d^2\lambda}{dt^2} = {} & \frac{1}{2}\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 \\ & -\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t}\right)\frac{d\lambda}{dt}+\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2} \\ & \times\left[\alpha+\beta\frac{t}{\lambda^2}+\gamma\frac{t-1}{(\lambda-1)^2}+\delta\frac{t(t-1)}{(\lambda-t)^2}\right].\end{aligned} \tag{3.1}$$

Historically, PVI was originated from the research on complex ODEs, led by many famous mathematicians including Poincaré, Picard, Painlevé and so on. The aim is to classify those nonlinear ODEs which possess the so-called *Painlevé property*, i.e. any solution has neither movable branch points nor movable essential singularities. Due to an increasingly important role in both mathematics and physics, PVI has been widely studied since the early 1970's. We refer to the text [23] for the introduction of Painlevé equations.

From the Painlevé property, any solution $\lambda(t)$ of (3.1) is a multi-valued meromorphic function in $\mathbb{C}\backslash\{0,1\}$. Therefore, it is natural to lift (3.1) to the covering space $\mathbb{H}=\{\tau \mid \operatorname{Im}\tau>0\}$ of $\mathbb{C}\backslash\{0,1\}$ by the following transformation:

$$t(\tau)=\frac{e_3(\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)}, \quad \lambda(t)=\frac{\wp(p(\tau);\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)},$$

where $e_k(\tau):=\wp(\frac{\omega_k}{2};\tau)$, $k\in\{1,2,3\}$. Consequently, $p(\tau)$ satisfies the following elliptic form (cf. [29])

$$\frac{d^2p(\tau)}{d\tau^2}=\frac{-1}{4\pi^2}\sum_{k=0}^{3}\alpha_i\wp'\left(p(\tau)+\tfrac{\omega_k}{2};\tau\right), \tag{3.2}$$

where $\wp'(z;\tau)=\frac{d}{dz}\wp(z;\tau)$ and $(\alpha_0,\alpha_1,\alpha_2,\alpha_3)=(\alpha,-\beta,\gamma,\frac{1}{2}-\delta)$. The advantage of (3.2) is that $\wp(p(\tau);\tau)$ is single-valued for $\tau\in\mathbb{H}$, although $p(\tau)$ has a branch point at those τ_0 such that $p(\tau_0)\in E_{\tau_0}[2]$.

Another important feature of PVI is that it governs the isomonodromic deformation of a second order Fuchsian ODE on $\mathbb{CP}^1$. Recently we proved in [8] that similar to this well-known case on $\mathbb{CP}^1$, the elliptic form (3.2) is closed related to the isomonodromy theory of the following generalized Lamé equation (GLE):

$$y''(z)=\begin{bmatrix}\sum_{k=0}^{3}n_k(n_k+1)\wp(z+\frac{\omega_k}{2};\tau)+\frac{3}{4}(\wp(z+p;\tau)\\+\wp(z-p;\tau))+A(\zeta(z+p;\tau)-\zeta(z-p;\tau))+B\end{bmatrix}y(z), \tag{3.3}$$

with $\pm p\notin E_\tau[2]$ and

$$B=A^2-\zeta(2p;\tau)A-\tfrac{3}{4}\wp(2p;\tau)-\sum_{k=0}^{3}n_k(n_k+1)\wp\left(p+\tfrac{\omega_k}{2};\tau\right), \tag{3.4}$$

where $\zeta(z)=\zeta(z;\tau):=-\int^z\wp(\xi;\tau)d\xi$ is the Weierstrass zeta function and the relation of parameters is given by

$$\alpha_k=\tfrac{1}{2}\left(n_k+\tfrac{1}{2}\right)^2 \text{ for } k\in\{0,1,2,3\}. \tag{3.5}$$

This GLE has regular singularities at $E_\tau[2]\cup\{\pm p\}$ with local exponents $-n_k$, n_k+1 at $\frac{\omega_k}{2}$ and $\frac{-1}{2},\frac{3}{2}$ at $\pm p$, respectively. Since the exponent difference at $\pm p$ is 2, (3.3) might have solutions with logarithmic singularity at $\pm p$. The formula (3.5) guarantees that $\pm p$ are *apparent singularities* (i.e. non-logarithmic). Define a completely integrable Hamiltonian system:

$$\frac{dp(\tau)}{d\tau}=\frac{\partial K}{\partial A}, \quad \frac{dA(\tau)}{d\tau}=-\frac{\partial K}{\partial p}, \tag{3.6}$$

where the Hamiltonian $K:=\frac{-i}{4\pi}(B+2p\eta_1(\tau)A)$. Here

$$\eta_1(\tau):=2\zeta(\tfrac{1}{2};\tau)=\zeta(z+1;\tau)-\zeta(z;\tau) \tag{3.7}$$

is a quasi-period of $\zeta(z;\tau)$. Then the Hamiltonian system (3.6) is equivalent to the elliptic form (3.2) with parameter (3.5). Furthermore, under the non-resonant condition $n_k\notin\frac{1}{2}+\mathbb{Z}$ for all k, $(p(\tau),A(\tau))$ satisfies the Hamiltonian system (3.6) if and only if GLE (3.3) with $(p(\tau),A(\tau))$ is monodromy preserving as τ deforms. See [8].

Now we turn back to the integer case $n_k\in\mathbb{Z}_{\geq 0}$. Let $p(\tau)$ be a solution of the elliptic form (3.2). We call that $p(\tau)$ is **a real solution** if the monodromy of the corresponding GLE (3.3) with $(p(\tau),A(\tau))$ is unitary. On the other hand, if $p(\tau_0)\equiv 0 \bmod \Lambda_{\tau_0}$ for some τ_0, then $t(\tau_0)\notin\{0,1\}$ is a pole of $\lambda(t)=\frac{\wp(p(\tau);\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)}$, and so we call τ_0 a pole of $p(\tau)$. We proved in [8] that the asymptotics at a pole τ_0 is

$$p(\tau)-p(\tau_0)=c_0(\tau-\tau_0)^{\frac{1}{2}}(1+h(\tau-\tau_0)+O(\tau-\tau_0)^2) \tag{3.8}$$

with $c_0^2=\pm i\frac{2n_0+1}{2\pi}$ and some constant $h\in\mathbb{C}$. Recall $n_0\geq 0$. As in [11] we call that a pole τ_0 of $p(\tau)$ is **a negative pole** (resp. **a positive pole**) if the asymptotics is given by (3.8) with $c_0^2=i\frac{2n_0+1}{2\pi}$ (resp. $c_0^2=-i\frac{2n_0+1}{2\pi}$). Now we can introduce the relation between GLE (3.3) and GLE (2.4). Let $p\to\frac{\omega_k}{2}$ in GLE (3.3) and suppose its potential converges, then GLE (3.3) will converge to GLE (2.4) with n_k replaced by $n_k\pm 1$. See [12] for the proof. In particular, if τ_0 is a negative pole (resp. a positive pole) of $p(\tau)$, then GLE (3.3) with $(p(\tau),A(\tau))$ will converge to GLE (2.4) with n_0 replaced by n_0+1 (resp. n_0-1) as $\tau\to\tau_0$. From here and the theory in Section 2, we obtain the following surprising connection between the mean field equation (1.1) and the elliptic form of Painlevé VI equation.

Theorem 3.1 ([13]). *Suppose $n_k\in\mathbb{Z}_{\geq 0}$ with $n_0\geq 1$. Then the mean field equation (1.1) has even solutions on E_{τ_0} if and only if τ_0 is a negative pole of some real solution $p(\tau)$ of the elliptic form (3.2) with parameter*

$$\alpha_0=\tfrac{1}{2}\left((n_0-1)+\tfrac{1}{2}\right)^2,\ \alpha_k=\tfrac{1}{2}\left(n_k+\tfrac{1}{2}\right)^2 \textit{ for } k\in\{1,2,3\}. \tag{3.9}$$

As mentioned before, the existence of even solutions of (1.1) depends essentially on the moduli τ of the underlying torus E_τ. Theorem 3.1 gives a precise characterization of those τ's such that (1.1) has even solutions from the viewpoint of Painlevé VI equation. This theory has interesting applications to Painlevé

VI equation. For example, if (1.1) has no even solutions on E_{τ_0}, then τ_0 can not be a negative pole of real solutions of (3.2) with parameter (3.9). Applying this idea, we proved in [14, 9] that all real solutions of PVI$(\frac{1}{8},\frac{-1}{8},\frac{1}{8},\frac{3}{8})$ and PVI$(\frac{9}{8},\frac{-1}{8},\frac{1}{8},\frac{3}{8})$ have no poles on the real line $\mathbb{R}$. We will prove this assertion for Painlevé VI equation with more general parameters $(\alpha,\beta,\gamma,\delta)$ in a subsequent one of [9].

We also study the connection between Painlevé VI equation and the mean field equation with additional singularities $\pm p\notin E_{\tau_0}$. A related result of Theorem 3.1 is following.

Theorem 3.2 ([13]). *Suppose $n_k\in\mathbb{Z}_{\geq 0}$, $\tau_0\in\mathbb{H}$ and $p\notin E_{\tau_0}[2]$. Then the following mean field equation with six singularities*

$$\Delta u+e^u=8\pi\sum_{k=0}^{3}n_k\delta_{\frac{\omega_k}{2}}+4\pi(\delta_p+\delta_{-p})\ \text{on}\ E_{\tau_0}, \tag{3.10}$$

has an even solution if and only if there exists a real solution $p(\tau)$ of the elliptic form (3.2) with parameter (3.5) such that $p=p(\tau_0)$.

Theorem 3.2 shows a new phenomena for the mean field equation: For a fixed underlying torus E_{τ_0}, the existence of even solutions depends essentially on the location of the additional singularities $\pm p$, i.e. (3.10) has even solutions if and only if $p\in\Omega_{\tau_0}$, where

$$\Omega_{\tau_0}:=\{p(\tau_0)\,|\,p(\tau)\text{ is a real solution of (3.2) with (3.5)}\}\setminus E_{\tau_0}[2].$$

We have explicit examples that $\Omega_{\tau_0}\neq E_{\tau_0}\setminus E_{\tau_0}[2]$. An interesting question is to study the geometry of this set Ω_{τ_0}. Besides, we remark that our assumption of evenness of solutions in Theorems 3.1 and 3.2 is not needed if three of n_k's are zeros, because in this case, we can prove that *once (3.10) (resp. (1.1)) has a solution, then it has an even solution*. We believe that this assertion holds for all $n_k\in\mathbb{Z}_{\geq 0}$, namely we conjecture that the assumption of evenness of solutions in Theorems 3.1 and 3.2 can be removed.

From the Aspect of Premodular Forms

In this section, we introduce the connection between the mean field equation (1.1) and premodular forms. First we briefly recall the spectral polynomial mentioned in Section 2. Denote $\mathbf{n}=(n_0,n_1,n_2,n_3)$ and consider the second symmetric product equation of GLE (2.4):

$$\Phi'''(z)-4I(z)\Phi'(z)-2I(z)\Phi(z)=0. \tag{4.1}$$

It is known (cf. [31]) that (4.1) has a solution $\Phi(z;B)$ which is a polynomial in B with coefficients being even elliptic functions. Multiplying Φ and integrating (4.1), we obtain that

$$\Phi'(z;B)^2-2\Phi(z;B)\Phi''(z;B)+4I(z)\Phi(z;B)^2 \tag{4.2}$$

is a polynomial of B which independent of z. Let $Q_{\mathbf{n}}(B)=Q_{\mathbf{n}}(B;\tau)$ denote the corresponding *monic polynomial* given by (4.2). This $Q_{\mathbf{n}}(B)$ is known as the *spectral polynomial* and

$$\Gamma_{\mathbf{n}}=\Gamma_{\mathbf{n}}(\tau):=\{(B,W)|W^2=Q_{\mathbf{n}}(B;\tau)\}$$

is called the *spectral curve* of the Treibich-Verdier potential.

To introduce the notion of premodular forms and its relation with $\Gamma_{\mathbf{n}}$, we recall that the monodromy representation ρ of (2.4) is a group homomorphism from $\pi_1(E_\tau)$ to $SL(2,\mathbb{C})$, because the Treibich-Verdier potential is a Picard potential in the sense of Gesztesy and Weikard (cf. [21, 22]). Since $\pi_1(E_\tau)$ is abelian, the monodromy group is always abelian. In terms of any linearly independent solutions $y_1(z)$ and $y_2(z)$, the monodromy group is generated by two matrices M_1, $M_2\in SL(2,\mathbb{C})$ satisfying

$$(y_1,y_2)(z+\omega_i)=(y_1(z),y_2(z))M_i,\ i=1,2, \tag{4.3}$$

and $M_1M_2=M_2M_1$. From here, M_1 and M_2 can be normalized to satisfy one of the followings.

a) If ρ is completely reducible, then

$$M_1=\begin{pmatrix}e^{-2\pi is}&0\\0&e^{2\pi is}\end{pmatrix},\quad M_2=\begin{pmatrix}e^{2\pi ir}&0\\0&e^{-2\pi ir}\end{pmatrix},\ (r,s)\in\mathbb{C}^2\setminus\tfrac{1}{2}\mathbb{Z}^2. \tag{4.4}$$

b) If ρ is not completely reducible, then

$$M_1=\varepsilon_1\begin{pmatrix}1&0\\1&1\end{pmatrix},\quad M_2=\varepsilon_2\begin{pmatrix}1&0\\C&1\end{pmatrix},\ \varepsilon_j=\pm1,\ C\in\mathbb{C}\cup\{\infty\}. \tag{4.5}$$

When $C=\infty$, the monodromy matrices are understood as

$$M_1=\varepsilon_1\begin{pmatrix}1&0\\0&1\end{pmatrix},M_2=\varepsilon_2\begin{pmatrix}1&0\\1&1\end{pmatrix}. \tag{4.6}$$

In particular, the monodromy is unitary if and only if Case a) occurs with $(r,s)\in\mathbb{R}^2\setminus\frac{1}{2}\mathbb{Z}^2$. See e.g. [12]. The aforementioned spectral polynomial $Q_{\mathbf{n}}(B;\tau)$ also plays a key role in the monodromy theory: *ρ is completely reducible if and only if $Q_{\mathbf{n}}(B;\tau)\neq 0$*. Obviously, not all 2×2 matrices of the form (4.4)–(4.5) are monodromy matrices of (2.4). Thus the following questions naturally arise:

(1) If $Q_{\mathbf{n}}(B;\tau)\neq 0$, how to determine the monodromy data (r,s)?
(2) If $Q_{\mathbf{n}}(B;\tau)=0$, how to determine the monodromy data C?

For the Lamé equation

$$y''(z) = [n(n+1)\wp(z;\tau) + B]y(z), \tag{4.7}$$

in [3, 26] Chai, Lin and Wang have constructed a *premodular form* $Z_{r,s}^{(n)}(\tau)$ such that the monodromy matrices M_1, M_2 of (4.7) at $\tau = \tau_0$ with some B are given by (4.4) if and only if $Z_{r,s}^{(n)}(\tau_0) = 0$. Therefore, the image of M_1, M_2 for ρ is $\{(r,s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2 | Z_{r,s}^{(n)}(\tau_0) = 0\}$. We note that $Z_{r,s}^{(n)}(\tau)$ is holomorphic in τ if $(r,s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$. Moreover, $Z_{r,s}^{(n)}(\tau)$ is a modular form of weight $\frac{n(n+1)}{2}$ with respect to the principal congruence subgroup $\Gamma(N)$ if $(r,s) = (\frac{k_1}{N}, \frac{k_2}{N})$ is a N-torsion point; see [26]. Thus $Z_{r,s}^{(n)}(\tau)$ is called a **premodular form.**

Here we want to extend the result in [26] to include the Trebich-Verdier potential. In [12] and a subsequent one, we will prove the following result.

Theorem 4.1. *There exists a premodular form $Z_{r,s}^{\mathbf{n}}(\tau)$ defined in $\tau \in \mathbb{H}$ for any pair of $(r,s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ such that the followings hold.*

- (a) *If $(r,s) = (\frac{k_1}{N}, \frac{k_2}{N})$ with $N \in 2\mathbb{N}_{\geq 2}$, $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ and $\gcd(k_1,k_2,N) = 1$, then $Z_{r,s}^{\mathbf{n}}(\tau)$ is a modular form of weight $\sum_{k=0}^{3} n_k(n_k+1)/2$ with respect to the principal congruence subgroup $\Gamma(N)$.*
- (b) *For $(r,s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ and $\tau_0 \in \mathbb{H}$ such that $r + s\tau_0 \notin \Lambda_{\tau_0}$, $Z_{r,s}^{\mathbf{n}}(\tau_0) = 0$ if and only if there is $B \in \mathbb{C}$ such that GLE (2.4) with $\tau = \tau_0$ has its monodromy matrices M_1 and M_2 given by (4.4).*
- (c) *The mean field equation (1.1) on E_{τ_0} has even solutions if and only if $Z_{r,s}^{\mathbf{n}}(\tau_0) = 0$ for some $(r,s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$.*

Theorem 4.1 generalizes the theory in [26] to the general case $n_k \in \mathbb{Z}_{\geq 0}$ for all k. As mentioned before, the existence of even solutions of (1.1) depends essentially on the moduli τ of the underlying torus E_τ. Theorem 4.1-(c) gives a precise characterization of those τ's such that (1.1) has even solutions from the viewpoint of zeros of certain premodular forms. Together with Theorem 3.1, we see that the zeros of premodular forms $Z_{r,s}^{\mathbf{n}}(\tau)$ with $(r,s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ coincide with the negative poles of real solutions of the elliptic form (3.2) with parameter (3.9). This phenomena strongly suggests the following conjecture.

Conjecture 4.2. *The premodular form $Z_{r,s}^{\mathbf{n}}(\tau)$ appears in the denominator of the expression of $\wp(p(\tau);\tau)$ for real solutions $p(\tau)$ of the elliptic form (3.2) with parameter (3.9).*

For the case $n_1 = n_2 = n_3 = 0$, Conjecture 4.2 is known to be true [11, 24], where we proved that the premodular form $Z_{r,s}^{(n)}(\tau)$ constructed by Lin and Wang [26] appears in the denominator of the expressions of real solutions of the elliptic form (3.2) with parameter $(\frac{1}{2}(n-\frac{1}{2})^2, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$. From here, we proved that for any $(r,s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$, *$Z_{r,s}^{(n)}(\tau)$ has only simple zeros in $\mathbb{H}$.* This simple zero property is crucial for us [11, 24] to confirm Dahmen and Beukers's conjecture (cf. [18]) of counting the number of the Lamé equation (4.7) with its monodromy group being Dihedral D_N. Therefore, Conjecture 4.2 is crucial to the following conjecture.

Conjecture 4.3. *For any $(r,s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$, the premodular form $Z_{r,s}^{\mathbf{n}}(\tau)$ has only simple zeros for $\tau \in \mathbb{H}$.*

For $n \leq 4$, the explicit expression of $Z_{r,s}^{(n)}(\tau)$ is known; see [26]. Define $\eta_k(\tau) := 2\zeta(\frac{\omega_k}{2};\tau)$, $k = 1,2$ to be two quasi-periods of $\zeta(z;\tau)$:

$$\eta_1(\tau) = \zeta(z+1;\tau) - \zeta(z;\tau), \quad \eta_2(\tau) = \zeta(z+\tau;\tau) - \zeta(z;\tau).$$

Define

$$Z = Z_{r,s}(\tau) := \zeta(r+s\tau;\tau) - r\eta_1(\tau) - s\eta_2(\tau).$$

Then it is known [26] that (write $\wp = \wp(r+s\tau;\tau)$ and $\wp' = \wp'(r+s\tau;\tau)$ for convenience): $Z_{r,s}^{(1)}(\tau) = Z_{r,s}(\tau)$,

$$\begin{aligned}
Z_{r,s}^{(2)}(\tau) &= Z^3 - 3\wp Z - \wp', \\
Z_{r,s}^{(3)}(\tau) &= Z^6 - 15\wp Z^4 - 20\wp' Z^3 + \left(\tfrac{27}{4}g_2 - 45\wp^2\right)Z^2 \\
&\quad - 12\wp\wp' Z - \tfrac{5}{4}(\wp')^2. \\
Z_{r,s}^{(4)}(\tau) &= Z^{10} - 45\wp Z^8 - 120\wp' Z^7 + (\tfrac{399}{4}g_2 - 630\wp^2)Z^6 \\
&\quad - 504\wp\wp' Z^5 - \tfrac{15}{4}(280\wp^3 - 49g_2\wp - 115g_3)Z^4 \\
&\quad + 15(11g_2 - 24\wp^2)\wp' Z^3 \\
&\quad - \tfrac{9}{4}(140\wp^4 - 245g_2\wp^2 + 190g_3\wp + 21g_2^2)Z^2 \\
&\quad - (40\wp^3 - 163g_2\wp + 125g_3)\wp' Z \\
&\quad + \tfrac{3}{4}(25g_2 - 3\wp^2)(\wp')^2.
\end{aligned}$$

For general n, the expression of $Z_{r,s}^{(n)}(\tau)$ is too complicate to be written down. Here are new examples for $Z_{r,s}^{\mathbf{n}}(\tau)$:

$$\begin{aligned}
Z_{r,s}^{(1,1,0,0)} &= Z^2 - \wp + e_1(\tau), \\
Z_{r,s}^{(1,0,1,0)} &= Z^2 - \wp + e_2(\tau), \\
Z_{r,s}^{(1,0,0,1)} &= Z^2 - \wp + e_3(\tau).
\end{aligned}$$

We believe that such premodular forms will have important applications. For example, we used $Z_{r,s}^{(2)}(\tau)$ to completely determine the critical points of the Eisenstein series $E_2(\tau)$ of weight 2. See [15] where we proved that $E_2'(\tau)$, as a quasimodular form, has at most one zero in each fundamental domain of $\Gamma_0(2)$.

Finally, we briefly explain the basic idea of proving Theorem 4.1. Following the ideas in [3, 26], the spectral curve $\Gamma_{\mathbf{n}}(\tau)$ can be embedded into $\mathrm{Sym}^N E_\tau := E_\tau^N / S_N$, the symmetric space of N-th copy of E_τ, where $N := \sum_{k=0}^{3} n_k$ from now on. Obviously, $\mathrm{Sym}^N E_\tau$ has a natural addition map to E_τ: $\{a_1, \cdots, a_N\} \mapsto \sum_{i=1}^{N} a_i$. Then the composition give arise to a finite morphism $\sigma_{\mathbf{n}}(\cdot|\tau)$:

$\overline{\Gamma_{\mathbf{n}}(\tau)} \to E_\tau$, still called the *addition map*. The degree of $\sigma_{\mathbf{n}}$ is defined as $\deg \sigma_{\mathbf{n}}(\cdot|\tau) = \#\sigma_{\mathbf{n}}^{-1}(z)$, $z \in E_\tau$, counted with multiplicity. Then the first key step of proving Theorem 4.1 is to prove the following result.

Theorem 4.4 ([12]). *Let $\tau \in \mathbb{H}$. Then the addition map $\sigma_{\mathbf{n}}(\cdot|\tau) : \overline{\Gamma_{\mathbf{n}}(\tau)} \to E_\tau$ has degree $\sum_{k=0}^{3} n_k(n_k+1)/2$.*

A corollary of Theorem 4.4 is that $\deg \sigma_{\mathbf{n}}(\cdot|\tau)$ (the same as the weight of the premodular form in Theorem 4.1) is independent of τ, which is not very obvious at the moment. When $n_1 = n_2 = n_3 = 0$, Theorem 4.4 was proved in [26] by applying *Theorem of the Cube* for morphisms between varieties in algebraic geometry, but this method seems not work in the general case. Our new strategy is to study GLE (3.3). Like GLE (2.4), we could associate a hyperelliptic curve $\Gamma_{\mathbf{n},p}(\tau) := \{(A,W) | W^2 = Q_{\mathbf{n},p}(A;\tau)\}$ and an addition map $\sigma_{\mathbf{n},p}$ for GLE (3.3). As mentioned in Section 3, when $p \to \omega_k/2, k = 0,1,2,3$, the limiting equation of GLE (3.3) would be GLE (2.4) with $\mathbf{n} = \mathbf{n}_k^{\pm}$, where $\mathbf{n}_k^{\pm}$ is defined by replacing n_k in $\mathbf{n}$ with $n_k \pm 1$. This relation motivates us to prove the following formula.

Theorem 4.5 ([12]). *For $p \notin E_\tau[2]$ and $k \in \{0,1,2,3\}$, there holds*

$$\deg \sigma_{\mathbf{n},p} = \deg \sigma_{\mathbf{n}_k^+} + \deg \sigma_{\mathbf{n}_k^-}.$$

Since Lin and Wang already proved in [26] that $\deg \sigma_{(n,0,0,0)} = \frac{n(n+1)}{2}$, Theorem 4.4 can be proved by applying Theorem 4.5 and an induction argument. In particular, this proof also implies $\deg \sigma_{\mathbf{n},p} = \sum_k n_k(n_k+1)+1$. Once Theorem 4.4 is proved, we can prove Theorem 4.1 by adopting the approach in [26]; see a forthcoming paper.

Acknowledgements

The research of the author was supported by NSFC (No. 11701312). I would like to take this opportunity to thank Professors Chang-Shou Lin, Chin-Lung Wang and Ting-Jung Kuo for sharing their ideas, insights and for their long-term help.

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