The Business of Hodge Theory and Algebraic Cycles*

by James D. Lewis† ‡

Abstract. These notes serve as a precursor to a 24 lecture mini course delivered at the USTC in Hefei, China, June 23–July 12, 2014. The 24 lectures will be published in the Communications in Mathematics and Statistics. They are meant to provide both an enticement and introduction to the aforementioned advanced notes.

MSC 2000 subject classifications: Primary: 14C35, 14C25, 19E15; Secondary: 14C30.
Keywords and phrases: Chow group, Hodge theory, algebraic cycle, regulator, Deligne cohomology, Beilinson-Hodge conjecture, Abel-Jacobi map, Bloch-Beilinson filtration.

Contents

1 Preface 37
2 The Ingredients 38
  2.1 Algebraic Cycles 38
  2.5 A Breezy Review of Hodge Theory 38
  2.19 Regulators 40
3 Bloch-Beilinson Filtration 43
  3.4 Arithmetic Normal Functions 45
4 The Beilinson-Hodge Conjectures 45
  4.3 A Key Observation 46
  4.5 Formulation of the Conjectures 47
5 Intermezzo: Milnor K-Theory and the Bloch-Kato Theorem 47
6 Integral Formulations and the Milnor Regulator 48
  6.2 Key Example I 48
  6.5 Alternate Take via a Morphism of Sites . 48
  6.6 Key Example II 49
  6.13 Consequences of the Bloch-Kato Theorem 50
References 50

1. Preface

Over the past 20 years, and following the works of Bloch, Suslin, Friedlander, Levine, *et al.*, the subject of algebraic cycles has grown immensely. Having connections to K-theory, the proof of the Milnor conjecture (Voevodsky), the same for the Bloch-Kato conjecture (Voevodsky-Rost), as well as Voevodsky's revolutionary version of motivic cohomology, which employs the sought for cohomological machinery, has shed new insights in algebraic cycles. At the same time, an explicit description of regulators associated to motivic cohomology in terms of polylogarithmic currents was fully developed ([KLM], [K-L], [BKLP]). These developments have certainly had an impact on motives, mathematical physics, number theory, as well as algebraic geometry itself. Apart from some omissions, certain choices of material had to be made to prepare the reader for [Lew6]. We hope that this will inspire a generation of mathematicians to work in this interesting subject. We assume the reader has had a course in algebraic geometry, and some knowledge of (co-)homology theory. Two sources that may be useful as reference material are [Lew1] and [Lew2]. One final point. If X/k is defined over a subfield $k \subset \mathbb{C}$, then singular cohomology $H^\bullet(X)$ is intended to mean $H^\bullet(X(\mathbb{C}))$, where $X(\mathbb{C})$ is the associated complex space.

* Dedicated to a future generation of Chinese scholars.
† 632 Central Academic Building, University of Alberta, Edmonton, Alberta T6G 2G1, Canada
E-mail: lewisjd@ualberta.ca
‡ Partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

2. The Ingredients

2.1 Algebraic Cycles

Let k be a field and W/k a quasi-projective variety of dimension d say. Notice that it could happen that $W(k)=\emptyset$, where $W(k)$ are the k valued points. For simplicity reasons alone, we assume $k=\overline{k}\subset\mathbb{C}$ is algebraically closed. Note again that $W(k)$ represents the codimension d points in W/k. The free abelian group generated by $W(k)$, is denoted by $z^d(W)=z_0(W)$, where $z_0(W)$ represents dimension. Any such point in $W(k)$ is the same thing as an irreducible subvariety of codimension d. Quite generally, for $0\le r\le d$, we put $z^r(W)=z_{d-r}(W)$ to be the free abelian group generated by irreducible subvarieties of codimension r $(=\dim d-r)$ in W. The group

$$z^\bullet(W):=\bigoplus_{r=0}^{d} z^r(W),$$

is too large for our interests, and lacks a ring structure under intersection. For instance, how does one compute the self-intersection of a cycle? Further, let us assume given $\xi_1\in z^i(W),\ \xi_2\in z^j(W)$ such that $\xi_1,\ \xi_2$ meet properly, viz., $i+j\ge d$. Then a definition of intersection multiplicity does not exist, unless W is smooth. Next, towards a solution to this problem, one can consider the divisor map

$$\text{(1)}\qquad \operatorname{div}:\bigoplus_{\substack{\mathrm{cd}_Z W=r-1\\ f\in k(Z)^\times}}(f,Z)\to z^r(W),$$

where div is induced by the following. If $Z\in z^{r-1}W)$ is irreducible and $f\in k(Z)^\times$, then $\operatorname{div}(f)=(f)_0-(f)_\infty$ (zeros minus poles of f on Z).

Definition 2.2. The cokernel of the map in (1), denoted by $\mathrm{CH}^r(W)$, is called the r-th Chow group of W.

One should think of $\mathrm{CH}^r(W)$ as a homology theory. We note in passing that Chow's moving lemma, together with the assumption that W is smooth, implies that $\mathrm{CH}^\bullet(W)$ has a ring structure. In the 1950's, Grothendieck defined algebraic K-theory for any given W. This was defined in terms of coherent sheaves on W, and is denoted by $K_0(W)'$. This led to Grothendieck's Riemann-Roch theorem, given by a Chern character map ch: $K_0(W)'\otimes\mathbb{Q}\xrightarrow{\sim}\mathrm{CH}^\bullet(W)\otimes\mathbb{Q}$. (If W is smooth, one could replace $K_0(W)'$ by $K_0(W)$, the latter only involving locally free sheaves.) There is a natural "γ" filtration on $K_0(W)'$, for which

$$\text{(2)}\qquad K_0^{(r)}(W)'\otimes\mathbb{Q}:=Gr_\gamma^r(W)\otimes\mathbb{Q}\xrightarrow{\sim}\mathrm{CH}^r(W)\otimes\mathbb{Q}.$$

It was Quillen who introduced the higher K groups $K_m(W)'$, which also involved a natural "γ" filtration. The search for a cycle theoretic interpretation of (2) led to the higher Chow groups, denoted by $\mathrm{CH}^\bullet(W,m)$, for which we now describe. The higher Chow groups were invented by S. Bloch [Blo1] (and independently by S. Landsberg). Let W/k be a quasi-projective variety. Consider the m-simplex:

$$\Delta^m=\operatorname{Spec}\left\{\frac{k[t_0,\dots,t_m]}{(1-\sum_{j=0}^m t_j)}\right\},\quad \Delta^m(k)\simeq k^m.$$

We set $z^r(W,m)=$

$$\left\{\xi\in z^r(W\times\Delta^m)\ \middle|\ \xi \text{ meets all faces } \{t_{i_1}=\cdots=t_{i_\ell}=0,\ \ell\ge 1\} \text{ properly}\right\}.$$

Note that $z^r(W,0)=z^r(W)$. Now set $\partial_j: z^r(W,m)\to z^r(W,m-1)$, the restriction to j-th face given by $t_j=0$, which turns out to be well-defined due to the fact that the faces in $W\times\Delta^m$ are local complete intersections. The boundary map

$$\partial:=\sum_{j=0}^m(-1)^j\partial_j: z^r(W,m)\to z^r(W,m-1),\ \text{satisfies } \partial^2=0.$$

Definition 2.3. $\mathrm{CH}^r(W,m)=$ homology of $\{z^r(W,\bullet),\partial\}$, at $\bullet=m$.

Based on norm and graph arguments, one can show that $\mathrm{CH}^r(W)=\mathrm{CH}^r(W,0)$. Further, analogous to (2), one has $K_m^{(r)}(W)'\otimes\mathbb{Q}\simeq\mathrm{CH}^r(W,m)\otimes\mathbb{Q}$.

Alternate take: Cubical version. Let $\square^m:=(\mathbb{P}^1\backslash\{1\})^m$ with coordinates z_i and 2^m codimension one faces obtained by setting $z_i=0,\infty$, and boundary maps

$$\partial=\sum(-1)^{i-1}(\partial_i^0-\partial_i^\infty),$$

where $\partial_i^0,\ \partial_i^\infty$ denote the restriction maps to the faces $z_i=0,\ z_i=\infty$ respectively. The rest of the definition is completely analogous except that one has to divide out by "degenerate" cycles. It is known that both complexes are quasi-isomorphic in the derived category sense (Bloch/Levine). For smooth W, one has a multiplicative structure $\mathrm{CH}^{r_1}(W,m_1)\otimes\mathrm{CH}^{r_2}(W,m_2)\xrightarrow{\cup}\mathrm{CH}^{r_1+r_2}(W,m_1+m_2)$.

The central goal of interest is:

Problem 2.4. Describe the "complexity" of $\mathrm{CH}^r(W,m)$.

In this paper, we will restrict ourselves to the more modest goal where $X:=W$ is smooth and projective over $k=\overline{k}\subseteq\mathbb{C}$, and consider $\mathrm{CH}^r(X,m;\mathbb{Q}):=\mathrm{CH}^r(X,m)\otimes\mathbb{Q}$. We will always assume $\dim X=d$. To approach this, we will need the techniques of Hodge theory.

2.5 A Breezy Review of Hodge Theory

Definition 2.6. Let $\mathbb{A}\subseteq\mathbb{R}$ be a subring. The cases of interest are $\mathbb{A}=\mathbb{Z},\ \mathbb{Q}$. While we are tempted to consider the case $\mathbb{A}=\mathbb{R}$, which is dealt with in [Lew6], this would lead us well beyond the scope of these notes. An $\mathbb{A}$-Hodge structure (HS) of weight $N \in \mathbb{Z}$ is given by the following datum:

- A finitely generated $\mathbb{A}$-module V, and either of the two equivalent statements:
  - $\bullet_1$ A decomposition $V_{\mathbb{C}} = \bigoplus_{p+q=N} V^{p,q}$, $\overline{V^{p,q}} = V^{q,p}$, where $^-$ is complex conjugation induced from conjugation on the second factor $\mathbb{C}$ of $V_{\mathbb{C}} := V \otimes \mathbb{C}$.
  - $\bullet_2$ A finite descending filtration $V_{\mathbb{C}} \supset \cdots \supset F^r \supset F^{r+1} \supset \cdots \supset \{0\}$, satisfying

$$V_{\mathbb{C}} = F^r \bigoplus \overline{F^{N-r+1}},\ \forall\, r \in \mathbb{Z}.$$

Remark 2.7. The equivalence of $\bullet_1$ and $\bullet_2$ can be seen as follows. Given the decomposition in $\bullet_1$, put

$$F^r V_{\mathbb{C}} = \bigoplus_{p+q=N,\, p\geq r} V^{p,q}.$$

Conversely, given $\{F^r\}$ in $\bullet_2$, put $V^{p,q} = F^p \bigcap \overline{F^q}$. This is all motivated by:

Example 2.8 (Hodge)**.** Let $X/\mathbb{C}$ be smooth, projective. Then $H^i(X,\mathbb{Z})$ is a $\mathbb{Z}$-Hodge structure of weight i. Indeed $H^i(X,\mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p+q=i} H^{p,q}(X)$, where $H^{p,q}(X)$ involves d-closed forms with p dz's and q $d\bar{z}$'s. Obviously $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

Example 2.9. $\mathbb{A}(r) := (2\pi \mathrm{i})^r \mathbb{A}$ is an $\mathbb{A}$-Hodge structure of weight $-2r$ and of pure Hodge type $(-r,-r)$, called the Tate twist.

Example 2.10. Let $X/\mathbb{C}$ be smooth, projective. Then $H^i(X,\mathbb{Q}(r)) := H^i(X,\mathbb{Q}) \otimes \mathbb{Q}(r)$ is a $\mathbb{Q}$-Hodge structure of weight $i - 2r$.

To extend these ideas to singular varieties, one requires the following terminology.

Definition 2.11. An $\mathbb{A}$-mixed Hodge structure ($\mathbb{A}$-MHS) is given by the following datum:

- A finitely generated $\mathbb{A}$-module $V_{\mathbb{A}}$,
- A finite descending "Hodge" filtration on $V_{\mathbb{C}} := V_{\mathbb{A}} \otimes \mathbb{C}$,

$$V_{\mathbb{C}} \supset \cdots \supset F^r \supset F^{r+1} \supset \cdots \supset \{0\},$$

- A finite increasing "weight" filtration on $V_{\mathbb{A}} \otimes \mathbb{Q} := V_{\mathbb{A}} \otimes_{\mathbb{Z}} \mathbb{Q}$,

$$\{0\} \subset \cdots \subset W_{\ell-1} \subset W_\ell \subset \cdots \subset V_{\mathbb{A}} \otimes \mathbb{Q},$$

  such that $\{F^r\}$ induces a (pure) HS of weight ℓ on $Gr_\ell^W := W_\ell / W_{\ell-1}$.

Theorem 2.12 (Deligne [De])**.** *Let Y be a complex variety. Then $H^i(Y,\mathbb{Z})$ has a canonical and functorial $\mathbb{Z}$-MHS, which agrees with the aforementioned (pure) Hodge structure in the case where Y is smooth and projective.*

Remark 2.13. (i) A morphism $h : V_{1,\mathbb{A}} \to V_{2,\mathbb{A}}$ of $\mathbb{A}$-MHS is an $\mathbb{A}$-linear map satisfying:

- $h\big(W_\ell V_{1,\mathbb{A}\otimes\mathbb{Q}}\big) \subseteq W_\ell V_{2,\mathbb{A}\otimes\mathbb{Q}}, \quad \forall\, \ell$,
- $h\big(F^r V_{1,\mathbb{C}}\big) \subseteq F^r V_{2,\mathbb{C}}, \quad \forall\, r$.

Deligne ([De] (Theorem 2.3.5)) shows that the category of $\mathbb{A}$-MHS is abelian; in particular if $h : V_{1,\mathbb{A}} \to V_{2,\mathbb{A}}$ is a morphism of $\mathbb{A}$-MHS, then $\ker(h)$, $\mathrm{coker}(h)$ are endowed with the induced filtrations. Let us further assume that $\mathbb{A}\otimes\mathbb{Q}$ is a field. Then Deligne (*op. cit.*) shows that h is strictly compatible[1] with the filtrations $W_\bullet$ and $F^\bullet$, and hence the functors $V \mapsto Gr_\ell^W V$, $V_{\mathbb{C}} \mapsto Gr_F^r V_{\mathbb{C}}$ are exact.

(ii) Roughly speaking, the functoriality of the MHS in Deligne's theorem translates to the following yoga: the "standard" exact sequences in singular (co)homology, together with push-forwards and pullbacks by morphisms (wherever permissible) respect MHS. In particular for a subvariety $Y \subset X$, the localization cohomology sequence associated to the pair (X,Y) is a long exact sequence of MHS. Here is where the Tate twist comes into play: Suppose that $Y \subset X$ is an inclusion of projective algebraic manifolds with $\mathrm{codim}_X Y = r \geq 1$. One has a Gysin map $H^{i-2r}(Y,\mathbb{Q}) \to H^i(X,\mathbb{Q})$ which involves Hodge structures of different weights. To remedy this, one considers the induced map $H^{i-2r}(Y,\mathbb{Q}(-r)) \to H^i(X,\mathbb{Q}(0)) = H^i(X,\mathbb{Q})$ via (twisted) Poincaré duality, which is a morphism of pure Hodge structures (hence of MHS). A simple proof of this fact can be found in §7 of [Lew1]. Note that the morphism $H^i_Y(X,\mathbb{Q}) \to H^i(X,\mathbb{Q})$ is a morphism of MHS, and that accordingly $H^i_Y(X,\mathbb{Q}) \simeq H^{i-2r}(Y,\mathbb{Q}(-r))$ is an isomorphism of MHS (with Y still smooth).

Example 2.14. Let $\overline{U}$ be a compact Riemann surface, $\emptyset \neq \Sigma \subset \overline{U}$ a finite set of points, and put $U := \overline{U} \backslash \Sigma$. According to Deligne, $H^1(U,\mathbb{Z}(1))$ carries a $\mathbb{Z}$-MHS. The Hodge filtration on $H^1(U,\mathbb{C})$ is defined in terms of a filtered complex of holomorphic differentials on U with logarithmic poles along Σ [De]. One can "observe" the MHS via weights as follows. Poincaré duality gives us $H^1_\Sigma(\overline{U},\mathbb{Z}) \simeq H_1(\Sigma,\mathbb{Z}) = 0$, and the localization sequence in cohomology below is a sequence of MHS:

$$0 \to H^1(\overline{U},\mathbb{Z}(1)) \to H^1(U,\mathbb{Z}(1)) \to H^0(\Sigma,\mathbb{Z}(0))^\circ \to 0,$$

where

$$H^0(\Sigma,\mathbb{Z}(0))^\circ := \ker\big(H^2_\Sigma(\overline{U},\mathbb{Z}(1)) \to H^2(\overline{U},\mathbb{Z}(1))\big) \simeq \mathbb{Z}(0)^{|\Sigma|-1}.$$

Put $W_0 = H^1(U,\mathbb{Z}(1))$, $W_{-1} = \mathrm{Im}\big(H^1(\overline{U},\mathbb{Z}(1)) \to H^1(U,\mathbb{Z}(1))\big)$, $W_{-2} = 0$. Then $Gr_{-1}^W H^1(U,\mathbb{Z}(1)) \simeq H^1(\overline{U},\mathbb{Z}(1))$ has pure weight -1 and $Gr_0^W H^1(U,\mathbb{Z}(1)) \simeq \mathbb{Z}(0)^{|\Sigma|-1}$ has pure weight 0.

[1] Strict compatibility means that $h(F^r V_{1,\mathbb{C}}) = h(V_{1,\mathbb{C}}) \cap F^r V_{2,\mathbb{C}}$ and $h(W_\ell V_{1,\mathbb{A}\otimes\mathbb{Q}}) = h(V_{1,\mathbb{A}\otimes\mathbb{Q}}) \cap W_\ell V_{2,\mathbb{A}\otimes\mathbb{Q}}$ for all r and ℓ. A nice explanation of Deligne's proof of this fact can be found in [St].

The following notation will be introduced:

Definition 2.15. Let V be an $\mathbb{A}$-MHS. We put

$$\Gamma_{\mathbb{A}} V := \hom_{\mathbb{A}-\mathrm{MHS}}(\mathbb{A}(0), V), \text{ and } J_{\mathbb{A}}(V) = \mathrm{Ext}^1_{\mathbb{A}-\mathrm{MHS}}(\mathbb{A}(0), V).$$

In the case where $\mathbb{A} = \mathbb{Z}$ or $\mathbb{A} = \mathbb{Q}$, we simply put $\Gamma = \Gamma_{\mathbb{A}}$ and $J = J_{\mathbb{A}}$.

Example 2.16. Suppose that $V = V_{\mathbb{Z}}$ is a $\mathbb{Z}$ (pure) HS of weight $2r$. Then $V(r) := V \otimes \mathbb{Z}(r)$ is of weight 0, and (up to the twist) one can identify ΓV with $V_{\mathbb{Z}} \cap F^r V_{\mathbb{C}} = V_{\mathbb{Z}} \cap V^{r,r} := \epsilon^{-1}(V^{r,r})$, where $\epsilon : V \to V_{\mathbb{C}}$.

Example 2.17. Let V be a $\mathbb{Z}$-MHS. There is the identification due to J. Carlson (see [Ca], [Ja2]),

$$J(V) \simeq \frac{W_0 V_{\mathbb{C}}}{F^0 W_0 V_{\mathbb{C}} + W_0 V},$$

where in the denominator term, $V := V_{\mathbb{Z}}$ is identified with its image $V_{\mathbb{Z}} \to V_{\mathbb{C}}$ (viz., quotienting out torsion). For example, if $\{E\} \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), V)$ corresponds to the short exact sequence of MHS: $0 \to V \to E \xrightarrow{\alpha} \mathbb{Z}(0) \to 0$, then one can find $x \in W_0 E$ and $y \in F^0 W_0 E_{\mathbb{C}}$ such that $\alpha(x) = \alpha(y) = 1$. Then $x - y \in V_{\mathbb{C}}$ descends to a class in $W_0 V_{\mathbb{C}} / \{F^0 W_0 V_{\mathbb{C}} + W_0 V\}$, which defines the map from $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), V)$ to $W_0 V_{\mathbb{C}} / \{F^0 W_0 V_{\mathbb{C}} + W_0 V\}$.

Remark 2.18 (Some useful facts about Poincaré, Serre duality, and twists). Let $X/\mathbb{C}$ be a smooth projective variety of dimension d. With regard to Betti cohomology, the following pairings are perfect:

$$\text{(i)} \quad H^i(X, \mathbb{C}) \times H^{2d-i}(X, \mathbb{C}) \xrightarrow{\cup} H^{2d}(X, \mathbb{C}) \simeq \mathbb{C}$$

$$\text{(ii)} \quad H^{p,q}(X) \times H^{d-p,d-q}(X) \xrightarrow{\cup} H^{d,d}(X) = H^{2d}(X, \mathbb{C}) \simeq \mathbb{C}$$

are perfect. The former and latter are Poincaré (resp. Serre) dualities. Note that this alone (being the same pairing) implies the compatibility of Poincaré with Serre duality. Notice that (i) and (ii) imply that

$$\text{(iii)} \quad \frac{H^i(X, \mathbb{C})}{F^r H^i(X, \mathbb{C})} \times F^{d-r+1} H^{2d-i}(X, \mathbb{C}) \xrightarrow{\cup} \mathbb{C},$$

is a perfect pairing. Hence

$$\text{(iv)} \quad \frac{H^i(X, \mathbb{C})}{F^r H^i(X, \mathbb{C})} \simeq \left\{F^{d-r+1} H^{2d-i}(X, \mathbb{C})\right\}^{\vee}.$$

Likewise from (i) we deduce:

$$\text{(v)} \quad H^i(X, \mathbb{C}) \simeq \left\{H^{2d-i}(X, \mathbb{C})\right\}^{\vee} \simeq H_{2d-i}(X, \mathbb{C}),$$

where the latter isomorphism is due to the de Rham isomorphism theorem. A generalization of (v) is the following. Let $W/\mathbb{C}$ be smooth and quasi projective of dimension d, with subvariety $Y \subset W$, and $\ell \in \mathbb{Z}$. Then there is an isomorphism of $\mathbb{A} - \mathrm{MHS}$:

$$\text{(vi)} \quad H^i_Y(W, \mathbb{A}(\ell)) \simeq H_{2d-i}(Y, \mathbb{A}(d-\ell)) := H_{2d-i}(Y, \mathbb{A}) \otimes \mathbb{A}(\ell - d).$$

Finally, combining (iv), (v) and (vi), we arrive at

$$\frac{H^i(X, \mathbb{C})}{F^r H^i(X, \mathbb{C}) + H^i(X, \mathbb{A}(\ell))} \simeq \frac{\left\{F^{d-r+1} H^{2d-i}(X, \mathbb{C})\right\}^{\vee}}{H_{2d-i}(X, \mathbb{A}(d-\ell))},$$

where $H^i(X, \mathbb{A}(\ell))$ is identified with its image in $H^i(X, \mathbb{C})$; accordingly $H_{2d-i}(X, \mathbb{A}(d-\ell))$ is identified with its image in $\left\{F^{d-r+1} H^{2d-i}(X, \mathbb{C})\right\}^{\vee}$ (called periods).

2.19 Regulators

To put it succinctly, a regulator is a generalization of the logarithm. Dirichlet used the logarithm to define a map from the multiplicative group of a ring of algebraic integers to a real vector space. Then Dirichlet proved the celebrated analytic class number formula which relates all the important number theoretic invariants of the number field to the covolume of the Dirichlet regulator. Since the 1960's Dirichlet's fundamental discovery has been found potentially to occur elsewhere in number theory, in algebraic geometry, in class field theory, in algebraic K-theory, in the theory of algebraic cycles and motives, and in Hodge theory. Regulators come in many different forms, according to the context.

For instance, the Borel regulator is the higher-dimensional analogue of the Dirichlet regulator, considered as a map on algebraic K-theory in dimension one. On the other hand, in Riemann surface theory, the regulators might involve Abelian integrals and Jacobians, extending the ideas of the 19th century analytic number theorists and geometers. Generally speaking, in its current incarnation, a regulator is a map (called a realization), from the algebraic K-theory of an algebraic variety to a "more computable" (co-)homology theory. Examples of regulators include Betti realizations, étale realizations, Hodge realizations, and so forth. Our main interests in this paper are the Hodge and Betti realizations. Due to the higher Chow cycle interpretation of the K-groups, we can think of regulators as realizations on the higher Chow groups.

Example 2.20. For a smooth projective X over $\bar{k} \subseteq \mathbb{C}$, there is the Betti cycle class map (where $\xi \in \mathrm{CH}^r(X,0)$ and $d = \dim X$), $\mathrm{cl}_{r,0} : \mathrm{CH}^r(X,0) \to \Gamma H^{2r}(X, \mathbb{Z}(r))$, $\xi \mapsto (2\pi \mathrm{i})^{r-d}\{\xi\} \in H_{2d-2r}(X, \mathbb{Z}(d-r))$,

$$\Gamma H_{2d-2r}(X, \mathbb{Z}(d-r)) \xrightarrow{\sim PD} \Gamma H^{2r}(X, \mathbb{Z}(r)),$$
$$(2\pi \mathrm{i})^{r-d}\{\xi\} \mapsto (2\pi \mathrm{i})^r [\xi],$$

where PD stands for Poincaré duality, $\{\xi\}$ is the fundamental class induced by a desingularization of the irreducible components of ξ. Observe that $H^{2r}(X, \mathbb{Z}(r)) = H^{2r}(X, \mathbb{Z}) \otimes \mathbb{Z}(r)$ has weight zero, likewise for $H_{2d-2r}(X, \mathbb{Z}(d-r)) := H_{2d-2r}(X, \mathbb{Z}) \otimes \mathbb{Z}(r-d)$. It is important to re-iterate that the PD isomorphism

$H_{2d-2r}(X,\mathbb{Z}(d-r)) \simeq H^{2r}(X,\mathbb{Z}(r))$ is an isomorphism of Hodge structures. Another point is that if we put $\mathcal{O}_X^\times$ to be the sheaf of nowhere vanishing regular functions on X, then there is a natural commutative diagram

$$\begin{array}{ccc} H^1(X,\mathcal{O}_X^\times) & \xrightarrow{\sim} & \mathrm{CH}^1(X,0) \\ d\log \downarrow & & \downarrow \mathrm{cl}_{1,0} \\ \Gamma H^2(X,\mathbb{Z}(1)) & = & \Gamma H^2(X,\mathbb{Z}(1)) \end{array}$$

which reinforces the need for twists.

Remark 2.21. Given the setting of Example 2.20, but with $\mathbb{Z}$ replaced by $\mathbb{Q}$, we arrive at the celebrated Hodge conjecture:

Conjecture 2.22. *$\mathrm{cl}_{r,0} : CH^r(X,0;\mathbb{Q}) \to \Gamma H^{2r}(X,\mathbb{Q}(r))$ is surjective.*

Note that $\Gamma H^{2r}(X,\mathbb{Q}(r)) = F^0H^{2r}(X,\mathbb{C}) \cap H^{2r}(X,\mathbb{Q}(r))$. Untwisting by $\mathbb{Q}(-r)$ allows us to identify $F^0H^{2r}(X,\mathbb{C}) \cap H^{2r}(X,\mathbb{Q}(r))$ with $F^rH^{2r}(X,\mathbb{C}) \cap H^{2r}(X,\mathbb{Q})$, $\xi \in \mathrm{CH}^r(X) \mapsto [\xi] \in F^rH^{2r}(X,\mathbb{C}) \cap H^{2r}(X,\mathbb{Q})$. Let $\xi \in \mathrm{CH}^r(X/\mathbb{C};\mathbb{Q})$. Notice that $X/\mathbb{C} = X_L \times \mathbb{C}$, where $L/\bar{k}$ is finitely generated. By enlarging L if necessary, we can assume that $\xi \in \mathrm{CH}^r(X_L;\mathbb{Q})$. It is clear that $L = \bar{k}(S)$ for some smooth variety $S/\bar{k}$. We can spread X_L to $\mathfrak{X} = S \times_{\bar{k}} X$, and likewise $\tilde{\xi} \in \mathrm{CH}^r(\mathfrak{X};\mathbb{Q})$. Since $S(\bar{k}) \neq \emptyset$, it follows from Betti rigidity (Künneth formula) that we can assume that ξ is defined over $\bar{k}$. Thus the Hodge conjecture[2] in it's classical form can be restated:

$$\mathrm{cl}_{r,0} : \mathrm{CH}^r(X/\mathbb{C},0;\mathbb{Q}) \twoheadrightarrow H^{r,r}(X;\mathbb{Q}) := F^rH^{2r}(X;\mathbb{Q}).$$

Example 2.23. We now put $\mathrm{CH}^r_{\mathrm{hom}}(X,0) = \ker \mathrm{cl}_{r,0}$. For notational simplicity, we will assume $X = X/\mathbb{C}$. Further, it is clear that

$$J\big(H^{2r-1}(X,\mathbb{Z}(r))\big) = \frac{H^{2r-1}(X,\mathbb{C})}{F^0H^{2r-1}(X,\mathbb{C}) + H^{2r-1}(X,\mathbb{Z}(r))},$$

which in untwisted form on the complex part, is really given by

$$\frac{H^{2r-1}(X,\mathbb{C})}{F^rH^{2r-1}(X,\mathbb{C}) + H^{2r-1}(X,\mathbb{Z}(r))}.$$

This is precisely the Griffiths Jacobian. The aforementioned compatibility of Poincaré and Serre duality yields

$$J(H^{2r-1}(X,\mathbb{Z}(r))) \simeq \frac{F^{d-r+1}H^{2d-2r+1}(X,\mathbb{C})^\vee}{H_{2d-2r+1}(X,\mathbb{Z}(d-r))}$$

The Abel-Jacobi map $\Phi_{r,0} : \mathrm{CH}^r_{\mathrm{hom}}(X,0) \to J(H^{2r-1}(X,\mathbb{Z}(r)))$ is defined as follows (Griffiths's prescription, modulo twist). Let $\xi \in \mathrm{CH}^r_{\mathrm{hom}}(X,0)$. Then $\xi = \partial\zeta$ bounds a $2d-2r+1$ real dimensional chain ζ in X. Let $\{w\} \in F^{d-r+1}H^{2d-2r+1}(X,\mathbb{C})$. Define:

$$\Phi_{r,0}(\xi)(\{w\}) = \frac{1}{(2\pi \mathrm{i})^{d-r}}\int_\zeta w \quad \text{(modulo periods)}.$$

That $\Phi_{r,0}$ is well-defined follows from the fact that $F^\ell H^i(X,\mathbb{C})$ depends only on the "complex structure" of X.

Alternate take for $\Phi_{r,0}$: Let $\xi \in \mathrm{CH}^r_{\mathrm{hom}}(X,0)$. First observe (and ignoring twists) that $H^{2r-1}_{|\xi|}(X,\mathbb{Z}) \simeq H_{2d-2r+1}(|\xi|,\mathbb{Z}) = 0$ as $\dim_{\mathbb{R}}|\xi| = 2d-2r$. Secondly, there is a fundamental class map $\xi \mapsto \{\xi\} \in H_{2d-2r}(|\xi|,\mathbb{Z}(d-r)) \simeq H^{2r}_{|\xi|}(X,\mathbb{Z}(r))$ (Poincaré duality). Further, since ξ is nulhomologous on X, we have by duality

$$[\xi] \in H^{2r}_{|\xi|}(X,\mathbb{Z}(r))^\circ := \ker\big(H^{2r}_{|\xi|}(X,\mathbb{Z}(r)) \to H^{2r}(X,\mathbb{Z}(r))\big).$$

Hence ξ determines a morphism of MHS, $\mathbb{Z}(0) \to H^{2r}_{|\xi|}(X,\mathbb{Z}(r))^\circ$. From the short exact sequence of MHS

$$\begin{aligned} 0 &\to H^{2r-1}(X,\mathbb{Z}(r)) \to H^{2r-1}(X\backslash|\xi|,\mathbb{Z}(r)) \\ &\to H^{2r}_{|\xi|}(X,\mathbb{Z}(r))^\circ \to 0, \end{aligned}$$

we can pullback via this morphism to obtain another short exact sequence of MHS,

$$0 \to H^{2r-1}(X,\mathbb{Z}(r)) \to E \to \mathbb{Z}(0) \to 0.$$

Then $\Phi_{r,0}(\xi) := \{E\} \in J\big(H^{2r-1}(X,\mathbb{Z}(r))\big)$. This class $\{E\}$ is easy to calculate in $J\big(H^{2r-1}(X,\mathbb{Z}(r))\big)$, in terms of a membrane integral. Note that via duality,

$$E \subset H^{2r-1}(X\backslash|\xi|,\mathbb{Z}(r)) \simeq H_{2d-2r+1}(X,|\xi|,\mathbb{Z}(d-r)),$$

and that if ζ is a real $2d-2r+1$ chain such that $\partial\zeta = \xi$ on X, then $\{\zeta\} \in H_{2d-2r+1}(X,|\xi|,\mathbb{Z})$. One can show that the class $x \in W_0E$ corresponding to the current

$$\frac{1}{(2\pi \mathrm{i})^{d-r}}\int_\zeta(-),$$

maps to $1 \in \mathbb{Z}(0)$. Now choose $y \in F^0W_0E_{\mathbb{C}}$ also mapping to $1 \in \mathbb{Z}(0)$. By Hodge type alone, the current corresponding to $x-y$ in the Poincaré dual description of $J\big(H^{2r-1}(X,\mathbb{Z}(r))\big)$ is the same as for $x = \frac{1}{(2\pi \mathrm{i})^{d-r}}\int_\zeta(-)$, which is precisely the aforementioned Griffiths prescription.

Remark 2.24. The constructions in Examples 2.20 and 2.23 can be combined in the following commutative diagram (with $X = X/\mathbb{C}$):

$$\begin{array}{ccccccc} 0 \to \mathrm{CH}^r_{\mathrm{hom}}(X,0) & \to & \mathrm{CH}^r(X,0) & \to & \mathrm{CH}^r(X,0)/\mathrm{CH}^r_{\mathrm{hom}}(X,0) \to 0 \\ \Phi_{r,0}\downarrow & & \downarrow\Psi_{r,0} & & \downarrow \mathrm{cl}_{r,0} \\ 0 \to J\big(H^{2r-1}(X,\mathbb{Z}(r))\big) & \to & H^{2r}_{\mathcal{D}}(X,\mathbb{Z}(r)) & \to & \Gamma H^{2r}(X,\mathbb{Z}(r)) \to 0 \end{array}$$

[2] It is false over $\mathbb{Z}$. See [Lew1].

where $H^{2r}_{\mathcal{D}}(X,\mathbb{Z}(r))$ is called Deligne cohomology, for which there is a cycle class map $\Psi_{r,0}$. Good references for Deligne cohomology[3] can be found in [EV] and [Ja1]. The key point is this: When $r=1$, $\Psi_{1,0}$ is an isomorphism, a fortiori $CH^1_{\hom}(X,0) \simeq J(H^1(X,\mathbb{Z}(1))$, and $\mathrm{cl}_{1,0} : \mathrm{CH}^1(X,0) \twoheadrightarrow \Gamma H^2(X,\mathbb{Z}(1))$, the latter also known as the Lefschetz $(1,1)$ theorem. Note that all torsion classes in $\Gamma H^{2r}(X,\mathbb{Z}(r))$ are algebraic for $r=1$, but that is not the case in general for $r>1$ (Atiyah-Hirzebruch). Thus on the other hand, if $d>r>1$ then $\mathrm{cl}_{r,0}$ need not be surjective, unless one tensors with $\mathbb{Q}$ (Hodge conjecture), and the image of $\Phi_{r,0}$ can be at most countable (Griffiths). Further, for $r>1$, the kernel of $\Phi_{r,0}$ can be "enormous" (Mumford; See [Lew1] (Ch. 15)).

Remark 2.25. Bloch [Blo2] has defined a cycle class map from a smooth variety over a field to any "good" cohomology theory; in particular, $\mathrm{cl}_{r,m} : \mathrm{CH}^r(X,m) \to H^{2r-m}_{\mathcal{D}}(X,\mathbb{Z}(r))$; moreover analogously there is a short exact sequence

$$0 \to J\big(H^{2r-m-1}(X,\mathbb{Z}(r))\big) \to H^{2r-m}_{\mathcal{D}}(X,\mathbb{Z}(r)) \to \Gamma H^{2r-m}(X,\mathbb{Z}(r)) \to 0,$$

as X is smooth and projective. It is easy to check that

$$\Gamma H^{2r-m}(X,\mathbb{Z}(r)) = H^{2r-m}_{\mathrm{tor}}(X,\mathbb{Z}(r)),\ \textit{for } m \geq 1,$$

where $H_{\mathrm{tor}}(-)$ stands for the torsion subgroup of $H(-)$. That makes Betti cohomology unsuitable for detecting higher K-theory classes, modulo torsion. Note however, if X is replaced by any smooth quasi-projective W, then $\Gamma H^{2r-m}(W,\mathbb{Z}(r))$ can be highly nontrivial.

Let $\Delta^m(\mathbb{R}_+) \subset \Delta^m$ be the real simplex in $\mathbb{R}^{m+1}$ and of course the analogous real m-cube $[-\infty,0]^m \subset \square^m$. Further, let $\{\xi\} \in \mathrm{CH}^r(X,m)$ be given. Then assuming a choice of representative ξ of $\{\xi\}$ in general position, the Betti cycle class map takes $\{\xi\}$ to $(2\pi \mathrm{i})^r[Pr_{X,*}(\xi \cap X \times \Delta^m(\mathbb{R}_+))]$, or equivalently to $(2\pi\mathrm{i})^r[Pr_{X,*}(\xi \cap X \times [-\infty,0]^m)]$. Now let $\mathrm{CH}^r_{\hom}(X,m) = \ker\big(\mathrm{CH}^r(X,m) \to \Gamma H^{2r-m}(X,\mathbb{Z}(r))\big)$. Via Deligne cohomology, this determines the generalized Abel-Jacobi map

$$\Phi_{r,m} : \mathrm{CH}^r_{\hom}(X,m) \to J\big(H^{2r-m-1}(X,\mathbb{Z}(r))\big).$$

Observe the isomorphism, where (3) is Poincaré duality:

$$J\big(H^{2r-m-1}(X,\mathbb{Z}(r))\big) = \frac{H^{2r-m-1}(X,\mathbb{C})}{F^rH^{2r-m-1}(X,\mathbb{C}) + H^{2r-m-1}(X,\mathbb{Z}(r))}$$
$$\simeq \frac{F^{d-r+1}H^{2d-2r+m+1}(X,\mathbb{C})^\vee}{H_{2d-2r+m+1}(X,\mathbb{Z}(d-r))}. \tag{3}$$

Following [KLM], we will describe $\Phi_{r,m}$ in terms of (3). Consider the cubical description of $\mathrm{CH}^r(X,m)$, and where $\log$ has the principal branch. Let $\{\xi\} \in \mathrm{CH}^r_{\hom}(X,m)$, and $\omega \in F^{d-r+1}H^{2d-2r+m+1}(X,\mathbb{C})$ be given. Further, let $\pi_1 : |\xi| \to X$, $\pi_2 : |\xi| \to \square^m$ be the obvious projections. One has $\gamma := \{\pi_2^{-1}[-\infty,0]^m\} \cap \xi = \partial\zeta$, and the Abel-Jacobi map is given by

(4)

$$\Phi_{r,m}(\xi) = \frac{1}{(2\pi\mathrm{i})^{d-r+m}} \times \Bigg[\int_{\xi \backslash \{\xi \cap \pi_2^{-1}([-\infty,0]\times\square^{m-1})\}} \pi_2^*\big((\log z_1) d\log z_2 \wedge \cdots \wedge d\log z_m\big) \wedge \pi_1^*(\omega)$$
$$- (2\pi\mathrm{i}) \int_{\{\xi\cap\pi_2^{-1}([-\infty,0]\times\square^{m-1})\}\backslash\{\xi\cap\pi_2^*([-\infty,0]^2\times\square^{m-2})\}} \pi_2^*\big((\log z_2) d\log z_3 \wedge \cdots \wedge d\log z_m\big) \wedge \pi_1^*(\omega) + \cdots$$
$$+ (-2\pi\mathrm{i})^{m-1} \int_{\{\xi\cap\pi_2^{-1}([-\infty,0]^{m-1}\times\square^1)\}\backslash\{\xi\cap\pi_2^{-1}([-\infty,0]^m)\}} \pi_2^*(\log z_m) \wedge \pi_1^*(\omega)$$
$$+ \left\{(-2\pi\mathrm{i})^m \int_\zeta \pi_1^*(\omega)\right\}\Bigg],$$

where the latter term is a membrane integral. There is also a simplicial version of this map, which we omit [BKLP]. In passing we should say that the Abel-Jacobi map can be defined for arbitrary varieties over $\mathbb{C}$ [K-L].

Example 2.26 ([KLM]). By a norm argument, a class $\xi \in \mathrm{CH}^r_{\hom}(X,1)$ can be expressed in the form $\sum_\alpha (f_\alpha, Z_\alpha)$ where Z_α is irreducible, $\mathrm{codim}_X Z_\alpha = r-1$, $f_\alpha \in k(Z_\alpha)^\times$, and $\sum_\alpha \mathrm{div}(f_\alpha) = 0$. (For a simple example, consider a rational elliptic curve $E \subset \mathbb{P}^2$ with say one node $P \in E$, and consider the canonical desingularization $\pi : \mathbb{P}^1 \to E$ with preimage $\pi^{-1}(P) = \{Q,R\}$. Choose $f \in k(\mathbb{P}^1)^\times$ such that $\mathrm{div}(f) = Q - R$. Since $k(\mathbb{P}^1) = k(E)$, it follows that $\mathrm{div}(f) = 0$ on E, a fortiori on $\mathbb{P}^2$. Hence $(f,E) \in \mathrm{CH}^2(\mathbb{P}^2,1)$.) Put $\gamma_\alpha = f_\alpha^{-1}[-\infty,0]$, and $\gamma = \sum_\alpha \gamma_\alpha$. Then $\partial\gamma = 0$ and since $\xi \in \mathrm{CH}^r_{\hom}(X,1)$, we have $\gamma = \partial\zeta$ on X. Let $\omega \in F^{d-r+1}H^{2d-2r+2}(X,\mathbb{C})$. The resulting current takes the form

$$\Phi_{r,1}(\xi) = \frac{1}{(2\pi\mathrm{i})^{d-r+1}} \times \sum_\alpha \left[\int_{Z_\alpha} \log(f_\alpha) \wedge (\omega) - 2\pi\mathrm{i}\int_\zeta (\omega)\right] \Big/ \text{periods}.$$

[3] In the case where $X/\mathbb{C}$ is smooth and projective, and in its simplest form, Deligne cohomology $H^i_{\mathcal{D}}(X,\mathbb{A}(j))$, $i,j \geq 0$, is given by the ith hypercohomology of the complex $\mathbb{A}(j)_{\mathcal{D}} := \{\mathbb{A}(j) \to \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^{j-1}_X\}$. The ith hypercohomology is computed by constructing a Leray open cover of X with respect to the associated sheaves in the complex $\mathbb{A}(j)_{\mathcal{D}}$, and forming an associated Cech double complex of $\mathbb{A}(j)_{\mathcal{D}}$. Taking the ith cohomology of the associated simple complex yields $H^i_{\mathcal{D}}(X,\mathbb{A}(j))$. In the case $X/\mathbb{C}$ is smooth and quasi-projective, the same definition of $H^i_{\mathcal{D}}(X,\mathbb{A}(j))$ applies and the resulting $H^i_{\mathcal{D}}(X,\mathbb{A}(j))$ is called analytic Deligne cohomology. One needs a little more machinery to construct Deligne-Beilinson cohomology, which yields an "algebraic" analog of the analytic $H^i_{\mathcal{D}}(X,\mathbb{A}(j))$. In this text, $H^i_{\mathcal{D}}(X,\mathbb{A}(j))$ will always mean Deligne-Beilinson cohomology, even if we call it Deligne cohomology.

The same formula appears in [Lev], albeit a sign change in front of the membrane integral due to a different branch of the log function.

Example 2.27 ([KLM]). Let $a,\ b \in \mathbb{C}^{\times}\backslash\{1\}$, and put:

$$V(a) = \left\{ \left(1-\frac{a}{t}, 1-t, t\right) \;\middle|\; t \in \mathbb{P}^1 \right\} \bigcap \square^3,$$
$$W(b) = \left\{ \left(1-\frac{b}{t}, t, 1-t\right) \;\middle|\; t \in \mathbb{P}^1 \right\} \bigcap \square^3,$$

and note that

$$\partial V(a) = (1-a, a), \quad \partial W(b) = (b, 1-b).$$

Therefore

$$\xi_a := V(a) - W(1-a) \in \mathrm{CH}^2(\mathrm{Pt}, 3).$$

The value $\Phi_{2,3}(\xi_a) \in \mathbb{C}/\mathbb{Z}(2)$ is easy to compute:

$$\Phi_{2,3}(\xi_a) = \mathrm{Li}_2(a) + \mathrm{Li}_2(1-a) + \log a \log(1-a), \tag{5}$$

where Li_2 is the dilogarithm. The RHS of (5) is independent of a by differentiation in a. (More generally due to Beilinson rigidity.[4])

$$\Phi_{2,3}(\xi_a) = \lim_{a\to 0} \Phi_{2,3}(\xi_a) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} \in \mathbb{C}/\mathbb{Z}(2),$$

which is a torsion class. Note that if we insist on $a \in \mathbb{Q}$, then it is well-known that ξ_a is a generator of $\mathrm{CH}^2(\mathrm{Spec}(\mathbb{Q}), 3) \simeq \mathbb{Z}/24\mathbb{Z}$.

3. Bloch-Beilinson Filtration

For simplicity of presentation, we will work with $\mathrm{CH}^r(X;\mathbb{Q}) := \mathrm{CH}^r(X,0;\mathbb{Q})$, viz., $m = 0$, and with $X = X/\mathbb{C}$ smooth and projective. A similar story holds for $\mathrm{CH}^r(X,m;\mathbb{Q})$, due to the work of M. Saito [A]. The purpose of $\mathbb{Q}$-coefficients is that this will involve a conjectural motivic decomposition of the diagonal class into Künneth projectors, which can only be expected to hold over $\mathbb{Q}$. We first set $F^0\mathrm{CH}^r(X;\mathbb{Q}) = \mathrm{CH}^r(X;\mathbb{Q})$. One has the cycle class map $F^0\mathrm{CH}^r(X;\mathbb{Q}) \to \Gamma H^{2r}(X,\mathbb{Q}(r))$, where we recall

$$\Gamma H^{2r}(X,\mathbb{Q}(r)) := \hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r}(X,\mathbb{Q}(r))) = \mathrm{Ext}^0_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r}(X,\mathbb{Q}(r))).$$

Put $F^1\mathrm{CH}^r(X;\mathbb{Q}) = \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q})$ to be the kernel. Correspondingly, we have the Abel-Jacobi map

$$F^1\mathrm{CH}^r(X;\mathbb{Q}) \to J\big(H^{2r-1}(X,\mathbb{Q}(r))\big) := \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-1}(X,\mathbb{Q}(r))).$$

[4] Notice that the equation in (5) is a functional equation. Vincent Maillot raised the fascinating possibility that there could be a connection between Beilinson rigidity and functional equations.

Let us denote the kernel of this map by $F^2\mathrm{CH}^r(X;\mathbb{Q})$. Proceeding in this direction, one may consider a map $F^2\mathrm{CH}^r(X;\mathbb{Q}) \to \mathrm{Ext}^2_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-2}(X,\mathbb{Q}(r)))$. But there is *a caveat* in order here. As first observed by Beilinson [Be1], for any two MHS $V_1,\ V_2$, $\mathrm{Ext}^{\nu}_{\mathrm{MHS}}(V_1,V_2) = 0$ for $\nu \geq 2$, and so this filtration has no way of truncating in the situation when $F^2\mathrm{CH}^r(X;\mathbb{Q}) \neq 0$ (see Remark 2.24). The fact that $\mathrm{Ext}^{\nu}_{\mathrm{MHS}}(V_1,V_2) = 0$ for $\nu \geq 2$ is due in part to J. Carlson's formula [Ca] for $\mathrm{Ext}^1_{\mathrm{MHS}}(V_1,-)$ as a right exact functor, together with this information: If the category of MHS were to have enough injectives, then this is formal homological algebra. In general, the precise idea works with a Yoneda-Ext argument. This leads to the notion of a Bloch-Beilinson (BB) filtration (originating from Bloch [Blo3], but later fortified by Beilinson), and in its paraphrased form, is that there is a descending filtration,

$$\mathrm{CH}^r(X;\mathbb{Q}) = F^0 \supset F^1 \supset F^2 \supset \cdots \supset F^r \supset \{0\},$$

where

$$Gr^{\nu}_F\mathrm{CH}^r(X;\mathbb{Q}) \simeq \mathrm{Ext}^{\nu}_{\mathcal{MM}}\big(\mathrm{Spec}(\mathbb{C}), h^{2r-\nu}(X)(r)\big),$$

and where $\mathcal{MM}$ is the conjectural category of mixed motives over $\mathbb{C}$, and $h^{2r-\nu}(X)(r)$ is motivic cohomology. The lack of an explicit description of $\mathcal{MM}/\mathbb{C}$ has not deterred others from presenting possible candidate BB filtrations. It is the personal prejudice of the author to consider the construction in [Lew3], based on the idea of $\overline{\mathbb{Q}}$-spreads. A similar story holds for [A], [GG]. To explain this, consider a smooth projective variety $X/\mathbb{C}$. One can think of $X = X_K \times \mathbb{C}$, where $K/\overline{\mathbb{Q}}$ is finitely generated. Clearly $K = \overline{\mathbb{Q}}(\mathcal{S})$ for some variety $\mathcal{S}/\overline{\mathbb{Q}}$. Let $\eta \in \mathcal{S}/\overline{\mathbb{Q}}$ be the generic point, and observe that $K \subset \mathbb{C}$ corresponds to an embedding $\overline{\mathbb{Q}}(\eta) \hookrightarrow \mathbb{C}$ over $\overline{\mathbb{Q}}$. Correspondingly, there is a spread $\rho : \mathcal{X} \to \mathcal{S}$ which we can assume is given by a smooth, proper morphism of smooth quasi-projective varieties, with the property that $X/\mathbb{C} = \mathcal{X}_\eta \times_{\overline{\mathbb{Q}}(\eta)} \mathbb{C}$. Likewise, if $\xi \in \mathrm{CH}^r(X/\mathbb{C};\mathbb{Q})$, then by possibly modifying $\mathcal{S}/\overline{\mathbb{Q}}$, there is a cycle $\tilde{\xi} \in \mathrm{CH}^r(\mathcal{X}/\overline{\mathbb{Q}};\mathbb{Q})$ for which $\tilde{\xi}_\eta = \xi$ in $\mathrm{CH}^r(X/\mathbb{C};\mathbb{Q})$. The following illustrates this process.

Example 3.1.

$$Y/\mathbb{C} = \mathrm{Spec}\left\{ \frac{\mathbb{C}[x,y]}{(\pi y^2 + (\sqrt{\pi}+4)x^3 + \mathrm{e}x)} \right\},$$
$$\mathcal{S}/\mathbb{Q} = \mathrm{Spec}\left\{ \frac{\mathbb{Q}[u,v,w]}{(u-v^2)} \right\},$$

Set:

$$\mathcal{Y}_{\mathcal{S}} = \mathrm{Spec}\left\{ \frac{\mathbb{Q}[x,y,u,v,w]}{(uy^2 + (v+4)x^3 + wx, u - v^2)} \right\}.$$

The inclusion

$$\frac{\mathbb{Q}[u,v,w]}{(u-v^2)} \subset \frac{\mathbb{Q}[x,y,u,v,w]}{(uy^2 + (v+4)x^3 + wx, u - v^2)},$$

defines a morphism $\mathcal{Y}_{\mathcal{S}} \to \mathcal{S}$, as varieties over $\mathbb{Q}$. Let $\eta \in \mathcal{S}$, be the generic point. Then

$$\mathbb{Q}(\eta) = \mathrm{Quot}\left(\frac{\mathbb{Q}[u,v,w]}{(u-v^2)}\right).$$

Note that the embedding

$$\mathbb{Q}(\eta) \hookrightarrow \mathbb{C}, \quad (u,v,w) \mapsto (\pi, \sqrt{\pi}, \mathrm{e}), \Rightarrow \mathcal{Y}_{\mathcal{S},\eta} \times \mathbb{C} = Y/\mathbb{C}.$$

At this point, we need a few ingredients; firstly the Bloch-Beilinson conjecture:

Conjecture 3.2 (BBC). *If W/k is smooth and projective over a number field k, then the Abel-Jacobi map Φ_r : $CH^r(W/k;\mathbb{Q}) \to J\big(H^{2r-1}(W,\mathbb{Q}(r))\big)$, is injective.*

One can argue that if the classical Hodge conjecture (HC) holds, then W in Conjecture 3.2 can be allowed to be a smooth quasi-projective variety. There are counterexamples to this conjecture if the transcendence degree of k over $\mathbb{Q}$ is > 0. Secondly, for any smooth quasi-projective $W/\mathbb{C}$, there is a variant of Deligne cohomology that incorporates weights, called Beilinson's absolute Hodge cohomology (see [Be1], [Ja1]). In particular, it applies to the term $J\big(H^{2r-1}(W,\mathbb{Q}(r))\big)$, which is not the same as in Deligne cohomology unless W is smooth and projective. There is a short exact sequence

$$0 \to J\big(H^{2r-1}(W,\mathbb{Q}(r))\big) \to H^{2r}_{\mathcal{H}}(W,\mathbb{Q}(r)) \to \Gamma H^{2r}(W,\mathbb{Q}(r)) \to 0.$$

Thus one can say that the BBC + HC is tatamount to saying

$$\mathrm{CH}^r(\mathcal{X}/\overline{\mathbb{Q}};\mathbb{Q}) \to H^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r)), \text{ is injective. We now state}$$

Theorem 3.3 ([Lew3]). *Let $X/\mathbb{C}$ be smooth projective of dimension d. Then for all r, there is a filtration,*

$$CH^r(X;\mathbb{Q}) = F^0 \supset F^1 \supset \cdots \supset F^\nu \supset F^{\nu+1} \supset \cdots \supset F^r \supset F^{r+1} = F^{r+2} = \cdots,$$

which satisfies the following:

(i) $F^1 = CH^r_{\mathrm{hom}}(X;\mathbb{Q})$.
(ii) $F^2 \subseteq \ker\Phi_r : CH^r_{\mathrm{hom}}(X;\mathbb{Q}) \to J\big(H^{2r-1}(X,\mathbb{Q}(r))\big)$.
(iii) $F^{\nu_1}CH^{r_1}(X;\mathbb{Q}) \bullet F^{\nu_2}CH^{r_2}(X;\mathbb{Q}) \subset F^{\nu_1+\nu_2}CH^{r_1+r_2}(X;\mathbb{Q})$, *where $\bullet$ is the intersection product.*
(iv) *F^ν is preserved under the action of correspondences between smooth projective varieties over $\mathbb{C}$.*
(v) *Let $\mathrm{Gr}^\nu_F := F^\nu/F^{\nu+1}$ and assume that the Künneth components of the diagonal class $[\Delta_X] = \oplus_{p+q=2d}[\Delta_X(p,q)] \in H^{2d}(X\times X,\mathbb{Q}(d)))$ are algebraic. Then*

$$\Delta_X(2d-2r+\ell,2r-\ell)_*\big|_{\mathrm{Gr}^\nu_F CH^r(X;\mathbb{Q})} = \delta_{\ell,\nu}\cdot \mathrm{Identity}.$$

[If we assume the conjecture that homological and numerical equivalence coincide, then (v) says that Gr^ν_F factors through the Grothendieck motive.]

(vi) *Let $D^r(X) := \bigcap_\nu F^\nu$. If the BBC + HC holds, then $D^r(X) = 0$.*

The idea of proof goes as follows. Recall the $\overline{\mathbb{Q}}$-spread $\rho : \mathcal{X} \to \mathcal{S}$, where ρ is smooth and proper. Let η be the generic point of $\mathcal{S}$, and put $K := \overline{\mathbb{Q}}(\eta)$. Write $X_K := \mathcal{X}_\eta$. From [Lew3] we introduced a decreasing filtration $\mathcal{F}^\nu\mathrm{CH}^r(\mathcal{X};\mathbb{Q})$, with the property that $Gr^\nu_{\mathcal{F}}\mathrm{CH}^r(\mathcal{X};\mathbb{Q}) \hookrightarrow E^{\nu,2r-\nu}_\infty(\rho)$, where $E^{\nu,2r-\nu}_\infty(\rho)$ is the ν-th graded piece of the Leray filtration on the lowest weight part $\underline{H}^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$ of Beilinson's absolute Hodge cohomology $H^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$ associated to ρ. That lowest weight part $\underline{H}^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r)) \subset H^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$ is given by the image $H^{2r}_{\mathcal{H}}(\overline{\mathcal{X}},\mathbb{Q}(r)) \to H^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$, where $\overline{\mathcal{X}}$ is a smooth compactification of $\mathcal{X}$. There is a cycle class map $\mathrm{CH}^r(\mathcal{X};\mathbb{Q}) := \mathrm{CH}^r(\mathcal{X}/\overline{\mathbb{Q}};\mathbb{Q}) \to \underline{H}^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$, which is conjecturally injective under the Bloch-Beilinson conjecture assumption, using the fact that there is a short exact sequence:

$$0 \to J\big(H^{2r-1}(\mathcal{X},\mathbb{Q}(r))\big) \to H^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r)) \to \Gamma H^{2r}(\mathcal{X},\mathbb{Q}(r)) \to 0.$$

(Injectivity would imply $D^r(X) = 0$.) Regardless of whether or not injectivity holds, the filtration $\mathcal{F}^\nu\mathrm{CH}^r(\mathcal{X};\mathbb{Q})$ is given by the pullback of the Leray filtration on $\underline{H}^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$ to $\mathrm{CH}^r(\mathcal{X};\mathbb{Q})$. It is proven in [Lew3] that the term $E^{\nu,2r-\nu}_\infty(\rho)$ fits in a short exact sequence:

$$0 \to \underline{E}^{\nu,2r-\nu}_\infty(\rho) \to E^{\nu,2r-\nu}_\infty(\rho) \to \underline{\underline{E}}^{\nu,2r-\nu}_\infty(\rho) \to 0,$$
$$\text{where } \underline{\underline{E}}^{\nu,2r-\nu}_\infty(\rho) = \Gamma H^\nu(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)),$$

$$\underline{E}^{\nu,2r-\nu}_\infty(\rho) = \frac{J(W_{-1}H^{\nu-1}(\mathcal{S},R^{2r-\nu}\rho_*\mathbb{Q}(r)))}{\Gamma(Gr^0_W H^{\nu-1}(\mathcal{S},R^{2r-\nu}\rho_*\mathbb{Q}(r)))} \subset J\big(H^{\nu-1}(\mathcal{S},R^{2r-\nu}\rho_*\mathbb{Q}(r))\big). \tag{6}$$

The latter inclusion in (6) is a result of the short exact sequence:

$$0 \to W_{-1}H^{\nu-1}(\mathcal{S},R^{2r-\nu}\rho_*\mathbb{Q}(r)) \to W_0H^{\nu-1}(\mathcal{S},R^{2r-\nu}\rho_*\mathbb{Q}(r)) \to Gr^0_W H^{\nu-1}(\mathcal{S},R^{2r-\nu}\rho_*\mathbb{Q}(r)) \to 0.$$

One then has (by definition)

$$F^\nu\mathrm{CH}^r(X_K;\mathbb{Q}) = \lim_{\substack{\to \\ U\subset\mathcal{S}/\overline{\mathbb{Q}}}} \mathcal{F}^\nu\mathrm{CH}^r(\mathcal{X}_U;\mathbb{Q}), \quad \mathcal{X}_U := \rho^{-1}(U)$$
$$F^\nu\mathrm{CH}^r(X/\mathbb{C};\mathbb{Q}) = \lim_{\substack{\to \\ K\subset\mathbb{C}}} F^\nu\mathrm{CH}^r(X_K;\mathbb{Q})$$

Further, since direct limits preserve exactness,

$$Gr^\nu_F\mathrm{CH}^r(X_K;\mathbb{Q}) = \lim_{\substack{\to \\ U\subset\mathcal{S}/\overline{\mathbb{Q}}}} Gr^\nu_{\mathcal{F}}\mathrm{CH}^r(\mathcal{X}_U;\mathbb{Q}),$$
$$Gr^\nu_F\mathrm{CH}^r(X_{\mathbb{C}};\mathbb{Q}) = \lim_{\substack{\to \\ K\subset\mathbb{C}}} Gr^\nu_F\mathrm{CH}^r(X_K;\mathbb{Q})$$

3.4 Arithmetic Normal Functions

We continue with the setting of a smooth and proper map $\rho : \mathcal{X} \to \mathcal{S}$ over $\overline{\mathbb{Q}}$, and after possibly shrinking $\mathcal{S}$, that $\mathcal{S}$ is affine, with $K = k(\mathcal{S})$. Let $V \subset \mathcal{S}(\mathbb{C})$ be a smooth, irreducible, closed subvariety of dimension $\nu - 1$ (note that $\mathcal{S}$ affine $\Rightarrow V$ affine). One has a commutative square

$$\begin{array}{ccc} \mathcal{X}_V & \hookrightarrow & \mathcal{X}(\mathbb{C}) \\ \rho_V \downarrow & & \downarrow \rho \\ V & \hookrightarrow & \mathcal{S}(\mathbb{C}), \end{array}$$

and a commutative diagram

$$\begin{array}{ccccccc} & & \xi \in Gr_{\mathcal{F}}^{\nu}\mathrm{CH}^r(\mathcal{X};\mathbb{Q}) \mapsto Gr_F^{\nu}\mathrm{CH}^r(X_K;\mathbb{Q}) & & & & \\ & & \downarrow & & & & \\ 0 \to \underline{E}_{\infty}^{\nu,2r-\nu}(\rho) & \to & E_{\infty}^{\nu,2r-\nu}(\rho) & \to & \underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\rho) & \to 0 \\ \downarrow & & \downarrow & & \downarrow & \\ 0 \to \underline{E}_{\infty}^{\nu,2r-\nu}(\rho_V) & \to & E_{\infty}^{\nu,2r-\nu}(\rho_V) & \to & \underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\rho_V) & \to 0 \\ & & & & || & \\ & & & & 0 & \end{array}$$

where $\underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\rho_V) = 0$ follows from the weak Lefschetz theorem for locally constant systems over affine varieties (see for example [Ar], and the references cited there). Thus for any $\xi \in Gr_{\mathcal{F}}^{\nu}\mathrm{CH}^r(\mathcal{X};\mathbb{Q})$, we have a "normal function" η_ξ with the property that for any such smooth irreducible closed $V \subset S(\mathbb{C})$ of dimension $\nu - 1$, we have a value $\eta_\xi(V) \in \underline{E}_{\infty}^{\nu,2r-\nu}(\rho_V)$. Here we think of V as a point on a suitable open subset of the Chow variety of dimension $\nu - 1$ subvarieties of $\mathcal{S}(\mathbb{C})$ and η_ξ defined on that subset. For example if $\nu = 1$, then we recover the classical notion of normal functions.

Definition 3.5. η_ξ is called an arithmetic normal function.

Example 3.6. If $\mathcal{S}$ is affine of dimension $\nu - 1$. Then in this case $V = \mathcal{S}$, and $\xi \in Gr_{\mathcal{F}}^{\nu}\mathrm{CH}^r(\mathcal{X};\mathbb{Q})$ induces a "single point" normal function

$$\eta_\xi(V) = \eta_\xi(\mathcal{S}) \in J\big(H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))\big).$$

Now let $\xi \in \mathcal{F}^{\nu}\mathrm{CH}^r(\mathcal{X};\mathbb{Q})$ be given, and let $[\xi] \in \underline{E}_{\infty}^{\nu,2r-\nu}(\rho)$ be its image via the composite $\mathcal{F}^{\nu}\mathrm{CH}^r(\mathcal{X};\mathbb{Q}) \to E_{\infty}^{\nu,2r-\nu}(\rho) \to \underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\rho)$.

Theorem 3.7 (see [K-L]). *The class* $[\xi]$ *depends only on* η_ξ, *and is called the topological invariant of* η_ξ.

Question 3.8.

(i) Can one characterize the BB filtration in terms of arithmetic normal functions?

(ii) What about the zero (or torsion) locus of such normal functions. I.e., are they defined over $\overline{\mathbb{Q}}$?

Remark 3.9. Special cases of Question 3.8(i) are worked out in [K-L]. Further, if both X and S are defined over $\overline{\mathbb{Q}}$, with $\mathcal{X} = S \times X$, and $\rho = \mathrm{Pr}_1$, then the answer is yes, as shown in [Lew5]. The use of arithmetic normal functions has played a role in [Lew7].

4. The Beilinson-Hodge Conjectures

In this section, we allow $X/\mathbb{C}$ to be smooth quasi-projective, and consider the integral and rationally defined Betti cycle class maps that factor through Deligne (or equivalently via Beilinson's absolute Hodge cohomology), viz.,

$$\mathrm{cl}_{r,m} : \mathrm{CH}^r(X,m) \to \Gamma H^{2r-m}(X,\mathbb{Z}(r)), \tag{7}$$

$$\mathrm{cl}_{r,m,\mathbb{Q}} : \mathrm{CH}^r(X,m;\mathbb{Q}) \to \Gamma H^{2r-m}(X,\mathbb{Q}(r)) \tag{8}$$

It was Jannsen [Ja1] (Cor. 9.11)) who first observed that the anticipated surjectivity of the map in (8) fails for some $X/\mathbb{C}$ (in the case $m = 1$). For motivational purposes, we recall the following situation discussed in [dJ-L]: Fix $m \geq 0$. Consider these 3 statements:

(S1) $\mathrm{cl}_{r,m,\mathbb{Q}} : \mathrm{CH}^r(X,m;\mathbb{Q}) \to \Gamma H^{2r-m}(X,\mathbb{Q}(r))$ is surjective for all smooth projective $X/\mathbb{C}$ and all r.

(S2) $\mathrm{cl}_{r,m,\mathbb{Q}} : \mathrm{CH}^r(X,m;\mathbb{Q}) \to \Gamma H^{2r-m}(X,\mathbb{Q}(r))$ is surjective for all smooth quasi-projective $X/\mathbb{C}$ and all r.

(S3) $\mathrm{cl}_{r,m,\mathbb{Q}}^{\lim} : \mathrm{CH}^r(\mathrm{Spec}(\mathbb{C}(X)),m;\mathbb{Q}) \to \Gamma H^{2r-m}(\mathbb{C}(X),\mathbb{Q}(r))$ is surjective for all varieties $X/\mathbb{C}$ and all r. (Hence for dimension reasons, $\Gamma H^{2r-m}(\mathbb{C}(X),\mathbb{Q}(r)) = 0$ for $r > m$).

Proposition 4.1. *For* $m = 0$, (S1) $\Leftrightarrow$ (S2) $\Leftrightarrow$ (S3); *moreover this is equivalent to saying that* $\Gamma H^{2r}(\mathbb{C}(X),\mathbb{Q}(r)) = 0$, *for all* X *and all* $r > 0$.

Proof. This is really the import of Deligne's mixed Hodge theory [De] and localization sequences. Let us show for example that statement (S1) is equivalent to $\Gamma H^{2r}(\mathbb{C}(X),\mathbb{Q}(r)) = 0$, for all X and all $r > 0$. Note that if $\xi \in \mathrm{CH}^r(X;\mathbb{Q})$ is given, with $r > 0$, then ξ vanishes on $X \backslash |\xi| \neq \emptyset$. Thus it is clear that (S1) $\Rightarrow \Gamma H^{2r}(\mathbb{C}(X),\mathbb{Q}(r)) = 0$. Conversely, let $\xi \in \Gamma H^{2r}(X,\mathbb{Q}(r))$, where $X/\mathbb{C}$ is smooth and projective. Then $\Gamma H^{2r}(\mathbb{C}(X),\mathbb{Q}(r)) = 0$ implies that $\xi \mapsto 0 \in H^{2r}(X \backslash Y,\mathbb{Q}(r))$ for some closed codimension one $Y \subset X$. Thus we have $\xi \in \mathrm{Image}\big(H_Y^{2r}(X,\mathbb{Q}(r)) \to H^{2r}(X,\mathbb{Q}(r))\big)$. Since $H^{2r}(X,\mathbb{Q}(r))$ is a pure Hodge structure, it follows from a weight argument [De] that $\xi \in \mathrm{Image}\big(\Gamma H^{2r-2}(\tilde{Y},\mathbb{Q}(r-1)) \to H^{2r}(X,\mathbb{Q}(r))\big)$, where $\tilde{Y} \to Y$ is a desingularization of Y. If $r - 1 = 0$, then ξ is algebraic. Otherwise $r - 1 > 0$, so we now use $\Gamma H^{2(r-1)}(\mathbb{C}(\tilde{Y}),\mathbb{Q}(r-1)) = 0$ and repeat the argument. □

In the case $m > 0$, it turns out that (S1), (S2) and (S3) are independent statements. For instance, by a weight argument $\Gamma H^{2r-m}(X, \mathbb{Q}(r)) = 0$ for X smooth projective, hence (S1) trivially holds. It turns out that (S2) fails, but (S3) is probably true. In fact, we believe in a stronger statement than (S3):

Conjecture 4.2 ([dJ-L-P]). *Let $\rho : \mathcal{X} \to \mathcal{S}$ be a smooth, proper map of varieties over $\mathbb{C}$. Then $\mathrm{cl}^{\lim}_{r,m,\mathbb{Q}} : CH^r(\mathcal{X}_\eta, m; \mathbb{Q}) \to \Gamma H^{2r-m}(\mathcal{X}_\eta, \mathbb{Q}(r))$ is surjective.*

Note that if $\mathcal{S} = \mathcal{X}$ in Conjecture 4.2, then we recover (S3). On the hand, if $\mathcal{S} = \mathrm{Spec}(\mathbb{C})$ and $m = 0$, we recover the HC.

4.3 A Key Observation

For this subsection, let $k = \bar{k} \subseteq \mathbb{C}$ be a subfield, X/k smooth projective, and $Y \subset X/k$ a proper subvariety. Let

$$H^{2r}_Y(X, \mathbb{Q}(r))^\circ = \ker\big(H^{2r}_Y(X, \mathbb{Q}(r)) \to H^{2r}(X, \mathbb{Q}(r))\big).$$

From the short exact sequence:

$$0 \to \frac{H^{2r-1}(X, \mathbb{Q}(r))}{H^{2r-1}_Y(X, \mathbb{Q}(r))} \to H^{2r-1}(X \backslash Y, \mathbb{Q}(r)) \to H^{2r}_Y(X, \mathbb{Q}(r))^\circ \to 0,$$

we arrive at:

$$\begin{array}{ccccc} \mathrm{CH}^r(X\backslash Y, 1; \mathbb{Q}) & \xrightarrow{\partial} & \mathrm{CH}^r_Y(X; \mathbb{Q})^\circ & \xrightarrow{\beta} & \mathrm{CH}^r_{\mathrm{hom}}(X; \mathbb{Q}) \\ \downarrow \mathrm{cl}_{r,1,\mathbb{Q}} & & \downarrow \lambda & & \downarrow \underline{\Phi}_r \\ \Gamma H^{2r-1}(X\backslash Y, \mathbb{Q}(r)) & \hookrightarrow & \Gamma H^{2r}_Y(X, \mathbb{Q}(r))^\circ & \to & J\left(\frac{H^{2r-1}(X,\mathbb{Q}(r))}{H^{2r-1}_Y(X,\mathbb{Q}(r))}\right) \end{array}$$

Here $\underline{\Phi}_r$ is the (reduced) Abel-Jacobi map. Observe that the HC implies that λ is onto. Indeed, via Poincaré duality, the surjectivity of λ is implied by a homological version of the HC for singular varieties, which is in fact equivalent to the HC (cohomological version) for smooth projective varieties (see [Ja2]). In general, λ onto together with the snake lemma implies that

$$\frac{\ker \underline{\Phi}_r|_{\mathrm{Im}(\beta)}}{\beta(\ker \lambda)} \simeq \mathrm{cok}(\mathrm{cl}_{r,1,\mathbb{Q}}).$$

Further, the HC implies:

$$\mathrm{cok}(\mathrm{cl}_{r,1,\mathbb{Q}}) \simeq \frac{\beta(\ker \lambda) + \ker \Phi_r|_{\mathrm{Im}(\beta)}}{\beta(\ker \lambda)}, \tag{9}$$

where again we recall $\Phi_r : \mathrm{CH}^r_{\mathrm{hom}}(X; \mathbb{Q}) \to J\big(H^{2r-1}(X, \mathbb{Q}(r))\big)$ is the Abel-Jacobi map. To see this, let us first assume for notational simplicity that Y has pure codimension $\ell \geq 1$ in X. Let $\tilde{Y} \to Y$ be a desingularization. Again by a weight argument [De], the images:

$$\mathrm{Gy} : H^{2r-1-2\ell}(\tilde{Y}, \mathbb{Q}(r-\ell)) \to H^{2r-1}(X, \mathbb{Q}(r)),$$

$$H^{2r-1}_Y(X, \mathbb{Q}(r)) \to H^{2r-1}(X, \mathbb{Q}(r)),$$

coincide. By semi-simplicity considerations, we can write

$$H^{2r-1}(X, \mathbb{Q}(r)) = \mathrm{Im}\ \mathrm{Gy} \oplus \{\mathrm{Im}\ \mathrm{Gy}\}^\perp,$$

$$H^{2r-1-2\ell}(\tilde{Y}, \mathbb{Q}(r-\ell)) = \{\ker \mathrm{Gy}\}^\perp \oplus \ker \mathrm{Gy}.$$

By the HC, the composite $H^{2r-1}(X, \mathbb{Q}(r)) \xrightarrow{\mathrm{Pr}_1} \mathrm{Im}\ \mathrm{Gy} \xrightarrow{\{\mathrm{Gy}|_{\{\ker \mathrm{Gy}\}^\perp}\}^{-1} \simeq} \{\ker \mathrm{Gy}\}^\perp$

$$\simeq \{\ker \mathrm{Gy}\}^\perp \oplus \{0\} \hookrightarrow H^{2r-1-2\ell}(\tilde{Y}, \mathbb{Q}(r-\ell)),$$

is induced by some $w \in \mathrm{CH}^{d-\ell}(X \times \tilde{Y}; \mathbb{Q})$, for which $\mathrm{Gy} \circ [w]_* = \mathrm{Pr}_1$. Now let $\xi \in \mathrm{Im}(\beta) \subset \mathrm{CH}^r_{\mathrm{hom}}(X)$ be given such that $\underline{\Phi}_r(\xi) = 0$. Then by semi-simplicity considerations, $\Phi_r(\xi) \in J\big(\mathrm{Im}(\mathrm{Gy})\big) \subset J\big(H^{2r-1}(X, \mathbb{Q}(r))\big)$. But $w_*(\xi) \in \mathrm{CH}^{r-\ell}_{\mathrm{hom}}(\tilde{Y}; \mathbb{Q})$, which maps to some $\xi_0 \in \ker(\lambda) \subset \mathrm{CH}^r_Y(X; \mathbb{Q})^\circ$ for which $\Phi_r(\xi - \beta(\xi_0)) = 0$. The isomorphism in (9) then follows.

Remark 4.4. We consider two extreme scenarios (the latter due to Jannsen [Ja1]):

- $k \subseteq \overline{\mathbb{Q}}$. Then BBC + HC $\Rightarrow \mathrm{cok}(\mathrm{cl}_{r,1,\mathbb{Q}}) = 0$.
- $k = \mathbb{C}$ and $\mathrm{cd}_X Y = r \Rightarrow \lambda$ is an isomorphism and $H^{2r-1}_Y(X, \mathbb{Q}(r)) = 0$. Hence $\mathrm{cok}(\mathrm{cl}_{r,1,\mathbb{Q}}) = \ker \Phi_r|_{\mathrm{Im}(\beta)}$, which need not be zero (Mumford).

Next, assuming $k = \mathbb{C}$ and taking the limit over all codimension one subvarieties $Y \subset X$, and assuming the HC, we arrive at

$$\mathrm{cok}(\mathrm{cl}^{\lim}_{r,1,\mathbb{Q}}) \simeq \frac{N^1\mathrm{CH}^r(X; \mathbb{Q}) + \ker \Phi_r}{N^1\mathrm{CH}^r(X; \mathbb{Q})}, \tag{10}$$

where $N^p\mathrm{CH}^r(X)$ are those cycles that are homologous to zero on subvarieties of codimension p in X. Jannsen, et al. [Ja3]) raised the important question as to whether the RHS in (10) is always zero, and in the case $r = 2$, this would imply that $\ker \Phi_2 \subset \mathrm{CH}^2_{\mathrm{alg}}(X; \mathbb{Q})$, where $\mathrm{CH}^r_{\mathrm{alg}}(X) \subset \mathrm{CH}^r(X)$ is the subgroup of cycles that are algebraically equivalent to zero (= the "continuous" part of $\mathrm{CH}^2(X)$). Thus there is a reasonable expectation that the RHS in (10) is always zero. Further, and for a connection between the vanishing of $\mathrm{cok}(\mathrm{cl}^{\lim}_{r,1,\mathbb{Q}})$ for all X and the field of definition of the zero locus of a cycle induced normal function, the reader can consult [Lew4].

4.5 Formulation of the Conjectures

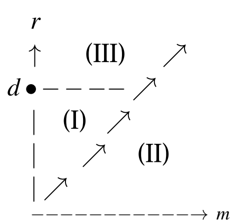
For any r, m and smooth quasi-projective $X/\mathbb{C}$, the approach to studying $\mathrm{cl}_{r,m,\mathbb{Q}} : \mathrm{CH}^r(X,m;\mathbb{Q}) \to \Gamma H^{2r-m}(X,\mathbb{Q}(r))$, involves a weight filtered spectral sequence associated to the simplicial complex $Y^{[\bullet]} \to Y \subset \overline{X}$, where $\overline{X}$ is a good compactification of X with NCD Y.

Theorem 4.6 (Vague form, see [dJ-L] (Thm 4.9)). *The obstruction to the surjectivity of* $\mathrm{cl}_{r,m,\mathbb{Q}}$ *is "explained" in terms of kernels of (higher) Abel-Jacobi maps.*

Remark 4.7. One can show that the BBC + HC implies that (S2) holds for all $X/\overline{\mathbb{Q}}$ [MSa2]; (also see [K-L] as well as [dJ-L] for different proofs). Remark 4.4 gives one instance of this (viz., the case $m=1$). Further, Remark 4.4 also gives a counterexample to (S2) in the case $m=1$.

Quite generally, we have the following picture for any smooth quasi-projective variety $X/\mathbb{C}$ and corresponding cycle class map

$$\mathrm{cl}_{r,m,\mathbb{Q}} : \mathrm{CH}^r(X,m;\mathbb{Q}) \to \Gamma H^{2r-m}(X,\mathbb{Q}(r)).$$



Region (I): $r \le d$, $m > 0$ and $r > m$.
Region (II): $r < m$.
Region (III): $r > m$ and $r > d$.

The map $\mathrm{cl}_{r,m,\mathbb{Q}}$ is trivially surjective in regions (II) and (III), since in those cases $\Gamma H^{2r-m}(X,\mathbb{Q}(r)) = 0$. In general surjectivity fails in region (I). Note that the r-axis corresponds to the classical Hodge conjecture.

Theorem 4.8 ([dJ-L] (Thm 5.1)). *Assume given* (r,m,d) *in region I above. Then there exists a smooth quasi-projective variety* $X/\mathbb{C}$ *of dimension* d *such that* $\mathrm{cl}_{r,m,\mathbb{Q}}$ *fails to be surjective.*

In summary, and outside of the situation involving varieties over number fields, we see that the following situation emerges:

Conjecture 4.9.

(i) $\mathrm{cl}_{m,m,\mathbb{Q}}$ *is surjective for all* m.
(ii) $\mathrm{cl}^{\lim}_{r,m,\mathbb{Q}}$ *is surjective for all* r *and* m.

Remark 4.10.

(1) The statement in (i) is generally referred to as the Beilinson-Hodge conjecture (see [A-K], [A-S], [Sa]). It seems more natural to also refer to it as the Beilinson-Milnor-Hodge conjecture.
(2) The present form of (ii) is indicative that there are numerator conditions in the definition of $\mathrm{CH}^r(X,m)$ for $m > 0$, that can either be removed after passing to the generic point, or the Hodge theory of the target becomes trivial.

5. Intermezzo: Milnor K-Theory and the Bloch-Kato Theorem

An excellent reference for this part is [B-T]. Let $\mathbb{F}$ be a field with multiplicative group $\mathbb{F}^\times \subset \mathbb{F}$. Consider the graded tensor algebra

$$T(\mathbb{F}) := \bigoplus_{m=0}^{\infty} \{\mathbb{F}^\times\}^{\otimes_{\mathbb{Z}} m} = \mathbb{Z} \oplus \mathbb{F}^\times \oplus \cdots,$$

and let $R(\mathbb{F})$ be the graded 2-sided ideal generated by

$$\big\{\tau \otimes (1-\tau) \bigm| \tau \in \mathbb{F}^\times \backslash \{1\}\big\}.$$

Recall that the Milnor K-theory of $\mathbb{F}$ is given by

$$K_\bullet^M(\mathbb{F}) := T(\mathbb{F})/R(\mathbb{F}) = \bigoplus_{m=0}^{\infty} K_m^M(\mathbb{F}).$$

Further, it is well-known that $K_m^M(\mathbb{F}) \simeq \mathrm{CH}^m(\mathrm{Spec}(\mathbb{F}),m)$, (Nesterenko/Suslin (1990), Totaro (1992)). We refer the reader to the discussion and terminology in subsection 6.5 below, where the Kummer short exact sequence and Hilbert 90 are discussed. For M prime to $\mathrm{char}(\mathbb{F})$, there is a map

$$\frac{\mathbb{F}^\times}{[\mathbb{F}^\times]^M} \to H^1_{\mathrm{et}}(\mathbb{F},\mu_M),$$

called the norm residue map. It naturally extends to

$$K_m^M(\mathbb{F})/M \to H^m_{\mathrm{et}}(\mathbb{F},\mu_M^{\otimes m})$$

We state the following.

Theorem 5.1 (Bloch-Kato theorem). *For a prime* $M \neq \mathrm{char}(\mathbb{F})$*, the norm-residue map*

$$K_m^M(\mathbb{F})/M \to H^m_{\mathrm{et}}(\mathbb{F},\mu_M^{\otimes m}),$$

is an isomorphism.

Remark 5.2.

(i) The proof of the Bloch-Kato theorem is due to V. Voevodsky and M. Rost (see [We]).
(ii) The case $m=1$ is due to the aforementioned Kummer short exact sequence and Hilbert 90, as will be seen in §6 below.
(iii) The case $M=2$ is the former Milnor conjecture, proven by V. Voevodsky.
(iv) The case $m=2$ is the Merkurjev-Suslin theorem.

6. Integral Formulations and the Milnor Regulator

Let X be a smooth quasi-projective variety. This section concerns the integrally defined map $\mathrm{cl}^{\lim}_{m,m} = d\log$:

$$\mathrm{cl}^{\lim}_{m,m} : \mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)), m) \to \Gamma\big(H^m(\mathbb{C}(X), \mathbb{Z}(m))\big),$$

$$\{f_1, \ldots, f_m\} \mapsto \bigwedge_1^m d\log f_j,$$

the formula being induced by $d\log : K_m^M(\mathbb{C}(X)) \to \Omega^m_{\mathbb{C}(X)/\mathbb{C}} =: \Omega^m_{\mathrm{Spec}(\mathbb{C}(X))/\mathbb{C}}$ is a rather obvious consequence of the commutative diagram:

$$\begin{array}{ccc} \bigotimes_{j=1}^m \mathrm{CH}^1(\mathrm{Spec}(\mathbb{C}(X)),1) & \overset{\cup}{\twoheadrightarrow} & \mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)),m) \\ \bigotimes_{j=1}^m d\log f_j \downarrow & & \downarrow \bigwedge_{j=1}^m d\log f_j \\ \bigotimes_{j=1}^m \Gamma H^1(\mathbb{C}(X),\mathbb{Z}(1)) & \overset{\cup}{\to} & \Gamma H^m(\mathbb{C}(X),\mathbb{Z}(m)) \end{array}$$

It turns out that $H^m(\mathbb{C}(X), \mathbb{Z}(m))$ is torsion-free (see Theorem 6.15 below). Thus

$$\Gamma H^m(\mathbb{C}(X), \mathbb{Z}(m)) = H^m(\mathbb{C}(X), \mathbb{Z}(m)) \cap F^m H^m(\mathbb{C}(X), \mathbb{C}),$$

makes sense.

Conjecture 6.1. $\mathrm{cl}^{\lim}_{m,m}$ *is surjective.*

Indeed the surjectivity of $\mathrm{cl}^{\lim}_{m,m}$ will be shown to be equivalent to the corresponding surjectivity statement for $\mathrm{cl}^{\lim}_{m,m,\mathbb{Q}}$ (Corollary 6.17).

6.2 Key Example I

For $X/\mathbb{C}$ smooth quasi-projective, we first consider the case $r = m = 1$:

$$\mathrm{cl}_{1,1} : \mathrm{CH}^1(X,1) \to H^1(X, \mathbb{Z}(1)).$$

We recall ([Blo1], [EV]) that

$$\mathrm{CH}^1(X,1) = \mathcal{O}_X^\times(X) \simeq H^1_{\mathcal{D}}(X, \mathbb{Z}(1)),$$

where the latter is Deligne-Beilinson cohomology. This leads to a commutative diagram of short exact sequences:

$$\begin{array}{ccccc} H^0(X,\mathbb{C}/\mathbb{Z}(1)) & \hookrightarrow & H^1_{\mathcal{D}}(X,\mathbb{Z}(1)) & \twoheadrightarrow & \Gamma H^1(X,\mathbb{Z}(1)) \\ \| \wr & & \| \wr & & \| \wr \\ \mathbb{C}^\times & \hookrightarrow & \mathrm{CH}^1(X,1) & \overset{\mathrm{cl}_{1,1}}{\twoheadrightarrow} & \Gamma H^1(X,\mathbb{Z}(1)) \end{array}$$

Corollary 6.3.

(i) $\mathrm{cl}_{1,1}$, $d\log$ *are surjective.*

(ii) $\ker(d\log)$, $\ker(\mathrm{cl}_{1,1})$ *are divisible.*

Thus for any integer $M \neq 0$,

$$\frac{\mathrm{CH}^1(\mathrm{Spec}(\mathbb{C}(X)),1)}{M\cdot \mathrm{CH}^1(\mathrm{Spec}(\mathbb{C}(X)),1)} \simeq \frac{\Gamma H^1(\mathbb{C}(X),\mathbb{Z}(1))}{M\cdot \Gamma H^1(\mathbb{C}(X),\mathbb{Z}(1))}.$$

Next, there is an exact sequence:

$$0 \to \frac{H^1(X,\mathbb{Z}(1))}{\Gamma H^1(X,\mathbb{Z}(1))} \to \frac{H^1(X,\mathbb{C})}{F^1H^1(X,\mathbb{C})} \to \mathrm{CH}^1(X).$$

Further,

$$\frac{H^1(X,\mathbb{C})}{F^1H^1(X,\mathbb{C})} \simeq H^1(\overline{X}, \mathcal{O}_{\overline{X}}),$$

using the fact that $F^1H^1(X,\mathbb{C})$ absorbs the pole singularities along $\overline{X}\backslash X$. Letting X shrink, and using the fact that $\mathrm{CH}^1(\mathrm{Spec}(\mathbb{C}(X)) = 0$, we arrive at

$$\frac{H^1(\mathbb{C}(X),\mathbb{Z}(1))}{\Gamma\big(H^1(\mathbb{C}(X),\mathbb{Z}(1))\big)} \simeq H^1(\overline{X}, \mathcal{O}_{\overline{X}}),$$

which is (uniquely) divisible. The upshot is that

$$\begin{aligned} \frac{\mathrm{CH}^1(\mathbb{C}(X),1)}{M\cdot \mathrm{CH}^1(\mathbb{C}(X),1)} &\simeq \frac{H^1(\mathbb{C}(X),\mathbb{Z}(1))}{M\cdot H^1(\mathbb{C}(X),\mathbb{Z}(1))} \\ &\simeq H^1\left(\mathbb{C}(X), \frac{\mathbb{Z}(1))}{M\cdot \mathbb{Z}(1)}\right), \end{aligned}$$

where the latter uses $H^2(\mathbb{C}(X),\mathbb{Z})$ torsion-free, which can be deduced from the Merkurjev-Suslin theorem (or more generally the Bloch-Kato theorem, see Theorem 6.15 below), or the Lefschetz $(1,1)$ theorem (see [dJ-L]). Indeed, the following result will suffice:

Lemma 6.4. *Let $V/\mathbb{C}$ be smooth, projective and let $Y \subset V$ be a pure codimension one subset such that the image $H^2_Y(V,\mathbb{Z}) \to H^2(V,\mathbb{Z})$ is precisely the Neron-Severi group $NS(V) = H^2_{\mathrm{alg}}(V,\mathbb{Z})$ of V. Let $U = V\backslash Y$. Then $H^2(U,\mathbb{Z})$ is torsion-free.*

Proof. By the Lefschetz $(1,1)$ theorem, $H^2(V,\mathbb{Z})/\mathrm{NS}(V)$ is torsion-free; moreover there is an exact sequence

$$0 \to \frac{H^2(V,\mathbb{Z})}{\mathrm{NS}(V)} \to H^2(U,\mathbb{Z}) \to H^3_Y(V,\mathbb{Z}).$$

It suffices to show that $H^3_Y(V,\mathbb{Z})$ is torsion-free. Let $d = \dim V$. Then by Poincaré duality (and ignoring twists), $H^3_Y(V,\mathbb{Z}) \simeq H_{2d-3}(Y,\mathbb{Z})$. Let $U_Y = Y\backslash Y_{\mathrm{sing}}$. For dimension reasons alone, one has an injection $H_{2d-3}(Y,\mathbb{Z}) \hookrightarrow H_{2d-3}(U_Y,\mathbb{Z}) \simeq H^1(U_Y,\mathbb{Z})$. The latter term is well known to be torsion-free. □

6.5 Alternate Take via a Morphism of Sites

To see this in another context, let us work in the étale topology on a smooth quasi-projective variety $V/\mathbb{C}$, and recall the sheaves $\mathbb{G}_m$, μ_M on V, where for $U \to V$ étale, $\mathbb{G}_m(U) = \Gamma(U, \mathcal{O}_U^\times)$, $\mu_M(U) = \{\xi \in \Gamma(U, \mathcal{O}_U) \mid \xi^M = 1\}$, and the corresponding Kummer short exact sequence:

$$0 \to \mu_M \to \mathbb{G}_m \xrightarrow{x \mapsto x^M} \mathbb{G}_m \to 0.$$

This gives us

$$(11) \qquad 0 \to \frac{\Gamma(V, \mathcal{O}_V^\times)}{M} \to H^1_{\text{et}}(V, \mu_M) \to \text{Pic}(V).$$

Likewise in the analytic topology, we have the exponential short exact sequence,

$$0 \to \mathbb{Z}(1) \to \mathcal{O}_{V,an} \to \mathcal{O}_{V,an}^\times \to 0,$$

which yields the (de Rham interpreted) connecting homomorphism:

$$(12) \qquad d\log : \Gamma(V, \mathcal{O}_{V,an}^\times) \to H^1(V, \mathbb{Z}(1)) \to H^1(V, \mathcal{O}_{V,an}).$$

Both sequences (11) and (12) have this is common:

1) $V = \text{Spec}(\mathbb{C}(X)) \Rightarrow \text{Pic}(V) = 0$ (Hilbert 90). Hence

$$H^1_{\text{et}}(\mathbb{C}(X), \mu_M) \simeq \mathbb{C}(X)^\times / [\mathbb{C}(X)^\times]^M,$$

where $[\mathbb{C}(X)^\times]^M = \{x^M \mid x \in \mathbb{C}(X)^\times\}$.

2) V Stein (e.g. affine) $\Rightarrow H^1(V, \mathcal{O}_{V,an}) = 0$.

We have this commutative diagram:

$$\begin{array}{ccccc}
\frac{\Gamma(V,\mathcal{O}_V^\times)}{M} & \to & H^1_{\text{et}}(V,\mu_M) & \to & \text{Pic}(V) \\
\downarrow & & \downarrow \wr & & \\
\frac{\Gamma(V,\mathcal{O}_{V,an}^\times)}{M} & \to & H^1(V,\mathbb{Z}(1)/M\mathbb{Z}(1)) & & \\
\uparrow & & \uparrow & & \\
\Gamma(V,\mathcal{O}_{V,an}^\times) & \to & H^1(V,\mathbb{Z}(1)) & \to & H^1(V,\mathcal{O}_{V,an}),
\end{array}$$

where the isomorphism $H^1_{\text{et}}(V, \mu_M) \to H^1(V, \mathbb{Z}(1)/M\mathbb{Z}(1))$ is induced by the morphism of sites (étale to analytic, see [Mi]). At the generic point, viz., $V = \text{Spec}(\mathbb{C}(X))$, we arrive at:

$$\begin{array}{ccc}
H^1_{\text{et}}(\mathbb{C}(X),\mu_M) & \simeq & H^1(\mathbb{C}(X),\mathbb{Z}(1)/M\mathbb{Z}(1)) \\
\wr| & & \wr| \\
\mathbb{C}(X)^\times/[\mathbb{C}(X)^\times]^M & \simeq & \text{CH}^1(\text{Spec}(\mathbb{C}(X)),1)/M
\end{array}$$

6.6 Key Example II

The following is essentially Lemma 3.1 in [Ja1], adapted to our situation through the lens of algebraic cycles. The details can be found in [Lew6]. First of all, for $X/\mathbb{C}$ smooth, projective, $\text{CH}^1(X,1) = \mathbb{C}^\times$, and we put $\text{CH}^r_{\text{dec}}(X,1) := \text{Image}\big(\text{CH}^1(X,1) \otimes \text{CH}^{r-1}(X,0) \xrightarrow{\cup} \text{CH}^r(X,1)\big)$.

Proposition 6.7. *Let $X/\mathbb{C}$ be a smooth, projective variety, and $Y \subset X$ a proper closed algebraic subset. Then the cycle class map*

$$CH^2_Y(X,1) \to H^3_{\mathcal{D},Y}(X, \mathbb{Z}(2)),$$

is an isomorphism.

Using Proposition 6.7, we deduce:

Proposition 6.8 ([dJ-L] (Cor. 6.5))**.**

$$\frac{\Gamma H^2(\mathbb{C}(X),\mathbb{Z}(2))}{d\log CH^2(\text{Spec}(\mathbb{C}(X)),2)} \simeq \ker\left[\frac{CH^2_{\text{hom}}(X,1)}{CH^2_{\text{dec}}(X,1)} \xrightarrow{\underline{\Phi}_{2,1}} J\left(\frac{H^2(X,\mathbb{Z}(2))}{H^2_{\text{alg}}(X,\mathbb{Z}(2))}\right)\right].$$

Remark 6.9. The Merkurjev-Suslin theorem implies that the former (hence latter) term is uniquely divisible (see Theorem 6.15). Thus $\underline{\Phi}_{2,1}$ is injective on torsion.

Proof. Let $U := X \backslash Y$ be given such that $H^2(U, \mathbb{Z}(2))$ is torsion-free. There is a diagram, where roughly speaking $\text{CH}^2_{Y,\text{dec}}(X,1)$ involves decomposable classes supported on Y.

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & \text{CH}^2_{Y,\text{dec}}(X,1) & & \\
 & & \downarrow & & \\
\text{CH}^2(U,2) & \to & \text{CH}^2_Y(X,1)^\circ & \to & \text{CH}^2_{\text{hom}}(X,1) \\
\downarrow & & \downarrow & & \downarrow \underline{\Phi}_{2,1} \\
\Gamma H^2(U,\mathbb{Z}(2)) & \hookrightarrow & \Gamma H^3_Y(X,\mathbb{Z}(2))^\circ & \to & J\left(\frac{H^2(X,\mathbb{Z}(2))}{H^2_Y(X,\mathbb{Z}(2))}\right)
\end{array}$$

Now apply the snake lemma and shrink U. □

In light of Remark 6.9, we mention the following:

Theorem 6.10 (Bruno Kahn [KaB])**.**

$$\left\{\frac{CH^2(X,1)}{CH^2_{\text{dec}}(X,1)}\right\}_{\text{tor}} \simeq H^2_{\text{tr}}(X, \mathbb{Q}/\mathbb{Z}(2)),$$

where

$$H^2_{\text{tr}}(X,\mathbb{Z}) = \text{cok}\big(H^2_{\text{alg}}(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})\big).$$

One then anticipates via the Abel-Jacobi map $\underline{\Phi}_{21}$ that

$$\left\{\frac{\text{CH}^2_{\text{hom}}(X,1)}{\text{CH}^2_{\text{dec}}(X,1)}\right\}_{\text{tor}} \simeq J\big(H^2_{\text{tr}}(X,\mathbb{Q}/\mathbb{Z}(2))\big)_{\text{tor}}.$$

Another application of the Merkurjev-Suslin theorem (see paragraph 6.13 below) gives:

Theorem 6.11 (Asakura. See [dJ-L] (Thm 11.1))**.** *Let U be a smooth complex variety. Then the cokernel of*

$$\mathrm{cl}_{2,2}: CH^2(U,2) \to H^2(U,\mathbb{Z}(2))$$

is torsion-free.

As an immediate consequence, we have:

Corollary 6.12 ([dJ-L] (Cor. 11.2))**.** *The following statements are equivalent:*

- $\mathrm{cl}_{2,2}: CH^2(U,2) \to \Gamma H^2(U,\mathbb{Z}(2))$ *is surjective.*
- $\mathrm{cl}_{2,2,\mathbb{Q}}: CH^2(U,2;\mathbb{Q}) \to \Gamma H^2(U,\mathbb{Q}(2))$ *is surjective.*

6.13 Consequences of the Bloch-Kato Theorem

In our situation, the Bloch-Kato theorem translates to saying,

Theorem 6.14. *The map*

$$\frac{CH^m(\mathrm{Spec}(\mathbb{C}(X)),m)}{M\cdot CH^m(\mathrm{Spec}(\mathbb{C}(X)),m)} \to H^m\left(\mathbb{C}(X),\frac{\mathbb{Z}(m))}{M\cdot\mathbb{Z}(m)}\right),$$

is an isomorphism, for any integer $M\neq 0$.

We have the following:

Theorem 6.15 ([dJ-L] (Thm 7.7))**.**

(i) $\forall i$, $H^i_{\mathrm{tor}}(\mathbb{C}(X),\mathbb{Z})=0$. *In particular, the torsion subgroup of $H^i(X,\mathbb{Z})$, where X is a smooth projective completion of X, is supported in codimension one, generalizing the case $i=2$ from the Lefschetz $(1,1)$ theorem.*

(ii) $\ker(\mathrm{cl}^{\lim}_{m,m})$ *is divisible.*

(iii) *The groups:*

$$\frac{H^m(\mathbb{C}(X),\mathbb{Z}(m))}{\Gamma H^m(\mathbb{C}(X),\mathbb{Z}(m))},\quad \frac{\Gamma H^m(\mathbb{C}(X),\mathbb{Z}(m))}{\mathrm{Image}(\mathrm{cl}^{\lim}_{m,m})},$$

are uniquely divisible.

Remark 6.16. Although part (i) of Theorem 6.15 is known among some experts, the knowledge of this fact doesn't seem to be universally known.

Proof. First observe that the map in Theorem 6.14 is given by the composite

$$\begin{array}{ccc} \frac{\mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)),m)}{M} & \to & H^m\left(\mathbb{C}(X),\frac{\mathbb{Z}(m)}{M}\right) \\ \searrow & & \nearrow \\ & \frac{H^m(\mathbb{C}(X),\mathbb{Z}(m))}{M} & \end{array}$$

Notice that the short exact sequence:

$$0\to\mathbb{Z}\xrightarrow{\times M}\mathbb{Z}\to\mathbb{Z}/M\mathbb{Z}\to 0,$$

induces the short exact sequence:

$$0\to\frac{H^m(\mathbb{C}(X),\mathbb{Z}(m))}{M}\to H^m\left(\mathbb{C}(X),\frac{\mathbb{Z}(m)}{M}\right)$$

$$\to H^{m+1}_{M-\mathrm{tor}}(\mathbb{C}(X),\mathbb{Z}(m))\to 0.$$

By Theorem 6.14, it follows that $H^{m+1}_{M-\mathrm{tor}}(\mathbb{C}(X),\mathbb{Z}(m))=0$, thus proving part (i). Next observe that

$$\Gamma H^m(\mathbb{C}(X),m)=F^mH^m(\mathbb{C}(X),\mathbb{C})\cap H^m(\mathbb{C}(X),\mathbb{Z}(m)),$$

and hence

$$\Gamma H^m(\mathbb{C}(X),m)\cap M\cdot H^m(\mathbb{C}(X),\mathbb{Z}(m))=M\cdot\Gamma H^m(\mathbb{C}(X),\mathbb{Z}(m)).$$

By Theorem 6.14, we have the commutative diagram of isomorphisms:

$$\begin{array}{ccc} \frac{\mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)),m)}{M\cdot\mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)),m)} & \simeq & H^m\left(\mathbb{C}(X),\frac{\mathbb{Z}(m)}{M\cdot\mathbb{Z}(m)}\right) \\ \wr\| & & \\ \frac{\mathrm{Image}(\mathrm{cl}^{\lim}_{m,m})}{M\cdot\mathrm{Image}(\mathrm{cl}^{\lim}_{m,m})} & & \wr\uparrow \\ \wr\| & & \\ \frac{\Gamma H^m(\mathbb{C}(X),\mathbb{Z}(m))}{M\cdot\Gamma H^m(\mathbb{C}(X),\mathbb{Z}(m))} & \simeq & \frac{H^m(\mathbb{C}(X),\mathbb{Z}(m))}{M\cdot H^m(\mathbb{C}(X),\mathbb{Z}(m))}, \end{array}$$

for which parts (ii) and (iii) easily follow. □

Corollary 6.17 ([dJ-L] (Cor. 7.8))**.**

$$\frac{\Gamma H^m(\mathbb{C}(X),\mathbb{Z}(m))}{\mathrm{Image}(\mathrm{cl}^{\lim}_{m,m})}=\frac{\Gamma H^m(\mathbb{C}(X),\mathbb{Q}(m))}{\mathrm{Image}(\mathrm{cl}^{\lim}_{m,m,\mathbb{Q}})}.$$

In particular

$$d\log: CH^m(\mathrm{Spec}(\mathbb{C}(X)),m)\to\Gamma H^m(\mathbb{C}(X),\mathbb{Z}(m)),$$

is surjective iff $d\log\otimes\mathbb{Q}$:

$$CH^m(\mathrm{Spec}(\mathbb{C}(X)),m;\mathbb{Q})\to\Gamma H^m(\mathbb{C}(X),\mathbb{Q}(m)),$$

is surjective.

References

[Ar] Arapura, D. *The Leray spectral sequence is motivic,* Invent. Math. **160** (2005), no. 3, 567–589.

[A] Asakura, M. *Motives and algebraic de Rham cohomology,* in: The Arithmetic and Geometry of Algebraic Cycles, Proceedings of the CRM Summer School, June 7–19, 1998, Banff, Alberta, Canada (Editors: B. Gordon, J. Lewis, S. Müller-Stach, S. Saito and N. Yui), NATO Science Series **548** (2000), Kluwer Academic Publishers.

[A-K] Arapura, D., Kumar, M. *Beilinson-Hodge cycles on semiabelian varieties.* Math. Res. Lett. **16**, (2009), no. 4, 557–562.

[A-S] Asakura, M., Saito, S. *Beilinson's Hodge conjecture with coefficients,* in Algebraic Cycles and Motives, Vol. 2, Edited by J. Nagel and C. Peters, LNS **344**, London Math. Soc., (2007), 3–37.

[B-T] Bass, H. and Tate, J. *The Milnor ring of a global field, in Algebraic K-theory II,* Lecture Notes in Math. **342**, Springer-Verlag, 1972, 349–446.

[Be1] Beilinson, A. *Notes on absolute Hodge cohomology,* in Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory, Contemporary Mathematics **55**, part 1, AMS, Providence 1986, 35–68.

[Blo1] Bloch, S. *Algebraic cycles and higher K-theory*, Adv. Math. **61** (1986), 267–304.

[Blo2] Bloch, S., *Algebraic cycles and the Beilinson conjectures,* Contemporary Math. Vol. **58**, Part I, (1986), 65–79.

[Blo3] Bloch, S., Lectures on algebraic cycles, Duke University Mathematics Series, IV. Duke University, Mathematics Department, Durham, N.C., 1980. 182 pp.

[BKLP] Burgos (Appendix), Kerr, Lewis, Lopatto, P. *Simplicial Abel-Jacobi maps and reciprocity laws.* To appear in the Journal of Algebraic Geometry.

[Ca] Carlson, J. *Extension of mixed Hodge structures.* In Journées de Géométrie Algébrique d'Angers 1979, Sijhoff and Nordhoff (1980), 107–127.

[dJ-L] de Jeu, R., Lewis, J. D. (with an appendix by M. Asakura), *Beilinson's Hodge conjecture for smooth varieties.* J. K-Theory **11** (2013), no. 2, 243–282.

[dJ-L-P] de Jeu, R., Lewis, J. D., Patel, D. *A relative version of the Beilinson-Hodge conjecture,* in Recent Advances in Hodge Theory, (M. Kerr and G. Pearlstein, Eds) London Math. Soc. Lecture Note Ser., **427**, Cambridge Univ. Press, Cambridge, 2016, 241–263,

[De] Deligne, P. *Théorie de Hodge, II, III,* Inst. Hautes Études Sci. Publ. Math. **40** (1971), 5–57; **44** (1974), 5–77.

[EV] Esnault, H. and Viehweg, E. *Deligne-Beilinson cohomology*, in Beilinson's Conjectures on Special Values of L-Functions, (Rapoport, Schappacher, Schneider, eds.), Perspect. Math. **4**, Academic Press, 1988, 43–91.

[GG] Green, M., Griffiths, P. A. *Hodge-theoretic invariants for algebraic cycles.* Int. Math. Res. Not. **9**, (2003), 477–510.

[Ja1] Jannsen, U. *Deligne homology, Hodge-$\mathcal{D}$-conjecture, and motives*, in Beilinson's Conjectures on Special Values of L-Functions, (Rapoport, Schappacher, Schneider, eds.), Perspect. Math. **4**, Academic Press, 1988, 305–372.

[Ja2] Jannsen, U., Mixed Motives and Algebraic K-Theory, Lecture Notes in Math. **1000** (1990), Springer-Verlag, Berlin.

[Ja3] Jannsen, U., *Equivalence relations on algebraic cycles,* in: The Arithmetic and Geometry of Algebraic Cycles, Proceedings of the CRM Summer School, June 7–19, 1998, Banff, Alberta, Canada (Editors: B. Gordon, J. Lewis, S. Müller-Stach, S. Saito and N. Yui), NATO Science Series **548** (2000), 225–260, Kluwer Academic Publishers.

[KaB] Kahn, B. *Groupe de Brauer et* $(2,1)$*-cycles indecomposables.* Preprint 2011.

[KLM] Kerr, M., Lewis, J. D., Müller-Stach, S. *The Abel-Jacobi map for higher Chow groups,* Compos. Math. **142** (2006), no. 2, 374–396.

[K-L] Kerr, M., Lewis, J. D. *The Abel-Jacobi map for higher Chow groups, II,* Invent. Math. **170** (2007), no. 2, 355–420.

[Lev] Levine, M. *Localization on singular varieties*, Invent. Math. **31** (1988), 423–464.

[Lew1] Lewis, J. D. A Survey of the Hodge Conjecture. Second edition. Appendix B by B. Brent Gordon. CRM Monograph Series, **10**. American Mathematical Society, Providence, RI, 1999. xvi+368 pp.

[Lew2] Lewis, J. D., *Lectures on algebraic cycles*, Bol. Soc. Mat. Mexicana (3) **7** (2001), no. 2, 137–192.

[Lew3] Lewis, J. D., *A filtration on the Chow groups of a complex projective variety,* Compositio Math. **128** (2001), no. 3, 299–322.

[Lew4] Lewis, J. D., *Abel-Jacobi equivalence and a variant of the Beilinson-Hodge conjecture,* J. Math. Sci. Univ. Tokyo **17** (2010), 179–199.

[Lew5] Lewis, J. D., *Arithmetic normal functions and filtrations on Chow groups,* Proc. Amer. Math. Soc. **140** (2012), no. 8, 2663–2670.

[Lew6] Lewis, J. D., *Lectures on Hodge theory and algebraic cycles,* Commun. Math. Stat. **4** (2016), no. 2, 93–188.

[Lew7] Lewis, J. D., *Cycles on varieties over subfields of* $\mathbb{C}$ *and cubic equivalence,* Motives and algebraic cycles, 233–247, Fields Inst. Commun., **56**, Amer. Math. Soc., Providence, RI, 2009.

[Mi] Milne, J. Étale Cohomology, Princeton Mathematical Series, **33**, Princeton University Press, Princeton, N.J., 1980.

[MSa2] Saito, M. *Hodge-type conjecture for higher Chow groups,* Pure and Applied Mathematics Quarterly Volume **5**, Number 3, (2009), 947–976.

[Sa] Saito, S. *Beilinson's Hodge and Tate conjectures,* in Transcendental Aspects of Algebraic Cycles, Edited by S. Müller-Stach and C. Peters, LNS **313**, London Math. Soc., (2004), 276–290.

[St] Steenbrink, J. H. M. *A summary of mixed Hodge theory*, Motives (Seattle, WA, 1991), 31–41, Proc. Sympos. Pure Math., **55**, Part 1, Amer. Math. Soc., Providence, RI, 1994.

[SJK-L] Kang, S-J., Lewis, J. D. *Beilinson's Hodge conjecture for* K_1 *revisited,* Cycles, motives and Shimura varieties, 197–215, Tata Inst. Fund. Res. Stud. Math., Tata Inst. Fund. Res., Mumbai, 2010.

[We] Weibel, C. *The norm residue isomorphism theorem,* Topology **2**, (2009), 346–372.