Open Problems

compiled by Kefeng Liu*

Note. The readers are welcome to propose the solutions. The authors should send their solutions to liu@math.ucla.edu and post their solutions in arXiv. The correct solutions will be announced and some souvenirs will be presented to the solvers.—*The Editors*

Problem 2017001 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Recently Neves and Marques [1] proved there are infinite number of minimal surfaces in a compact Riemannian manifold with positive Ricci curvature and dimension at most seven. It will be interesting to know the Euler number of such surfaces. Are they bounded by the index linearly? In [2, 3], Grigor'yan, Netrusov and I proved this if the three manifold has positive Ricci curvature. One can ask similar question for codimension one minimal hypersurface in higher dimensions. Can one bound the sum of Betti number in terms of the index in a linear manner?

[1] F. Marques and A. Neves, *Existence of infinitely many minimal hypersurfaces in positive Ricci curvature*, arXiv:1311.6501.

[2] A. Grigor'yan and S.-T. Yau, *Isoperimetric properties of higher eigenvalues of elliptic operators*. Amer. J. Math. 125 (2003), no. 4, 893–940.

[3] A. Grigor'yan, Y. Netrusov and S.-T. Yau, *Eigenvalues of elliptic operators and geometric applications*. Surveys in differential geometry. Vol. IX, 147–217, Surv. Differ. Geom., IX, Int. Press, Somerville, MA, 2004.

Problem 2017002 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Classify all complete Ricci-flat manifolds that are stable or unstable with respect to the Ricci flow. In particular, check those that arise from Kähler geometry and those from Wick rotation of stationary solutions of the Einstein equation.

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Problem 2017003[1] (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Robert Bryant [1] constructed a Calabi-Yau metric in a neighborhood of the complex tube of a real analytic three manifold. So that the real analytic manifold becomes a special Lagrangian submanifold. Bryant assumed the manifold to be parallelizable. His method should also work in higher dimensions. A higher dimensional version of Bryant's theorem was recently studied by Doicu [2], but he assumes the Euler number of the real manifold to be zero. Is the condition necessary?

If you start with any real analytic Kähler manifold, Feix [3] constructed a hyperkähler metric in a neighborhood of the zero section of the cotangent bundle.

[1] R. Bryant, *Calibrated embeddings in the special Lagrangian and coassociative cases*. Ann. Glob. Anal. Geom. **18** (2000), 405–435.

[2] A. Doicu, *Calabi-Yau structures on cotangent bundles*, arXiv:1310.7394.

[3] B. Feix, *Hyperkähler metrics on cotangent bundles*. J. Reine Angew. Math. **532** (2001), 33–46.

Problem 2017004[2] (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Given a minimal regular algebraic surface of general type such that $2c_2 = c_1^2$, is the universal cover given by polydisk? There is counterexample due to Moishezon and Teicher [1].

[1] We thank Claude LeBrun for helpful comments on Problem 2017003.

[2] We thank Ludmil Katzarkov, Calude Lebrun, Damin Wu and Fangyang Zheng for helpful comments on a previous version of Problem 2017004.

We conjecture that when the Euler number is large, the minimal surface of general type with index zero is either covered by polydisk or is a compactification of a ball quotient.

Note that the complement $X-C$ of a surface X of some curve C satisfies the Chern number equality $3\chi(X-C)=(K+C)^2$ implying that the complement is covered by the ball, which follows from the existence of complete Kähler-Einstein metric [2–5].

In [6], we showed that for a compact algebraic surface M of general type, if TM is the sum of two line bundles, then M is a finite quotient of the product of two compact curves.

If we know the surface is topologically equivalent to the quotient of polydisk by discrete group, Jost and Yau [7] proved that complex structures on such manifolds must be obtained by quotient of polydisk by discrete groups. In fact, they also proved that if a Kähler surface admits a continuous map to a curve of genus greater than one, the map can be deformed to a holomorphic map to an algebraic curve of the same genus.

[1] B. Moishezon and M. Teicher, *Existence of simply connected algebraic surfaces of general type with positive and zero indices*, Proc. Nat. Acad. Sci. U.S.A. **83** (1986), 6665–6666.

[2] S.-T. Yau, *Nonlinear analysis in geometry*, Enseign. Math. (2) **33** (1987), 109–158.

[3] S.-Y. Cheng and S.-T. Yau, *Inequality between Chern numbers of singular Kähler surfaces and characterization of orbit space of discrete group of SU(2,1).* Complex differential geometry and nonlinear differential equations (Brunswick, Maine, 1984), 31–44, Contemp. Math., 49, Amer. Math. Soc., Providence, RI, 1986.

[4] G. Tian and S.-T. Yau, *Complete Kähler manifolds with zero Ricci curvature. I*, J. Amer. Math. Soc. **3** (1990), 579–609.

[5] G. Tian and S.-T. Yau, *Complete Kähler manifolds with zero Ricci curvature. II*, Invent. Math. **106** (1991), 27–60.

[6] S.-T. Yau, *A splitting theorem and an algebraic geometric characterization of locally Hermitian symmetric spaces*, Comm. Anal. Geom. **1** (1993), 473–486.

[7] J. Jost and S.-T. Yau, *A strong rigidity theorem for a certain class of compact complex analytic surfaces*, Math. Ann. **271** (1985), 143–152.

Problem 2017005 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

There are two problems I learnt from a lecture given by Daniel Gardiner.

Viterbo [1] proved that for a convex set C in $\mathbb{R}^{2n}$, then

$$\frac{d(C)}{d(B)} \leq \gamma_n \left(\frac{\mathrm{vol}(C)}{\mathrm{vol}(B)}\right)^{1/n}$$

where B is the unit ball, d is any symplectic capacity. Viterbo conjectured that γ_n is equal to one and is only achieved by the ball. Moreover he speculated that the result could be extended to domains bounded by hypersurfaces of restricted contact type.

In [2], Viterbo's conjecture was reformulated as

$$\mathrm{vol}(K) \geq \frac{1}{n!}\ell_1(\partial K)^n,$$

where K is a convex set with smooth boundary and $\ell_1(\partial K)$ is the infimum of the actions of all closed characteristics on ∂K.

Viterbo's conjecture implies the following Mahler's conjecture (see Remark 1 after Theorem 5.1 of [1]).

Given a symmetric bounded convex set K in $\mathbb{R}^n$, its dual convex set K^* is defined by

$$K^\circ = \{x \in \mathbb{R}^n \mid \langle x, y\rangle < 1, \forall y \in K\},$$

which is another symmetric bounded convex set.

The Mahler volume $M(K)$ for K is defined to be

$$M(K) := V(K)V(K^\circ),$$

where $V(K)$ denotes the Lebesgue measure of K.

Mahler (1939) conjectured that for any symmetric bounded convex set K with nonempty interior in $\mathbb{R}^n$,

$$M(K) \geq \frac{4^n}{n!}.$$

The lower bound is achieved by unit cube. The conjecture has been verified by Mahler when $n=2$ and still open in higher dimensions.

[1] C. Viterbo, *Metric and isoperimetric problems in symplectic geometry*. J. Amer. Math. Soc. **13** (2000), 411–431.

[2] J. Álvarez Paiva and F. Balacheff, *Contact geometry and isosystolic inequalities*. Geom. Funct. Anal. **24** (2014), 648–669.

Problem 2017006 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Let W be a $(n-1,n-1)$ form which is $\partial\bar{\partial}$-closed in an n-dimensional complex manifold M. Suppose W is numerically positive in the sense that when integrating over any effective divisor of M, the integral is positive. Given a holomorphic bundle V over M, we can define the degree of V by taking the first Chern form of V wedge with W and integrating over M. It will be independent of the connection on V that defines the Chern form. Hence we can define the bundle to be slope stable in the standard sense.

If V is stable and $\det V$ is trivial, can we find a Hermitian connection on V so that the curvature of V wedge with W is identically zero? This connection can be interpreted as Hermitian-Yang-Mills connection. When M is birational to algebraic manifold, and V is stable with $\det V$ trivial, will the second Chern form of V wedge with w^{n-2} gives a positive number unless the bundle is holomorphically trivial. Here w is a smooth $(1,1)$ form whose $n-1$ power is W. What happens if we only assume W to be numerically nonnegative instead of numerically positive, but that volume integral of w^n is positive. For Calabi-Yau manifolds, there is the concept of mirror symmetry. It is accepted that the mirror of the Hermitian-Yang-Mills equation is the special Lagrangian equation. What is the analogue of this generalized Hermitian-Yang-Mills connection?

Problem 2017007 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Can we classify compact complex manifolds whose tangent bundle admits a flat Hermitian connection? H. C. Wang proved that if the tangent bundle is holomorphically trivial, then the manifold is covered by a complex Lie group. More generally, if the tangent bundle of a complex manifold admits some connection with holonomy group sitting in some special subgroup of $GL(n,\mathbb{C})$, can we make conclusions about the structure of the complex manifold. What happens if we put some condition on the torsion of the connection?

If we have a compact Hermitian manifold (M,J,g) and a flat g-Hermitian connection ∇ (here I mean that $\nabla g = 0$ and $\nabla J = 0$ too), then if we furthermore assume that the torsion of ∇ has vanishing $(1,1)$-part (when viewed as a TM-valued 2-form), then it follows that ∇ is the Chern connection of g, and since it is flat it is then easy to see that the manifold must be complex parallelizable, i.e., TM is holomorphically trivial.

Problem 2017008 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

If a codimension 1 minimal embedded hypersurface of S^n is diffeomorphic to the product of two (or more) spheres of positive dimension, is it the standard embedding? When $n = 3$, it is the Lawson's conjecture which has been proved by Brendle.

Problem 2017009 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

This is a question about resolution of singularities of Kähler metrics. Let us look at the following class of metrics:

Take a complex variety M and a subvariety S of M, we consider Kähler metrics g defined in $M - S$ that satisfies the following condition: at each point $x \in S$, there is a neighborhood U of x so that a nonsingular manifold O and a subvariety D of O and a holomorphic map $F : O \to U$ which maps D into S so that each component of the inverse of $S \cap U$ is a compact subvariety of D. (In fact, a component of the inverse image of a compact neighborhood is compact.)

The map is locally invertible on every point in $O - D$, and the pullback of the metric g (defined on $M - S$) under F can be extended to be a smooth nonsingular metric on O. We also allow the pullback metric to be a Kähler metric defined on $O - D$, complete towards D and its curvature and covariant derivatives are bounded.

The Kähler metric g is said to admit resolution of singularities if a system of maps $\{F\}$ exists at every point $x \in S$. A good example is the orbifold metric where O can be taken to be the ball and the map F is the map from the ball to its quotient space which maps the origin to the quotient singularity. Note that the singular behavior of the metric depends on the system $\{F\}$ which is defined in holomorphic category.

I conjecture that if the curvature and the covariant derivatives of the curvature of this Kähler metric are bounded in each neighborhood of x of S, the resolution system $\{F\}$ exists. Such a statement may be called resolution of singularities of Kähler metric. Note that if we fix a holomorphic system $\{F\}$, there is only one complete Kähler-Einstein metric with negative Ricci curvature resolved by $\{F\}$.

On the other hand, there can be distinct Kähler-Einstein metrics if we choose different system to resolve the singularities of the metric. One can define two systems of resolutions to be equivalent if the holomorphic map from $O - D$ to $O' - D'$ can be extended to be a nonsingular map from O to O' and the same is true for the inverse map from $O' - D'$ to $O - D$.

This concept appeared in my works in the late 1970s with S.-Y Cheng on the construction of Kähler-Einstein metrics on singular varieties. The existence of Kähler-Einstein metrics can be readily generalized to this class of singular metrics. (Basically the same argument I used with Cheng in [1].) Is it true that algebraic manifolds of general type admits such Kähler metrics with negative Ricci curvature? It is certainly true for algebraic surface of general type.

Since the arguments of Ricci flow largely depend only on maximal principle, Kähler-Ricci flow works well with class of singular metrics. Note that such Kähler metric includes a class of Kähler metrics which can be degenerate along S.

[1] S.-Y. Cheng and S.-T. Yau, *Inequality between Chern numbers of singular Kähler surfaces and characterization of orbit space of discrete group of $SU(2,1)$*. Complex differential geometry and nonlinear differential equations (Brunswick, Maine, 1984), 31–44, Contemp. Math., 49, Amer. Math. Soc., Providence, RI, 1986.

Problem 2017010 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Given a compact Kähler manifold M with a nonsingular anticanonical divisor D whose normal bundle is trivial. Prove that all complete Ricci-flat metric in $M \backslash D$ must be asymptotically cylindrical. The existence part was proved by me, the final form appeared in Tian-Yau [1, 2]. The question here is uniqueness.

Prove that a complete Ricci-flat Kähler manifold with linear growth volume is asymptotic to a cylindrical manifold. In my paper with Tian, when the anticanonical divisor at infinity is nonsingular and ample along the divisor, we constructed complete Ricci-flat metric whose volume growth is at most $R^{2n/(n+1)}$, where R is the radius of the geodesic ball. Conversely if a complete irreducible Ricci-flat manifold has that volume growth, would it be compactified to a compact manifold with nonsingular anticanonical divisor.

Calabi asked whether the Bernstein theorem for Complex Monge-Ampère equations is true or not. This turns out not to be true as was observed by LeBrun [3], using the Taub-NUT metric. The counterexample does not have maximal volume growth. Hence one can still ask if a complete Kähler metric defined on the n-dimensional complex Euclidean space has volume growth of R^{2n}, is it flat?

[1] G. Tian and S.-T. Yau, *Complete Kähler manifolds with zero Ricci curvature. I.* J. Amer. Math. Soc. **3** (1990), 579–609.

[2] G. Tian and S.-T. Yau, *Complete Kähler manifolds with zero Ricci curvature. II.* Invent. Math. **106** (1991), 27–60.

[3] C. LeBrun, *Complete Ricci-flat Kähler metrics on $\mathbb{C}^n$ need not be flat.* Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), 297–304, Proc. Sympos. Pure Math., 52, Part 2, Amer. Math. Soc., Providence, RI, 1991.

Problem 2017011 (Differential Geometry). *Proposed by Damin Wu, University of Connecticut and Shing-Tung Yau, Harvard University.*

Let M be a compact Kähler manifold such that over the projective tangent bundle of M, the first Chern class of the hyperplane line bundle is negative in the tautological direction. Prove that the first Chern class of M is negative.

The differential geometric version of this was proved by the authors in [1]. This is the algebraic geometric version and perhaps can be looked at as the Finsler generation of the previous theorem of Wu-Yau.

[1] D. Wu and S.-T. Yau, *Negative holomorphic curvature and positive canonical bundle.* Invent. Math. **204** (2016), 595–604.