Life and Work of Alexander Grothendieck

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And every science, when we understand it not as an instrument of power and domination but as an adventure in knowledge pursued by our species across the ages, is nothing but this harmony, more or less vast, more or less rich from one epoch to another, which unfurls over the course of generations and centuries, by the delicate counterpoint of all the themes appearing in turn, as if summoned from the void.

Introduction

Alexander Grothendieck was one of the great mathematicians in the twentieth century. After his death in 2014 we look back at his life, and at his mathematical achievements.

In his relatively short active mathematical life, say from 1948 to 1970, he revolutionized several branches of mathematics. The mathematical landscape changed under his hands.

In this note we try to convey the foremost aspects of his endeavors: to consider concepts through their most essential properties, discarding all unnecessary assumptions. In his capable hands fundamental properties were revealed, with spectacular applications.

His personal life is full of beautiful but also sorrowful aspects. As he continuted to receive both praise and prizes for his mathematics, his personality was becoming progressively unstable. We are grateful for the rich theory he left us. It was painful to see his growing struggles with many aspects of life.

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In many aspects of the work of Grothendieck we see how previously understood theory, examples and proofs were generalized. This "quantum leap" from existing theory to deep and new insight we consider as one of the important aspects of the heritage of ideas of Grothendieck. Many ideas of Grothendieck's generalizations have been internalized by later generations, to the extent that it is difficult to imagine how to think now about mathematics without these natural concepts.

At several occasions Grothendieck asked the right question, and expected mathematics to be as beautiful as possible. If indeed the general structure would be of the kind he expected, he would find a way to describe it. When the situation was more complex, more complicated, and not as beautiful as expected at first sight, Grothendieck gave up at several occasions, where other mathematicians were finally able to sort out special cases and described in detail the situation (perhaps not as beautiful as previously expected).

Sources

Most material contained in the first section "A short biography of Alexander Grothendieck" of this note is taken from one of the following sources:

http://www.grothendieckcircle.org/ (where further material can be found); the three books by Winfried Scharlau [25], [27]; see also [30], [26], [29], [28].

The paper [21] (in two parts) gives an excellent survey; also see [16], [17], [15].

For some details about the mathematical work of Grothendieck not already contained in his published work, see [32], [10], [23], [31], [33].

Perhaps it is much better to read those sources instead of the material below in Section 1.

1. A Short Biography of Alexander Grothendieck

1.1

We will give a short survey of some of the most important bibliographical details about the life of Alexander Grothendieck. This material can be found in various sources. For details see http://www.grothendieckcircle.org/.

The subsections 1.1–1.10 are merely excerpts taken from various sources. The subsections 1.11 and 1.12 are our (partial) personal impressions about Alexander Grothendieck as a mathematician and as a person.

- Alexander Grothendieck was born in Berlin on 28 March 1928, and he died in Saint-Lizier (France) on 13 November 2014.
- His mother was Hanka Grothendieck (28 March 1900–16 December 1957). The name of his father probably was Alexander Schapiro (6 August 1890–1942).
- Childhood 1928–1945 (Berlin, Hamburg, Nîmes, internment Rieucros, Chambon sur Ligne).
- Studies in Mathematics in Montpellier, 1944–1948.
- PhD research 1948–1953 (Paris, Nancy) (functional analysis).
  The contribution to this field by Grothendieck is discussed in: J. Diestel, *Grothendieck and Banach space theory*; see [32], pp. 1–12.
- Brasil, Chicago, 1953–1955 (homological algebra), see [1].
  The genesis of K-theory by Grothendieck and his contribution in this field is described in M. Karoubi, *L'influence d'Alexandre Grothendieck en K-theorie*; see [32], pp. 13–23.
- In the period 1957–1970 his impressive work on a new approach to algebraic geometry appeared, initially with the aim of proving the Weil conjectures (Weil formulated these in 1949; Grothendieck told us about his plan in 1958, see [2]). During most of these years he was at the IHES, Bures-sur-Yvette, France. In 1966 at the International Congress of Mathematicians in Moscow, Grothendieck receives the Fields Medal. However, for political reasons, he refused to travel to Moscow.
- Grothendieck resigned his position at IHES in 1970, lived in Paris, Orsay, Montpellier (1973–1984), Villecun (1973–1979), worked at the CNRS (1984–1985), Les Aumettes (1980–1991); in 1991 he left and it was unclear where he was living; he moved to Lasserre (French Pyrenees) at some time.

See [25], [27], see http://www.scharlau-online.de/DOKS/cv_groth.pdf.

Grothendieck was born on 28 March 1928 as Alexander Raddatz: the husband of his mother Johanna Grothendieck was Alf Raddatz at that time. His biological father (Alexander Schapiro) declared in 1929 this boy was his child, [25, p. 77], and the name was changed into Alexander Grothendieck. Many interesting things can be said about his parents, see below.

People close to him called him Schurik. Though his first name was occasionally spelled "Alexandre", we will stick to "Alexander". When we write Grothendieck in this note, we are referring to Alexander Grothendieck."

1.2 Hanka Grothendieck

Johanna Grothendieck came from a Protestant family. Several stories and documents raise the question whether she was the right person to give guidance and love to her children.

Growing up in Hamburg, she was an intelligent person with many possibilities. She did not feel happy in that bourgeois surroundings already at a young age. After a marriage with Alf Raddatz and a difficult delivery the couple lived in gradually more and more sorrowful circumstances; they walked with their baby Maidi in a stroller from Hamburg to Berlin, where they had an unconventional life in poverty; see [25, §11, §12].

Her life in poverty, full of anarchistic ideas, does not seem a very happy one. All her life she opposed a bourgeois life, e.g. as her daughter Maidi said, see [25, page 79]: *dass ihre Mutter einerseits streng und willkürig war, anderseits die Kinder aber zu ganz unbürgerlichen Verhaltem angehalten hat: Sie sollen nicht grüßen und nicht die Hand geben.* (Her mother was strict and haphazard, on the other hand she demanded her children to conduct themselves in ways different from the bourgeois norm: they were not to greet, nor to shake hands with people.)

She lived in Berlin until 1933, and afterwards in France. She wrote a book, 1500 typed pages long, which has not (yet) appeared; see [28], [29]. The last years of her life were full of conflicts; see e.g. [25, §27].

We see a gifted person, with bad luck and a difficult character. She seemed very attached to the father of Alexander.

1.3 Alexander Schapiro

The life of the father of Alexander, 1890–1942, reads like an adventure story. His name was Alexander (Sascha) Schapiro, but in order to hide his Jewish roots he sometimes used the name "Sascha Tanaroff". (Like the famous author Isaac Babel, 1894–1940, who hid his Jewish roots in his Red Cavalry period by adopting the name Kirill Vasilievich Lyutov.) At the age of 14 he joined an anarchist group, where he was involved in an unsuccessful attempt to murder Czar Nicholas II in 1905. All other members of the group were executed; Sascha Schapiro was spared because of his age. He was send to prison, freed, captured again, escaped, lost an arm, and finally with the collapse of the Czarist regime in Russia in 1917 he was released from prison. In 1921 he fled using a forged name "Alexander Tanarov". After many breathtaking adventures he arrived in Berlin. Once Alexander Grothendieck told Cartier *...with pride and exaltation, that his father had been a political prisoner under* 16 *different regimes.* See [32, page 277].

Sascha was impressed by Hanka; their son Alexander was born in 1928. Hanka, Sascha, Maidi and Alexander lived as a family of sorts in Berlin until 1933. However as a revolutionary Sascha Schapiro thought he should not have children, [25, p. 88].

As Sascha was both Jewish and an anarchist, his life was in danger under the growing Nazi regime in Germany. (Sascha could "smell" this at an early stage; not many other people did.) Sascha left Berlin for France. He tried to make a living as a photographer, despite having only one arm. He travelled and went to Spain to join the Civil War fighters. At the beginning of World War II (1940–1945) Sascha was interned in the camp of Le Vernet, from where he was deported to Auschwitz in 1942, where he probably died in August 1942.

Hanka Grothendieck and her family lived a characteristically anarchistic life in Berlin under poverty. It is interesting to compare their stories with the description of Sebastian Haffner—*Geschichte eines Deutschen: Die Erinnerungen 1914–1933,* [47] (*Defying Hitler: A Memoir.*). Setting Haffner's description of Germany at that time and the sorrowful circumstances of Hanka Grothendieck and her family side by side, it is hard to imagine these happened in the same country at the same time. But we must remember how different their social circles were.

The "other" Schapiro. It was not easy to pinpoint the right identity of the father of Grothendieck. There was another anarchist with the same name, and the two Schapiro's were easily confused.

Cartier wrote: *As Alexander told me, his father's political career constitutes a "Who's Who of the European revolution from 1900 to 1940".* See p. 390 of [16].

Winfried Scharlau communicated to me:

> *If I remember correctly, Grothendieck wrote somewhere that his father is mentioned in John Reed's book "Ten days that shook the world". In this book only the "other" Schapiro is mentioned.*

Question. Did Grothendieck know there were (at least) two persons with the same name?

> *He learned this only after I had told him, 2003 or later. I am not sure that he ever really realized the confusion. But probably he knew very little, perhaps almost nothing about "other" Schapiro.*

Grothendieck was devoted to his parents. In difficult times he took care of his mother, although this was not easy (especially see §27 of [25]: "Hanka Grothendiecks letzte Jahre".) He had few memories of his father; it is said that Grothendieck had his head shaved in memory of him. A painting portraying Sascha in his last year was carried by Grothendieck through his many moves.

In [13] (somewhere 1983–1985) he wrote: *Ich brachte meinem Vater und meiner Mutter eine grenzenlose Bewunderung und Liebe entgegen. Ihre Personen waren für mich das Maß aller Dinge.* (For my father and my mother I had an unlimited admiration and love. For me they were the measure and extent of everything.)

1.4 Childhood 1928–1945

During the period 1928–1933 the family lived in Berlin.

In the Fall of 1933 dramatic things happened. Sascha Schapiro went to France.

As Hanka wanted to follow Sascha, she sent her daughter Maidi to a boarding school for handicapped children (although the girl wasn't handicapped at all). She negotiated with Wilhelm and Dagmar Heydorn in Hamburg-Blankenese as to whether they could take care of her 5-year-old child. She told all kinds of stories about herself and the father, and she offered 100 Marks per month. The Heydorns were taking care of several young children, and they were willing to accept also this 5-year old boy under the above conditions.

In December 1933 Hanka brought Alexander to the Heydorn family near Hamburg. Here is a summary of this heart-breaking story, as told by Dagmar Heydorn, [25, pp. 92/93]. Hanka showed up at the door with her young boy, and said right away that she was very poor and could not afford to pay anything, contrary to what the Heydorns had been led to believe. She admitted that she had lied in other parts of her story as well. She had only three conditions on her son's education:

1) do not talk to him about God;
2) do not send him to school, and instead let Wilhelm Heydorn do the teaching;
3) do not cut his hair.

In this confusing situation the Heydorns showed little Schurik what would be his room. When they came down again Hanka the mother had disappeared. The child never asked for his mother, as Dagmar Heydorn reported. But they did send him to school. He went to a Volksschule for primary education, and later to a Gymnasium. In later years Grothendieck kept contact with the Heydorn family, and we have the impression he had very good memories of that period in his life.

In 1939 difficulties arose. It was unclear to the Heydorn family whether they would be able to keep the children, and the fact that Alexanders father was a Jew might cause problems. They decided to send the child to his mother in France.

Hanka and Alexander were interned as "undesirables" in the Rieucros Camp near Mende. From there he went to school in the village four or five kilometers away. After the Rieucros camp was dissolved, the inmates were transferred to Gurs. Alexander was sent to the village Le Chambon sur Lignon, and attended the famous Collège Cévénol. In 1945 his rather chaotic school-career ended by his successful baccalauréat examination.

The following is a short description of Alexandre Grothendieck, written by the woman who ran La Guespy, apparently written down shortly after the war. M. Steckler was the surveyor at La Guespy (of whom the notes say: "Je me le rappelle jouant férocement aux échecs avec Alex.")

> *Alexandre Grothendieck, dit Alex le Poète, allemand, russe? Mère au camp de Gurs. Enfant très intelligent, toujours plongé dans ses réflexions, ses lectures, écrivant, très bon joueur d'échecs—parties acharnées avec M. Steckler. Réclame le silence pour écouter la musique. Sinon enfant tapageur, nerveux, brusque.*
>
> (Alexandre Grothendieck, called Alex the poet, German, Russian? Mother at the camp at Gurs. Very intelligent child, always plunged in his thoughts, his books and what he was writing. Very good chess player—chess matches set against Mr. Steckler. Demands silence for listening to music. Otherwise a noisy, nervous and brusk child.)

http://www.grothendieckcircle.org/.

1.5 Studies in Mathematics. PhD Research

Copied from http://www.grothendieckcircle.org/.

1945–1948: Grothendieck and his mother moved to a small village near Montpellier; he worked irregularly on a farm while studying mathematics at the university of Montpellier.

1948/49: Having obtained his degree, Grothendieck went to Paris with the goal of obtaining a doctorate. He took several courses and met most of the famous mathematicians of the time, above all H. Cartan.

1949–1953: Cartan advised Grothendieck to do his doctorate in Nancy with Laurent Schwartz. He went there, finished his doctorate within two years, and created a new theory of nuclear spaces (topological vector spaces with many of the good properties of finite-dimensional vector spaces). See [19] for a description of Grothendieck's contribution in this area.

1.6 Homological Algebra

1953/1954: Grothendieck spent time in São Paulo in Brazil, teaching topological vector spaces. At the end of 1954, he was ready to "leave the field of topological vector spaces with no regrets and start seriously working on algebraic topology".

1955–1957: Grothendieck spent part of the year 1955 at the University of Kansas in Lawrence, Kansas and then visited Chicago. He had hoped to find a position in France, but difficulties arose because of his foreign nationality.

During this period of times his interests shifted from topological vector spaces to algebraic topology and algebraic geometry. Rapid progress was being made in these areas. It was only natural that Grothendieck began with homological algebra, and he wanted to learn it well. However the influential book [41] by Cartan and Eilenberg was not yet available. (It appeared in 1958). Instead of reading preliminary versions of [41] that already circulated at the time, Grothendieck decided to develop the theory from scratch. The result is the memoir [1], a grand synthesis which incorporate homological algebra for modules (over rings) and sheaf theory under a common framework. Ideas in this paper have had a large impact on the development of modern "abstract" mathematics. We see again and again in later work by Grothendieck the hallmark ability to create such abstract structures that captured the essence of the problem, from which more progress followed naturally.

In December 1955 Grothendieck discovered the formula $\mathrm{H}^p(X,\mathscr{F})' = \mathrm{Ext}^{n-p}_{\mathscr{O}}(X,\mathscr{F},\Omega^n)$ for a coherent sheaf $\mathscr{F}$ on an n-dimensional projective smooth variety X; see [10, pp. 19–20]. The left-hand-side of the equality, $\mathrm{H}^p(X,\mathscr{F})'$, denotes the dual vector space of $\mathrm{H}^p(X,\mathscr{F})$. This formula is a special case of the Grothendieck duality theorem; an early version appeared in [5], exp. 149, 1957.

1.7 Riemann–Roch

The first salvo after the memoir [1] came in the form of the Grothendieck–Riemann–Roch theorem, announced in the opening sentence

> "*Ci-joint une démonstration très simple de Riemann–Roch, indépendante de la caractérisque.*" (You will find enclosed a very simple proof of the Riemann–Roch independent of the characteristic.)

of the letter [10, p. 57] dated November 1st, 1957. The first published proof [39] (with an appendix [7] by Grothendieck) is the published version of the notes of a seminar at the IAS Princeton in the fall of 1957, on the paper *Classes de faisceaux et théorème de Riemann–Roch* enclosed in the 11.1.1957 letter. The latter was eventually published in SGA6, pp. 20–77. We refer to [22] for an account of Grothendieck's influence on K-theory.

1.8 A New Approach to Algebraic Geometry

...this seemed like black magic. Mumford, [32], page 78.

In 1949 André Weil formulated his conjectures about the *Riemann hypothesis in characteristic* p (we will refer to this by pRH, in order to avoid confusion with the classical RH). This was initiated earlier and partial results were achieved by Emil Artin, F. K. Schmidt and H. Hasse in the period 1924–1936. Weil proved the pRH for curves and for abelian varieties over finite fields. He had the incredible insight that an analogue of the Lefschetz fixed point theorem could finish the proof in the general case of arbitrary varieties over finite fields. The "only problem" was to define an appropriate cohomology theory and prove this analogue of the Lefschetz fixed point theorem in that context. For a historical survey of this material see [60], [54]

This challenge was exactly suited to Grothendieck's talent; he was a master of developing general structures that naturally resolved concrete problems. At the Edinburgh 1958 ICM Grothendieck gave a talk; only one person in the audience seemed to understand what was going on—that was before I (FO) knew that was Serre, and it was the beginning of a new era in algebraic geometry.

Indeed this cohomology was constructed, and almost all aspects of the Weil conjectures were proved by Grothendieck and co-workers. Only one aspect remained unproved: the fact that the eigenvalues of Frobenius had the correct (complex) absolute values. This was proved by Deligne, for which he received the Fields medal in 1978. Grothendieck was not very satisfied; he thought that the "standard conjectures" (1969) should be proved, and the Weil conjectures would follow without much effort. Below is a comment on page 214 of [32], about a passage on pages 125–126 of [10]: *Grothendieck could not prevent himself, later, from expressing bitter disapproval of Deligne's method for finishing the proof of the Weil conjectures, which did not follow his own grander and more difficult plan.* However these more general conjectures seem out of reach for the time being, even now after more than 50 years.

It might be that Grothendieck started considering this question when he began to study algebraic geometry around 1956. In 1958 Grothendieck told us he was starting to put *algebraic geometry* on a new footing. Just imagine a respectable part of "old mathematics", where we thought we had most ideas and methods at our disposal. Then a completely new setup is constructed, which was not immediately accepted by everyone. But now we know this is "the best" way of looking at these objects. In Section 3 we portray some of these ideas. We hope you are convinced this really is a revolution. The parents of our hero wanted a revolution that would change the world. Their son Alexander made the necessary blueprint of a revolution in a superb way, which reshaped people's perspective in (some part of) mathematics. In this note we hope to show some aspects of this new way of looking at mathematics.

In the period 1957–1970 Grothendieck, assisted by many colleagues and students, wrote thousands of pages, mostly on general theory. In 1966 at the International Congress of Mathematicians in Moscow, Grothendieck received the Fields Medal. However, for political reasons, he refused to travel to Moscow.

1.9 Political and Ecological Activities

He seemed to believe that social issues can be settled with the same kind of proofs as mathematical ones, and in fact often ended up actually irritating people even when they were aware of this his importance as a mathematician and perfectly receptive to the ideas he was expressing. See [32, pp. 283/284].

For Grothendieck's ecological activities see www.grothendieckcircle.org.

Adhering to his character in a clear and very active way, he was upset about violence and the use of weapons. In November 1967 he travelled to Hanoi, Vietnam, and in several talks he described the horror of the Vietnam war.

He started a "movement" called "Survivre et vivre" (Struggle for the survival of human mankind).

- *survivre*: survive and oppose the effects of our industrialized life, including pollution and the destruction of our environment and natural resources;
- *vivre*: live and abolish the contradiction between scientific research and the indiscriminate use of science and technology, especially the proliferation of military equipment by the arms industry.

The momentum was there. Many people were convinced something like this had to be done. I (FO) remember a meeting at the ICM in Nice, 1970, the year "Survivre et vivre" was founded. The audience in a packed lecture hall was eagerly awaiting to hear from Grothendieck what one should expect and what was to be done. The meeting ended in chaos. Grothendieck asked (if I remember well) whether anybody had any ideas. Some people in the first rows starting to shout, to discuss, and we did not get anywhere. While Grothendieck had great ideas about mathematics it seems that on this topic he could not make his concern and his emotions concrete in well directed and fruitful actions.

1.10 After 1970

> *The last time I saw Grothendieck in person was I believe in Kingston, Ontario, in 1971. At that time, when asked to give a mathematical talk, he would request equal time to speak about his peace work and his organization Survivre. He was drifting away from mathematics and applying his brilliant mind to problems of humanity. No matter how logically persuasive, I thought his efforts in that direction were politically naïve.* R. Hartshorne, [32, page 173]

The fruitful period of active mathematics ended in the case of Grothendieck in 1970. We can speculate why such drastic actions were taken. Grothendieck resigned from the IHES, because he found out that (a very small portion of) the financial means of the institute came form military sources. However we can speculate there were other reasons.

- His personality became more and more unstable.
- His desire to fight social mechanisms in the style of "Survivre" did not produce effective outcomes.

We might also think that the burden he had taken of revolutionizing algebraic geometry was heavy: in 1960 he thought he would finish his 13 chapters of *Éléments de géométrie algébrique* in a few years; on August 18, 1959, he wrote to Serre about his schedule for finishing EGA in the next four years, see [10, page 83]. (The plan for the thirteen chapters can be found on page 6 of EGA I.) In 1966 Grothendieck was more realistic, writing to Mumford that probably chapters V to VIII would take another eight more years,

> *...and by then we will have a clearer picture of what would be most useful to do next—and maybe to decide whether we should push the treatise further at all.*

See his letter to Mumford on 4 November 1966, [23], pp. 719–720. In 1970 only 4 chapters (in 8 volumes) had appeared ("only"? What a rich source of beautiful theory!) Material for the remaining chapters appeared in (more or less) preliminary form in the SGA seminars [4], and also in the collection [5] of Bourbaki seminars. In [11], on page 51, he wrote that finally he could start doing research ("m'élacer dans l'inconnu").

We know from the above letter to Mumford that in November 1966 Grothendieck was still considering going on and perhaps finishing EGA. The fact that a small part of the funding for the IHES came from military sources is often mentioned as the reason why Grothendieck resigned from his position (and severed later ties with the mathematical community). In January 1970 Grothendieck wrote that he stopped working at the IHES, [32, page 745]; however a few day later he communicated also to Mumford that *things got arranged, as I was backed by my colleagues from IHES for demanding no military funds should be used for the budget ... Thus I have taken up my job at IHES again, ...* [32, page 747].

After Grothendieck left the scene, we had to move on; we could use the beautiful ideas by Grothendieck, but his grand plan for a comprehensive foundation of algebraic geometry remain unfinished.

Our hero during the period 1970–2014 was a disturbed person, still with great ideas, but also with haunted patterns in his thoughts. He moved several times. Quite often we had no idea where he was. He wrote long texts, part of which contain profound mathematical ideas (not all of them yet understood). Sour memories, and phantasies which are difficult to follow, are also found in these texts.

Here is a list of texts he wrote during that period, copied from [26]:

1979: *L'Eloge de l'Inceste* (In Praise of Incest) (January to July 1979, perhaps lost).

1981: *La Longue Marche à Travers la Théorie de Galois* (The Long March through Galois Theory) (January to June 1981, about 1,600 pages, plus about the same amount of commentary and supplementary material; unpublished, but since 2004 parts have been available on the Internet).

1983: *À la Poursuite des Champs* (Pursuing Stacks) (approximately 650 pages, started as a "letter" to D. Quillen, unpublished). Associated with this is an extensive correspondence with Ronnie Brown and Tim Porter.

1984: *Esquisse d'un Programme* (Sketch of Program) (January 1984). This is still a rich source of ideas. Grothendieck wrote this as an application for a position. Published in [62, pp. 5–18].

1983–1985: *Récoltes et Semailles: Réflexions et Témoignage sur un Passé de Mathématicien* (Reapings and Sowings: Reflections and Testimony on the Past of a Mathematician) (1252 pages plus approximately 200 pages of introduction, commentary, and summaries; produced as photocopies; available on the Internet).

1987: *La Clef des Songes* (The Key to Dreams) (315 pages, unpublished).

1987–88: *Notes pour la Clef des Songes* (Notes on the Key to Dreams) (691 pages, unpublished); includes a freestanding work, *Les Mutants*.

1990: *Développements sur la Lettre de la Bonne Nouvelle* (Developments on the Letter of Good News)

(82+2 pages, unpublished; written February 18–March 15, 1990).

1990: *Les Dérivateurs* (about 2,000 pages, unpublished, but parts available on the Internet).

Just an example: one of the lines in "The Key to dreams" reads: *Der Einzige Gott schweigt. Und wenn Er spricht, dann mit so tiefer Stimme, dass ihn niemand jemals versteht.* (The only God is silent. And when he speaks it is with such a low voice, that no one can understand him.)

1.11 Some Characteristic Aspects of the Work of Grothendieck

... mon attention systématiquement était ...
dirigée vers les objets de généralité maximale ...
Grothendieck on page 3 of [11]; see [62, page 8]

In all of his work in mathematics Grothendieck was original and fundamental. Already in his PhD thesis he made a fundamental contribution and gave new insights to a field, and in such a way that had eluded other mathematicians

Going to a new field he rewrote the whole theory when others were just starting to understand these structures.

Then around 1956–1958 Grothendieck began to work in algebraic geometry, a field with a rich history, and with already many existing theory. He told us that this was because he was planning to solve the Weil conjectures. However, do not interpret this as "problem solving". On many occasions Grothendieck told us that finding the structure involved was the only essential thing to do and then the solution would come out by itself: "immerse a large nut in a softening fluid, and the nut opens just by itself". We have seen many instances where going to the very roots and pure thought gave insight and solved difficult problems.

For Grothendieck mathematics seemed elegant and results should come just by pure thought. Hence his amazement: *I found it kind of astonishing that you should be obliged to dive deep and so far in order to prove a theorem whose statement looks so simple-minded.* (In his letter to Mumford, [23, page 717].)

When an idea showed up Grothendieck could develop a whole structure, much more than needed for the application at hand. Other mathematicians trying to solve a problem could be satisfied seeing the general pattern, using it and moving on to another topic. Grothendieck would try to describe the idea in the most general situation, applications already out of sight; quite often it turned out that such a general theory had many unexpected applications to other problems.

A new concept of "coverings in some topology" was needed; the new idea in the guise of the "etale topology" needed for direct applications can be written down in just a few pages. It took Grothendieck several hundred pages (three volumes in Lect. Notes Math. **269, 270, 305**) to describe in SGA4 the much more general notion of a "Grothendieck topology", an impressive and useful theory. We discuss Grothendieck topologies in Section 5 and étale cohomology in Section 6.

However, sometimes in mathematics a solution does not "come by itself". The comment *... obtaining even good results "the wrong way"—using clever tricks to get around deep theoretical obstacles—could infuriate Grothendieck* in [31, p. 64] describes Grothendieck's stance. There are examples where a direct approach in the hands of Grothendieck did not work out and he left the question, while others did proceeded with a combination of theory (often developed by Grothendieck) and direct verification of special cases, making non-obvious choices or other "non-canonical" choices and eventually succeeded. For some of these examples see [24, §8].

Grothendieck had the good fortune to have Serre and Mumford at his side. These colleagues often answered questions and gave examples needed for Grothendieck to see whether a certain theoretical approach would be reasonable. In [10] and [23] we see many examples along these lines.

There were some instances Grothendieck had no interest in work done by others following his ideas, or supplementing possible approaches. Possibly this was part of his character. Also it gave Grothendieck the chance to follow his own deep thoughts without losing time on other approaches.

1.12 Some Aspects of the Person Alexander Grothendieck

Sometimes aspects of a childhood are used to "explain" a character; we will refrain from this.

Grothendieck could feel and work in an intense way. For instance, consider his description, as an adult, of his childhood feelings for his parents: *For my father and my mother I had an unlimited admiration and love.*

In his active years in mathematics he worked for many hours a day. He wrote hundreds of pages of flawless mathematics.

Later he wrote long manuscripts, progressively mixing mathematics with religious feelings, philosophical ideas and strange phantasies more frequently. We wonder how one person could produce this many pages.

After 1970 until the end of his life he made several drastic decisions: he stopped trusting other people, and he destroyed manuscript and letters (including those of his parents). A picture of a tormented, disturbed individual emerges.

Some students and colleagues have nice memories to Grothendieck. Difficult relations, and a sour and a hateful opinion of Grothendieck of some others makes us sad. His feelings and actions were absolute and forceful.

What he lacked in parental love in his childhood, he could not give to his own children.

2. Tohoku

For my own sake, I have made a systematic (as yet unfinished) review of my ideas of homological algebra. I find it agreeable to stick all sorts of things, which are not much fun when taken individually, together under the heading of derived functors. [10, p. 6], [32, p. 197]

2.1

From the paragraph containing the above passage on page 6 of [10] written in Lawrence, Kansas on February 18, 1955, we know that Grothendieck had already made substantial progress on his synthesis of homological algebra. He wanted to teach a course on homological algebra based on the not-yet-published book [41] by Cartan and Eilenberg, which was destined to be enormously influential, but he couldn't get hold of a copy of the manuscript. His solution was to work out by himself everything he presumed would be in [41]. The result was the memoir [1], often sited as *Tohoku*. A preliminary version was sent to Serre on June 4, 1955 for the coming Bourbaki Congress, where it was read carefully and "converted everyone"; cf. [10, 16–18]. The final version, which Grothendieck called his *multiplodoque d'algèbre homologique*, was written in the months of September–October 1956 and was accepted by Tannaka in November 1956 for the Tohoku Journal.[1]

The Cartan–Eilenberg book [41] itself is a great work of synthesis of the developments of algebraic methods in topology and their applications during the period 1935–55. The book [41] is mostly about the theory of derived functors between categories of modules over rings. However theory of sheaves, conceived by Leray and developed in the late 1940's (e.g. Séminaire Henri Cartan 1950/51) with spectacular applications [63], [64], [65], [66] in the hands of Serre, does not fit in [41].

[1] See p. 43 and p. 49 of [10]; the following tidbits are also found in [10, pp. 43–49]. Grothendieck asked Serre for his opinions about his *multiplodoque* and asked for suggestions where to publish it. He didn't want to comply with the editorial requests for the *Transactions* from Eilenberg, which he called "tabous de rédaction tres sévères de Sammy". Serre was unsympathetic, telling Grothendieck that his objections were "idiotes", and that all Eilenberg demanded was that a manuscript should be "*lisible* sans effort d'intelligence", which Serre considered the least one could ask ("c'et bien le moins").

Grothendieck began studying homological algebra in 1954 when he was in São Paulo. In less than a year, he succeeded in formulating a notion of *abelian categories*, and produced a general theory of derived functors on abelian categories, with cohomology of sheaves and the homological algebra for modules as special cases; see [10, pp. 13–14]. Serre brought an early draft of [1] to a Bourbaki meeting in the summer of 1955; later he wrote to Grothendieck that "your paper on homological algebra was read carefully and converted everyone (even Dieudonné, who seems to be completely functorized!) to your point of view", [10, p. 17].

2.2

The idea of doing general homological algebra in a category with properties similar to that of the category of all modules over a ring was known to Cartan and Eilenberg; c.f. [10, p. 15], [40], [48]. Grothendieck defined an *abelian category* to be an additive category satisfying the following two axioms:

(AB 1) Every morphism has a kernel and a cokernel

(AB 2) For every morphism $u : A \to B$, the morphism $\mathrm{Coim}(u) \to \mathrm{Im}(u)$ induced by u is an isomorphism, where $\mathrm{Coim}(u) := \mathrm{Coker}(\mathrm{Ker}(u) \to A)$ and $\mathrm{Im}(u) = \mathrm{Ker}(B \to \mathrm{Coker}(u))$. Grothendieck formulated an important property that an abelian category may satisfy:

(AB 5) Filtered limits of objects exist, and filtered limits of exact sequences are exact.

The standard notion of injective and projective modules generalize immediately to abelian categories. Just as in the case of modules, if an abelian category $\mathcal{C}$ *has enough injectives* in the sense that every object of $\mathcal{C}$ can be embedded in an injective object, then one can use injective resolutions in $\mathscr{C}$ to define *right derived functors* $\mathrm{R}^i F$, $i \geq 0$, of a left exact functor $F : \mathcal{C} \to \mathcal{C}'$ between abelian categories.

Theorem 1.10.1 of [1] says that if an abelian category $\mathcal{C}$ satisies (AB 5), and there exists an object U such that every object in $\mathcal{C}$ is a quotient of $U^{\oplus I}$ for some index set I, then $\mathcal{C}$ has enough injectives. It follows that the category of sheaves of abelian groups on a given topological space X has enough injectives. So one can define derived functors of the global section functor $\mathscr{F} \mapsto \Gamma(X, \mathscr{F})$ for sheaves $\mathscr{F}$ of abelian groups on X; they are the cohomology groups $\mathrm{H}^i(X, \mathscr{F})$ of $\mathscr{F}$.

Theorem 2.4.1 of [1] provides a generalization of the Leray spectral sequence, which include many spectral sequences used in algebraic topology and algebraic geometry: Suppose $F : \mathscr{C} \to \mathscr{C}'$ and $G : \mathscr{C}' \to \mathscr{C}''$ are two additive functors between abelian categories. Assume that $\mathscr{C}, \mathscr{C}'$ have enough injectives. Assume moreover that G is left exact and $\mathrm{R}^i G(F(I)) = 0$ for all

$i > 0$ for all injective objects I in $\mathscr{C}$, then there is a functorial first quadrant E_2 spectral sequence

$$E_2^{i,j}(A) = \mathrm{R}^i(\mathrm{R}^j(F(A)) \implies \mathrm{R}^{i+j}(G \circ F)(A) \quad i,j \in \mathbb{N}$$

for all objects A in $\mathscr{C}$.

2.3 Examples

(a) A sheaf $\mathscr{S}$ on a topological space X is said to be *flabby* if every section of $\mathscr{S}$ over an open subset of X extends to a global section of $\mathscr{S}$ over X. One can verify directly that if $\mathscr{S}$ is flabby, and

$$0 \to \mathscr{S} \to \mathscr{F}_1 \to \mathscr{F}_2 \to 0$$

is a short exact sequence of sheaves of abelian groups, then the map $\Gamma(X, \mathscr{F}_1) \to \mathscr{F}_2$ is surjective. From his lemma plus the fact that every injective sheaf is flabby, it is not difficult to show that $\mathrm{H}^i(X, \mathscr{S}) = 0$ for all $i > 0$.

(b) ([1, p. 160]) $\mathrm{H}^i(X, \mathcal{A}) = 0$ for all $i > 0$ if $\mathcal{A}$ is the constant sheaf attached to an abelian group A and X is an *irreducible* topological space X in the sense that the intersection of any two non-empty open subsets is non-empty. Note that the Zariski topology of any variety in the sense of 3.5 is irreducible. So constant sheaves on a variety have only trivial cohomologies. This is what one expects, because the Čech complex attached to every finite (or locally finite) open covering of an irreducible topological space is contractible.

(c) ([1, 3.4.1]) The cohomology groups $\mathrm{H}^i(X, \mathcal{O}_X^\times)$ for the Zariski topology of a variety X over a field with coefficients in the group of units of the structure sheaf $\mathcal{O}_X$ vanish for all $i \geq 2$.

(d) ([1, 3.6.5]) If X is a Noetherian topological space of combinatorial dimension at most n, then for any sheaves of abelian groups $\mathcal{F}$ on X, we have $\mathrm{H}^i(X, \mathcal{F}) = 0$ for all integers $i \geq n+1$. This theorem was proved in January 1956; see [10, p. 26]

We recall that a topological space is *Noetherian* if every decreasing sequence of closed subsets is stationary. A Noetherian topological space has combinatorial dimension (or Krull dimension) at most n if every strictly decreasing chain of *irreducible* closed subsets has at most $n+1$ elements. Varieties over fields with Zariski topology (see 3.5) are typical examples of Noetherian topological spaces; the combinatorial dimension of a variety V over a field K is equal to the dimension of V, i.e. the transcendence degree of the function field $K(V)$ over K.

2.3.1 Remark

(i) Three years after the discoveries in [1], Grothendieck applied the formalism in [1] to algebraic geometry as he promised on the second page of [1], proving a finiteness theorem for cohomologies of coherent sheaves relative to a *proper* morphism of schemes, including a cohomological proof of *Zariski's Main Theorem*; see his 1958 letter to Zariski, [23, pp. 633–634]. The statement (d) above played a crucial role in Grothendieck's proof through *downward* induction.

(ii) The case $i = 2$ of the statement (c) above implies that any Zariski-locally-trivial algebraic fibration of a variety whose fibers are projective spaces $\mathbb{P}^n$ is in fact trivial: twisted forms of $\mathbb{P}^n$ which appear "in nature" are not locally trivial for the Zariski topology. This observation may serve as part of the motivation for a "finer topology" on algebraic varieties. It turns out that the for the étale topology, to be discussed in §5.1, there are many locally trivial $\mathbb{P}^n$ fibrations.

In contrast because of Hilbert's theorem 90, the Zariski topology is quite adequate for studying principle homogeneous spaces for GL_n, or equivalently vector bundles, over varieties.

3. Varieties and Schemes

I want to talk about how Grothendieck's revolution profoundly affected my own understanding of algebraic geometry.
David Mumford, [32, page 75]

Some references: see [71], [70] for the theory of varieties, [65] for the theory of coherent sheaves on varieties over an algebraically closed field. The multivolume EGA is a comprehensive treatment of the theory of schemes; [46] is a standard textbook of manageable size.

3.1

Algebraic geometry describes geometric object with the help of algebraic equations (polynomials). In almost every part of mathematics these methods are applied, but especially in number theory and in complex geometry. In this section we give a survey of some notions developed in the last two centuries, and we indicate where Grothendieck fits into this picture (but—alas—this is not a complete history of algebraic geometry).

- In the 19-th century these topics were first developed. We mention Hurwitz and Klein, but especially Riemann who showed us (in the case of Riemann surfaces) that we can define geometric structure by local charts, where the transition functions describe the context in which we are working (complex analytic, topological, algebraic, etc.).
- In the beginning of the 20-th century Italian geometers proved a wealth of geometric theorems; among them we see Luigi Cremona (1830–1903), Guido Castelnuovo (1865–1952),

Federigo Enriques (1871–1946), Francesco Severi (1879–1961) and Beniamino Segre (1903–1977). Later their beautiful geometric approaches were shown in some cases to lack algebraic rigor.

- Algebraic foundations were laid in the middle of the 20-th century (Bartel L. van der Waerden, Oscar Zariski, André Weil, and many others).
- In 1955 Serre took up methods of sheaf theory in order to unify these concepts.
- From 1958 Grothendieck revolutionized algebraic geometry. He showed us a unified way to describe these concepts.

In this section we will describe the notion of an algebraic variety as was done in Weil's Foundations [71]. We will indicate the introduction of sheaf theory as in [65] by Serre, and describe the notion of schemes as introduced by Grothendieck. We show in which way this generalizes the concept of algebraic varieties, and how it opens up new applications.

A historical remark. It seems that the terminology "scheme" was introduced by Chevalley; see [32, page 275, footnote 10]. It took Grothendieck three years to arrive at the insight that the one should drop both the Noetherian and Jacobson condition on commutative rings and include all prime ideals in the affine spectrum. According to Tate [33, p. 43]: *Schemes were already in the air, though always with restrictions on the rings involved. In February 1955 Serre mentions that the theory of coherent sheaves works on the spectrum of commutative rings in which every prime ideal is an intersection of maximal ideals. A year later, Grothendieck tells of Cartier's introducing the technically useful notion of quasicoherent sheaf on arithmetic varieties made by gluing together the spectra of noetherian rings. But it would take two more years before Grothendieck realized that the noetherian condition should be dropped and one should include all prime ideals in the spectrum, that in the end, as Serre puts it in the notes, the best category of commutative rings is the category of all commutative rings.*

3.2 Affine Varieties

We consider a field K and $K \subset k$, where k is an algebraically closed field. We write $K[\underline{T}] = K[T_1, \cdots, T_n]$ for some $n \in \mathbb{Z}_{>0}$. For an ideal $I \subset K[\underline{T}]$ we write $\mathscr{Z}(I) = V$ for the "set of zeros of I" defined by: for every field extension $K \subset L$

$$\mathscr{Z}(I)(L) := \{t = (t_1, \cdots, t_n) \in L^n \mid g \in I \Rightarrow g(t) = 0\}.$$

One can "visualize" this as "the set" $V(k) \subset k^n$. However we should be careful in identifying an algebraic variety with the underlying set of points (in a given field).

In case $I \subset K[\underline{T}]$ has the property that $I \cdot k[\underline{T}] \subset k[\underline{T}]$ is a prime ideal, we say that $\mathscr{Z}(I) = V$ is an *affine algebraic variety*.

3.3 Abstract Varieties

In this section we discuss *affine varieties* and *affine schemes*. In general we need more general notions such as abstract varieties, varieties locally given by an affine variety, with algebraic transition functions, and notions such as (quasi)-projective varieties. Certainly for algebraic geometry these more general notions are of vital importance. However for a first understanding and for comparison with schemes it suffices to describe the affine variants.

Below we discuss affine schemes (and omit descriptions of arbitrary schemes).

3.4 Complex Varieties

Consider the case $K = k = \mathbb{C}$. For a variety V the set $V(\mathbb{C})$ has a natural topology called the *complex topology*, given by the norm on $\mathbb{C}$. This seems a natural choice. For many years we worked with *complex varieties* defined this way. For example the book [45] is completely devoted to this transcendental approach, with many impressive results.

Clearly there are difficulties in applying such transcendental methods to number theory. Analytic functions and rational numbers are objects that live in different worlds. For such applications we need other techniques.

A lot of work was done, and this topic is now well understood: compare properties of an algebraic variety V over $\mathbb{C}$ with analytic and topological properties of $V(\mathbb{C})$; which analytic varieties are algebraizable? Earlier work by Lefschetz and Chow addressing the question of algebraicity were satisfactorily completed in [66]. It inspired Grothendieck to one of his influential theorems, the Lefschetz–Chow–Grothendieck "existence theorem" (theorem of GAGA type); for a description and references see Chapter 8 by L. Illusie in [6].

3.5 The Zariski Topology

Zariski constructed a topology on an arbitrary algebraic variety. As in Section 3.2 a closed set in V is defined as the set of all zeros of the system of polynomial equations given by an ideal in $K[\underline{T}]/I$.

Example. Consider $V = \mathbb{A}^1_K$, given by $I = (0) \subset K[T]$, the "affine line over K". We see that the empty set and the whole of V are closed; all closed sets consist of the zeros of some polynomial; hence such a closed set is a finite set in $V(k)$ or equal to $V(k)$. Is this interesting?

E.g. for $K = \mathbb{C}$ this topology is very different from the complex topology; the Zariski topology at first sight seems much too coarse to be interesting. However experience has shown us that this opens the door to deep notions in algebraic geometry.

3.6 Varieties and Sheaves

Let V be an affine variety over an algebraically closed field $K = k$ as in Section 3.2 given by $I \subset K[T_1, \cdots, T_n]$ with $V = \mathscr{Z}(I)$; over an algebraically closed ground field we can "identify" V and $V(k)$ as we will do here. Let us suppose that I is a prime ideal. We write

$$A = A_V = K[T_1, \cdots, T_n]/I,$$

called the *coordinate ring of* V. Elements of A_V can be interpreted as functions on V with values in k. Moreover, in this special case, $f, g \in A_V$ are equal if and only if these associated functions are equal. For a point $t \in V$ we write $P_t \subset A_V$ for the set

$$P_t = \{f \in A_V \mid f(t) = 0\}.$$

We see this is a prime ideal. We write $\mathscr{O}_{V,t}$ for the localization of A_V with respect to P_t:

$$\mathscr{O}_{V,t} = (A \setminus P_t)^{-1}A = \left\{\frac{g}{h} \mid g, h \in A, \quad h \notin P_t\right\}.$$

This is a local ring; every element of this ring is regular (has no poles) in a Zariski open neighborhood of t in V. The union

$$\mathscr{O}_V = \bigcup_{t \in V} \mathscr{O}_{V,t}$$

is what is now called "the sheaf of germs of regular functions on V", as was done in [65]. The importance of this concept is that local and global properties of V are combined in one description. For this definition and much more concerning sheaves we refer to the influential paper [65]; Serre received the Fields medal in 1954 for his "investigations in the homotopy theory and the theory of sheaves". From now on we use sheaves without further explanation.

3.7

At this moment (of reading this paper, or of considerations in the history of algebraic geometry) just contemplate what can be done after laying foundations and introducing the concepts described above. The "classical algebraic geometry" serves well for the purpose of describing algebraic varieties over algebraically closed fields. However there are drawbacks, as one soon discovers. We will describe some of them.

3.7.1 An Intersection of Two Varieties Need Not Be a Variety

Examples are easy to give. For example intersect a plane conic C with a line Z. If Z is not tangent to C we obtain two points (either defined over the base field or "conjugate"). However if Z is a tangent line we feel the interaction as a "point with multiplicity two"; how can we describe this in the theory of varieties?

3.7.2 Different Ideals Can Describe the Same Zero-Set

Consider the ideals

$$(X,Y), \quad (X,Y^2) \quad (X^3,Y) \subset K[X,Y].$$

Cleary they all define the set $\{(0,0)\}$ as zero set, but we like to feel these three sets as different: the first is just a point, the second is a "double point" in the vertical direction, and the last one is a "triple point" in the horizontal direction. We like to distinguish these (and many more variations on the same theme), both as different objects and as different subobjects of $\mathbb{A}^2$.

3.7.3 Over a Non-Algebraically Closed Field We Are Sometimes in Trouble

It may happen that V is defined over K, but $V(K)$ is empty, or perhaps non-empty but not Zariski dense in V. How do we describe "points on V"?

What happens if I is a prime ideal in $K[T_1, \cdots, T_n]$ but $I \cdot k[T_1, \cdots, T_n]$ is not a prime ideal? What do we study as "the set of zeros of I"?

3.7.4

Consider a non-perfect field K of characteristic p; let $a \in K$ such that $\sqrt[p]{a} \notin K$. Consider

$$I = (T^p - a) \subset K[T].$$

In this case $\mathscr{Z}(I)$ is not "a variety over K" in the terminology of Section 3.2: although $I \subset K[T]$ is a prime ideal, $I \cdot k[T]$ is not a prime ideal. How should we handle "$\mathscr{Z}(I)$" in this case?

3.7.5 Families of Algebraic Varieties Are Difficult to Handle in the Theory of Varieties; Algebraic Geometry in Mixed Characteristic Is Difficult to Describe Well

Many attempts were made to describe such methods and questions, quite natural in algebraic geometry. Let us give one example, a question investigated in the influential paper [57] by Néron. Suppose we are given an abelian variety A (e.g. an elliptic curve) either over the generic point of a non-singular algebraic curve Γ, or over the field of fractions of a discrete valuation ring R (two aspects of the same kind of situations). Is there a "best way" to extend this abelian variety A to a family (of some sort) over Γ, or an object with R as "ring of constants"? Once this is done we can then study the fiber over any closed point in Γ, or we can study the situation after the reduction of constants $R \to \kappa$ over the residue class field; this is a way to find "the best degeneration" of A. What would be a satisfactory theory to describe such situations and to decide which degenerations are possible? In [57] we find a technical, valuable theory that has had an impact on algebraic geometry; however these techniques were difficult to understand and to use.

For example consider an elliptic curve E over Q, and let $R = (\mathbb{Z} \setminus (p))^{-1}\mathbb{Z}$ be the ring of fractions with denominator not divisible by a given prime number p. What is the "best way" of extending E to the ring of constants, and deriving the "reduction mod p" of E? A rich theory, that of *Weierstrass minimal models*, gives some answer; but then you discover drawbacks of this, and e.g. you see that the *Néron minimal model* in general is not described by the Weierstrass minimal model; you see the technical difficulties. How do you describe such phenomena in the old theory of varieties?

Questions and examples of this kind were well-known and studied by sometimes rather ad-hoc methods. These ideas were put on a unified footing once Grothendieck developed the theory of schemes.

3.8 Affine Schemes

One aspect of the way Grothendieck approached mathematics is the following: work in a situation as general as possible, delete all assumptions not strictly necessary. Thinking about "algebraic varieties", why would we work over a base field? Why do we consider coordinate rings only in the case of integral domains? Once you adopt a more general idea, there is a clear definition.

Definition. Let A be a commutative ring with $1 \in A$. Let X be the set of prime ideals in A (N.B. a prime ideal is not the whole ring). Consider the Zariski topology on X. For $x \in X$, corresponding with $P_x \subset A$, define

$$\mathscr{O}_{X,x} = (A \setminus P_t)^{-1}A, \quad \text{and} \quad \mathscr{O}_X = \cup_{x \in X}\ \mathscr{O}_{X,x},$$

considered as a sheaf on the topological space X. The pair $(X, \mathscr{O}_X)$ is called an *affine scheme*. We write $\mathrm{Spec}(A) = (X, \mathscr{O}_X)$, or sometimes $\mathrm{Spec}(A) = X$.

One can define morphisms $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ by having a continuous $f : X \to Y$ map plus "pulling back functions", a section in $\mathscr{O}_Y$ over some open set $U \subset Y$ is mapped onto a section in $\mathscr{O}_X$ over the inverse image $f^{-1}(U) \subset X$. In this theory it is proved that

$$\mathrm{Hom}(\mathrm{Spec}(A), \mathrm{Spec}(B)) = \mathrm{Hom}(B, A),$$

where the first "Hom" ($= \mathrm{Hom}_{\mathrm{Sch}}$) stands for morphisms in the category of schemes, and the second "Hom" ($= \mathrm{Hom}_{\mathrm{ring}}$) stands for ring homomorphisms.

In the beginning we had difficulties accepting this theory. Here is a simple example. Let $B = k[S, T]$. The points of the algebraic variety $V = \mathbb{A}^2_k$ defined by this coordinate ring are of the form (s, t), and $V(k) = k^2$; we remark that points in V coincide with maximal ideals in B. It seems more intuitive to consider *maximal ideals* in A rather than all prime ideals in B.

In the theory of schemes we see that $X = \mathrm{Spec}(k[S, T])$ consist of objects of three different kinds: (a) points corresponding to a maximal ideal in $B = k[S, T]$, (b) points in X corresponding to irreducible algebraic curves in k^2, namely ideals J, non-zero and non-maximal, and (c) one point η in X corresponding to the zero ideal in B. Why is this necessary?

Suppose we have a morphism of schemes $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$. This should correspond to a homomorphisms of rings $A \leftarrow B$ by "pulling back functions",

$$(\mathrm{Spec}(A) \to \mathrm{Spec}(B)) \quad \longleftrightarrow \quad (A \leftarrow B):$$

conversely a homomorphism $A \leftarrow B$ should correspond to a morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$. As an example, consider the map

$$A = k(S, T) \leftarrow k[S, T]$$

induced by the inclusion map of $k[S, T]$ in its field of fractions; does this give a morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$? We see that $\mathrm{Spec}(k(S, T))$ consists of one point, and this should map to a point in $\mathrm{Spec}(k[S, T])$, indeed to η. Or consider a non-maximal prime ideal $0 \neq J$, e.g. $J = (S)$; let C be the field of fractions of B/J, and consider the natural homomorphism $C \leftarrow k[S, T]$. We see that $\mathrm{Spec}(C)$ consists of one point and the morphism $\mathrm{Spec}(C) \to \mathrm{Spec}(B)$ maps this to the point $c \in X$ corresponding with the prime ideal J. We need all prime ideals of B in order to obtain a coherent theory.

Note that in the theory of algebraic varieties considering $V = \mathbb{A}^2_k$ as above the "generic point of the variety V" was considered in the theory by Weil, but $\{\eta\}$ is not "a variety defined over k"; however, $\eta : \mathrm{Spec}(k(S, T)) \to \mathrm{Spec}(k[S, T])$ is a morphism of schemes; we see that previous theory was incorporated in a more general context.

Under a morphism of schemes a closed point of the source may go to the generic point of the target. For instance let $\mathrm{Spec}(\mathbb{C}[t]) \to \mathrm{Spec}(\mathbb{Q}[t])$ be the morphism of affine schemes corresponding to the inclusion $\mathbb{Q}[t] \hookrightarrow \mathbb{C}[t]$ of rings. The image of the maximal ideal $(t - \pi)$ of $\mathbb{C}[t]$, corresponding to the point "$t = \pi$" of the affine line over $\mathbb{C}$, is sent to the generic point of $\mathrm{Spec}(\mathbb{Q}[t])$, which correspond to the prime ideal (0) of $\mathbb{Q}[t]$. Here π is the transcendental number $\pi = 3.14159\cdots$.

As mentioned before, we discussed affine varieties, but not more general concepts such as abstract varieties. We have defined affine schemes. More general concepts are needed in general, but for our discussion here and a first understanding of this topic it is not of main importance.

Grothendieck emphasized that we should think not only about absolute objects (such as a scheme); rather the relative objects, i.e. morphisms, should be the focus of our attention. Thinking in relative terms is second nature to him. A talk by Grothendieck usually started by writing X, a vertical arrow, and S while saying "Let X be a scheme over S". Note two aspects.

First, every scheme admits a (unique) morphism to $\mathrm{Spec}(\mathbb{Z})$; if you are interested in a special situation (geometry over $\mathbb{C}$, or over $\mathbb{Z}$ or whatever), you inform the audience / the reader what is the base scheme.

Second, $X \to S$ and $T \to S$ give rise to

$$X \times_S T = X_T \longrightarrow T;$$

note that X_T over T is an object different from $X \to S$. In classical algebraic geometry often an object over K and "the same" object over a field containing K are denoted by the same symbol. You will see this leads to confusion and mistakes. In the Grothendieck theory the "circle" defined by the equation $\mathscr{Z}(S^2+T^2-1)$ over $\mathbb{Q}$ is not the same object as the circle defined by this equation say over $\mathbb{C}$. In the notation above, the scheme X on the one hand and X_T on the other hand should not be identified if $T \to S$ is not an isomorphism.

3.8.1

Consider $\mathrm{Spec}(\mathbb{Z})$. This is a regular scheme (every localization is a regulars local ring) of Krull dimension one (every chain of prime ideals has length at most one, and there is a chain of length one, e.g.: $(0) \subset (17)$.) Hence this scheme has properties analogous to those of a regular affine algebraic curve. This is the classical concept of the analogy between rings of integers in number fields and coordinate rings of algebraic curves as noted by Kronecker, Weil and many others. Grothendieck's scheme theory provides a general frame work for unifying considerations.

3.8.2

Let $X \to S$ and $Y \to S$ be schemes over a base scheme and let

$$X \times_S Y \longrightarrow S$$

be their fibered product. This is a scheme over S. This notion generalizes the concept a product of varieties, and of "intersection"; indeed if $X \subset S$ and $T \subset S$ then $X \times_S Y \subset S$ is the scheme-theoretic intersection. Observe, that even if X and Y "are varieties" (e.g. a like and a conic in a plane), their intersection need not be; scheme theory repairs such defects, even in very simple situations.

3.8.3

For an algebraic variety every local ring has no non-zero nilpotent elements (a nilpotent element $f \in A$ is an element $f \neq 0$ such that there exists $n \in \mathbb{Z}_{>0}$ with $f^n = 0$). However for the more general definition of schemes this is not excluded (and we gain a lot). However you have to get accustomed that a regular function on a scheme gives a "function"

$$f \in A, \quad f \mapsto f(x) \in \kappa(x), \quad x \in X,$$

where $\kappa(x) = \mathscr{O}_{X,x}/\mathfrak{m}_{X,x}$ is the residue class field of the local ring $cO_{X,x}$ at $x \in X$, but the target of these functions may vary with x (consider $A = \mathbb{Z}$), and it may happen that $f \neq 0$ but $f(x) = 0$ for every $x \in X$.

3.8.4

Here is a **basic example.** Let K be a field, and $A = K[\varepsilon]/(\varepsilon^2)$ (an object not present in the theory of varieties). Observation:

every K-morphism $\mathrm{Spec}(A) \to X$ selects a point $x \in X$ with $\kappa(x) = K$
and a *tangent vector* to X attached at x.

Indeed, $\mathrm{Spec}(A)$ consists of one point, the image of this morphism is one point in X, and moreover we obtain a K-algebra homomorphism

$$A = K[\varepsilon] \leftarrow \mathscr{O}_{X,x}, \quad \text{hence } K \leftarrow \kappa(x) \leftarrow K,$$

and a K-linear map

$$A \cdot \varepsilon = K \cdot \varepsilon \leftarrow \mathfrak{m}/\mathfrak{m}^2 \leftarrow \mathfrak{m}, \quad \mathfrak{m} = \mathfrak{m}_{X,x}.$$

In classical deformation theory, e.g. see [53] such situations were studied, however $\mathrm{Spec}(K[\varepsilon]/(\varepsilon^2))$ "did not exist", and difficult description were needed. In the Grothendieck scheme theory they find a natural surrounding for describing such descriptions.

3.8.5

Instead of "a family of varieties over a base" we consider a scheme over a scheme. If you want to be convinced of the elegance of this formulation compare the essence of the paper [57] and the definition of the Néron minimal model given there, and scheme theoretic definition, e.g. see [38], 1.1, and see discussions in [69].

3.8.6

Consider the example as in Section 3.7.4: with $A = K[T]/(T^p - a)$ and $\sqrt[p]{a} \notin K$ as above. The ideal $(T^p - a) \cdot K[T] \subset K[T]$ is a prime ideal. The scheme $X = \mathrm{Spec}(K[T]/(T^p - a))$ does exist, the ring A is a field, and this scheme consists of one point; however there is no K-morphism $\mathrm{Spec}(K) \to X$ (in classical language "this point is not rational over K"). For any extension field $K \subset L$ containing $\sqrt[p]{a}$ the ring

$$A \otimes_K L \quad \cong \quad L[\varepsilon]/(\varepsilon^p);$$

we see that

$$X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$$

is a one-point scheme with nilpotents in its structure sheaf, but these nilpotents only "show up" after an appropriate field extension. An object like this X was not considered in the theory of algebraic varieties.

3.9

Schemes with nilpotents in the structure sheaf can appear in a natural way. Before the introduction of schemes this was sometimes mysterious. We mention one example. In constructing Picard varieties we were accustomed that the dimension of $\mathscr{P}ic(V)$ should be equal to the dimension of $\mathrm{H}^1(V, \mathscr{O}_V)$ (in whatever disguise). And in general this is true for algebraic curves and for abelian varieties.

However in 1955 Igusa constructed an example of an algebraic surface V over a field K in positive characteristic with

$$\dim_K \left(\mathrm{H}^1(V, \mathscr{O}_V)\right) > \dim\left(\mathscr{P}ic(V)\right),$$

see [50]. The construction is easy: consider an algebraically closed field k of characteristic two; take two elliptic curves E_1 and E_2 over k such that E_2 has a k-point of order two. (An elliptic curve over a an field of characteristic $p > 0$ which has a point of order p is called an "ordinary elliptic curve"; such a point of order p is unique up to multiplication by $(\mathbb{Z}/p\mathbb{Z})^\times$ if it exits.) Say $a \in E_2(k)$ has order two; define

$$\iota : E_1 \times E_2 \longrightarrow E_1 \times E_2, \quad \iota(x,y) = (-x, y+a);$$

clearly ι is an involution; define $V := (E_1 \times E_2)/\{\mathrm{id}, \iota\}$, the quotient taken by the action of $\mathbb{Z}/2 \cong \{id, \iota\}$; it can be shown the Picard *variety* of V does exist, and it has dimension one; however $\dim_k(\mathrm{H}^1(V, \mathscr{O}_V)) = 2$. More precisely:

(i) There exists an open subgroup scheme $\mathscr{P}ic^\tau(V)$ of the Picard scheme $\mathscr{P}ic(V)$ such that the quotient $\mathscr{P}ic(V)/\mathscr{P}ic^\tau(V)$ is a torsion free constant group scheme over k.

(ii) We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{P}ic^0(V)_{\mathrm{red}} \arrow[r] \arrow[d, hook] & \mathscr{P}ic^\tau(V) \arrow[r] \arrow[d, "="] & E_1[2] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{P}ic^0(V) \arrow[r] & \mathscr{P}ic^\tau(V) \arrow[r] & E_1[2]_{\mathrm{et}} \arrow[r] & 0
\end{tikzcd}$$

with exact rows, where

- $\mathscr{P}ic^0(V)$ is the neutral component of the Picard scheme $\mathscr{P}ic(V)$,
- $\mathscr{P}ic^0(V)_{\mathrm{red}}$ is $\mathscr{P}ic^0(V)$ with reduced structure sheaf, i.e. the "Picard variety",
- $E_1[2]$ is the subgroup scheme of 2-torsion points of E_1, and
- $E_1[2]_{\mathrm{et}}$ is the maximal étale quotient of $E_1[2]$.

The above commutative diagram induces an isomorphism $E_1[2]^0 \xrightarrow{\sim} \mathscr{P}ic^0(V)/\mathscr{P}ic^0(V)_{\mathrm{red}}$, where $E_1[2]^0$ is the neutral component of $E[2]$. The latter is local and non-reduced, of rank 2 (respectively 4) if E_1 is ordinary (respectively not ordinary).

(iii) Let $f : V \to E_3 := E_2/\{0, a\}$ be the map induced by the second projection $\mathrm{pr}_2 : E_1 \times E_2 \to E_2$. The morphism $f^* : \mathscr{P}ic^0(E_3) \to \mathscr{P}ic^0(V)$ induces an isomorphism $\mathscr{P}ic^0(E_3) \xrightarrow{\sim} \mathscr{P}ic^0(V)_{\mathrm{red}}$.

(iv) Let $V_1 \to \mathrm{Spec}(R)$ be the universal first order equicharacteristic deformation of V, where R is an Artinian equicharacteristic local ring with residue field k. Then $\mathscr{P}ic^\tau(V_1/R)$ is *not flat* over R.

The example (iv) of a non-flat $\mathscr{P}ic^\tau$ is due to Mumford; see footnote (19) on [23, page 648] a discussion and more consequences.

It was a puzzling situation. Later we found that group schemes in characteristic zero are reduced (contain no nilpotents in their structure sheaf), but in positive characteristic this is not true; moreover using the method of computing the tangent space explained above, a simple calculation shows that

the tangent space $\mathfrak{t}_{Q,0}$ at 0
to the *Picard scheme* $Q = \mathscr{P}ic(V)$

is canonically isomorphic with

$$\mathfrak{t}_{Q,0} \cong \mathrm{H}^1(V, \mathscr{O}_V).$$

We indicate a proof that $\mathscr{P}ic(V)$ is non-reduced. Consider $A = K[\varepsilon]/(\varepsilon^2)$ and the exact sequence

$$0 \to K \cong I = \varepsilon{\cdot}A \longrightarrow A \longrightarrow A/I = K \to 0.$$

Tensoring with $\mathscr{O}_V$ and taking units we derive

$$1 \to (1 + I{\cdot}\mathscr{O}_V)^\times \longrightarrow (A \otimes \mathscr{O}_V)^* \longrightarrow \mathscr{O}_V^* \to 1.$$

Using $\varepsilon^2 = 0$ we obtain

$$(\mathscr{O}_V)^+ \xrightarrow{\sim} (1 + I{\cdot}\mathscr{O}_V)^\times, \quad f \mapsto 1 + \varepsilon{\cdot}f.$$

Also we suppose that global functions on V are constant. Moreover

$$\mathrm{Ker}\left(\mathrm{H}^1(V, (A \otimes \mathscr{O}_V)^*) \to H^1(V, \mathscr{O}_V^*)\right) = \mathfrak{t}_{Q,0};$$

here we use the observation in Section 3.8.4 and the definition of the Picard functor. We obtain the exact sequences

$$1 \to \Gamma(V,(1+I\cdot\mathscr{O}_V)^{\times})) \longrightarrow \Gamma(V,(A\otimes\mathscr{O}_V)^{*}) \longrightarrow \Gamma(V,\mathscr{O}_V^{*}) \\ = K^{*} \to 1,$$

and

$$0 \to \mathrm{H}^1(V,\mathscr{O}_V) \longrightarrow \mathrm{H}^1(V,(A\otimes\mathscr{O}_V)^{*}) \longrightarrow H^1(V,\mathscr{O}_V^{*}).$$

Hence we prove the statement:

$$\mathrm{H}^1(V,\mathscr{O}_V) \xrightarrow{\sim} \mathrm{t}_{Q,0} = \mathrm{Ker}\left(\mathrm{H}^1(V,(A\otimes\mathscr{O}_V)^{*}) \to H^1(V,\mathscr{O}_V^{*})\right).$$

Note that we did not even assume that the Picard scheme $Q = \mathscr{P}ic(V)$ exists; we just considered the Picard functor directly.

The explanation of Igusa's example is that although the *Picard variety* P of V exists (in the classical theory of varieties), its Picard scheme Q (in the theory of schemes) is different, and the ideal of nilpotent elements in the structure sheaf of Q defines $P = Q_{\mathrm{red}} \subset Q$.

3.10 Hidden Nilpotent Elements

Sometimes nilpotents do not show up over a small field (as we saw in Section 3.8.6); hence there is confusion possible in the theory of algebraic varieties, but in the context of schemes these phenomena are easily explained.

We know that the following phenomenon can happen: an irreducible zero-set becomes reducible after extension of base field. For example $X^2+Y^2 \in \mathbb{R}[X,Y]$ defines an irreducible zero-set, which however becomes reducible over $\mathbb{C}$ as $X^2+Y^2 = (X+\sqrt{-1}\cdot Y)(X-\sqrt{-1}\cdot Y) \in \mathbb{R}[X,Y]$. Here is an analogous, natural example for nilpotents.

Example. Consider $K = \mathbb{F}_2(t)$, a transcendental extension of $\mathbb{F}_2$. Let $E \subset \mathbb{P}^2_K$ be given as

$$E = \mathscr{Z}(Y^2Z+XYZ+X^3+tZ^3).$$

This is a non-singular curve of genus one. As is usual, we take the point whose projective coordinates are $[x=0:y=1:z=0]$ as the unity element for the group law, and we obtain an elliptic curve E. As a group scheme we can consider $E[2]$, the 2-torsion on this abelian variety of dimension one. It is the scheme-theoretic kernel of the endomorphism $[2]_E : E \to E$, mulpliplication by 2 for the group law of E. We see that as a scheme $E[2]$ is a *disjoint* union $\mu_2 \sqcup T$, where $\mu_2 \cong \mathrm{Spec}(K[\tau]/\tau^2)$ and $T \subset E$ is a *reduced* subscheme (reduced means its structure sheaf has no nilpotents), with $T \cong \mathrm{Spec}(K[Y]/(Y^2+t)$. However $T \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K[\sqrt{t}]) \cong \mathrm{Spec}(K[\sqrt{t}][Y]/((Y+\sqrt{t})^2)$: after base change nilpotents show up.

This example works for every prime number; we just took $p=2$ in order to have simpler equations.

3.11 Varieties in the Language of Schemes

Sometimes we want to go back and forth between notions in classical algebraic geometry and the language of schemes. Let K be a base field. A variety V over K is an algebraic scheme over K (i.e. of finite type) that is geometrically irreducible and geometrically reduced; this last condition means that for every field extension $K \subset L$ we have that $V \otimes L := V \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ is irreducible and has no nilpotents in its structure sheaf.

For example, if $K \subsetneqq K'$ is a finite extension, then $\mathrm{Spec}(K')$ is not a K-variety.

See Section 3.12 for some examples of schemes.

3.11.1

Recall that we consider only commutative rings as base-rings. We say that A is a Λ-algebra if A and Λ are (commutative) rings and a ring homomorphism (the structure map) $\Lambda \to A$ is given (here we consider the relative situation, not the absolute situation). Note that any ring is a $\mathbb{Z}$-algebra.

We say A is a Λ algebra of finite type if there exists $n \in \mathbb{Z}_{\geq 0}$ and a surjective Λ-algebra homomorphism

$$\Lambda[T_1,\cdots,T_n] \twoheadrightarrow A.$$

A Λ-algebra that is finitely generated as a Λ-module is of finite type; however there are many Λ-algebras of finite type not finitely generated as a Λ-module; e.g. $\Lambda[T]$ is of finite type but not finitely generated as a Λ-module.

Note that for any affine algebraic variety V with affine coordinate ring $K \to A$ we know that A is a K-algebra is of finite type.

3.11.2

Suppose we have a variety over $\mathbb{Q}$ and we want to consider a possibility of "reduction modulo p" for a given prime number. We would like to develop this method, so effective in number theory, also in algebraic geometry. In history we have seen many struggles to lay foundations for this; often with beautiful results, described in complicated methods.

Note that the ring

$$R = (\mathbb{Z}\setminus(p))^{-1}\mathbb{Z} = \{\frac{a}{b} \mid a,\ b \in \mathbb{Z},\ b \text{ not divisible by } p\},$$

the ring of fractions with denominator not divisible by p, is not a $\mathbb{Z}$-algebra of finite type (left as exercise to the reader). However this is a natural choice for a "base-ring" doing geometry in mixed characteristic. Hence we include such algebras in our considerations.

3.11.3

Let us go back to classical geometry. In the beautiful paper [53] we find a long and strong description of deformation theory. In retrospect we see that descriptions would have been more transparent if more general base schemes would have been allowed (as Grothendieck proposed, not only in the algebraic category, but also in the complex analytic theory). For example in Definition 11.1 in [53] on "the number of moduli", the notions "effectively parametrized", Def. 6.2 on page 366, and "complete families", Def. 1.7 on paged 337 are used. Only first-order deformations were considered, and families are given over a global base. We see the way this can be incorporated and generalized in modern theory were arbitrary base schemes are allowed, and where "(pro-)representable functors" lead the way to transparent descriptions.

Many more examples can be given were scheme theory replaces classical complicated descriptions by elegant and powerful methods.

3.12

We discussed some examples of schemes. We see how shortcomings of the theory of varieties are repaired. Sometimes we want to use the word "(algebraic) variety" and compare that concept with the corresponding object in the theory of schemes. Here is comparison. Consider a field K, and

- *varieties* defined over K (i.e. given as zeros of polynomials and chart-changes with coefficients in K) that are absolutely irreducible over K, and regular maps as their morphism;
- *schemes* over K, of finite type, and absolutely integral and irreducible, and their morphisms as ringed spaces.

These categories are equivalent. However, the notion of "points" of the objects in questions differs. Already in the case of an affine scheme $\mathrm{Spec}(R)$, where R is a finitely generated integral domain over K such that $R \otimes_K K^{\mathrm{alg}}$ is integral: (a) points of the corresponding variety are K-linear ring homomorphisms $R \to \Omega$, where Ω is an algebraically closed field containing K; (b) points of $\mathrm{Spec}(R)$ are prime ideals of R, and for any K-algebra B, and B-valued points of $\mathrm{Spec}(R)$ are K-linear ring homomorphisms $R \to B$.

Some examples of schemes not considered in this comparison:

(i) Suppose $K \subsetneq L$ is a finite algebraic extension; in this case $X = \mathrm{Spec}(L)$ is of finite type over $\mathrm{Spec}(K)$; however X is not absolutely integral; e.g. if K/L is separable,
$$X \times_{\mathrm{Spec}(K)} \mathrm{Spec}\,\overline{K}$$
is reducible, and if K/L is purely inseparable this product is non-reduced. These schemes should not be considered as "varieties".

(ii) Suppose $X = \mathrm{Spec}(K((t)))$. In this case X is not of finite type over K.

(iii) Suppose $X = \mathrm{Spec}(K[\varepsilon]/(\varepsilon^2))$. In this case X is non-reduced.

These three examples each play an important role in modern algebraic geometry (even in considerations over a base-field).

3.13

We describe an example of a non-reduced moduli scheme of space curves and a related non-reduced moduli scheme of threefolds due to Mumford. Details can be found in [56]. We begin by describing a particular type of smooth projective curves in $\mathbb{P}^3$ over an algebraically closed base field k of characteristic 0. These smooth space curves C are characterized by the following conditions:

(1) C has degree 14, genus 24, and lies on a smooth cubic surface S in $\mathbb{P}^3$. Note that S is the unique smooth cubic, for if C lies on another smooth cubic surface S', then $\deg(S \cdot S') = 9 < 14 = \deg(C)$.

(2) There exists a line E (i.e. a smooth $\mathbb{P}^1$ of degree 1) on S such that $\mathcal{O}_S(C) \cong \mathcal{O}_S(H+2E)$. Here H stands for a hyperplane section on the cubic surface S. It is well known that there are exactly 27 lines on S, and the self-intersection number of each line on S is -1.

The dimensions of some coherent cohomology groups are given below. They can be computed using suitable exact sequences.

(3) $\deg(C \cdot C)_S = 60$

(4) $\dim \mathrm{H}^0(S, \mathcal{O}_S(C)) = 38$; $\mathrm{H}^i(S, \mathcal{O}_S(C)) = (0)$ for $i = 1, 2$.

(5) $\dim \mathrm{H}^0(S, \mathcal{N}_{C/S}) = 37$, and $\mathrm{H}^1(S, \mathcal{N}_{C/S}) = (0)$, where $\mathcal{N}_{C/S})$ is the normal bundle of $C \hookrightarrow S$.

(6) $\dim \mathrm{H}^0(C, \mathcal{N}_{C/\mathbb{P}^3}) = 57$, and $\dim \mathrm{H}^1(C, \mathcal{N}_{C/\mathbb{P}^3}) = 1$ where $\mathcal{N}_{C/\mathbb{P}^3}$ is the normal bundle of $C \hookrightarrow \mathbb{P}^3$.

(7) $\dim \mathrm{H}^0(C, \mathcal{N}_{F \hookrightarrow \mathbb{P}^3} \otimes_{\mathcal{O}_F} \mathcal{O}_C) = 20$, and $\dim \mathrm{H}^1(C, \mathcal{N}_{F \hookrightarrow \mathbb{P}^3} \otimes_{\mathcal{O}_F} \mathcal{O}_C) = 1$, where $\mathcal{N}_{F \hookrightarrow \mathbb{P}^3}$ is the normal bundle of $F \hookrightarrow \mathbb{P}^3$.

To show the existence of curves satisfying conditions, let E be one of the 27 lines on S, and consider the short exact sequences
$$0 \to \mathcal{O}_S(4H+E) \to \mathcal{O}_F(4H+2E) \to \mathcal{O}_E(2) \to 0$$
and
$$0 \to \mathcal{O}_S(4H) \to \mathcal{O}_F(4H+E) \to \mathcal{O}_E(3) \to 0.$$
From the associated long exact sequences one gets
$$\mathrm{H}^i(\mathcal{O}_S(4H)) = \mathrm{H}^i(\mathcal{O}_S(4H+E) = \mathrm{H}^i(\mathcal{O}_S(4H+2E) = (0)$$
for $i = 1, 2$,

and from Riemann–Roch for surfaces one gets

$$\dim \mathrm{H}^0(\mathcal{O}_S(4H+2E)) = 38.$$

Because $\mathscr{P}ic^0(S) = (0)$, the collection of all curves on a fixed smooth cubic surface satisfying conditions (1) and (2) form a 37-dimensional family. Since the family of all smooth cubics in $\mathbb{P}^3$ is 19-dimensional, one sees that the collection of all curves in $\mathbb{P}^3$ satisfying (1) and (2) is parametrized by a 56-dimensional variety, classically called a Chow variety.

On the other hand, the cohomology group $\mathrm{H}^0(C, \mathscr{N}_{C/\mathbb{P}^3})$ parametrizes first order deformations of C. The fact that its dimension (57) is strictly bigger than the dimension of the Chow variety (56) says that the question on the *completeness of the characteristic map* for space curves has a negative answer. In the language of schemes, the statement corresponding Hilbert *scheme* $\mathscr{H}ilb$ is non-reduced.

For a smooth space curve C in $\mathbb{P}^3$ satisfying conditions (1) and (2) above, consider the blowing-up $\mathrm{bl}_C : X_C \to \mathbb{P}^3$ of $C \hookrightarrow \mathbb{P}^3$. Mumford remarked on page 643 of [56], citing [52], that the local moduli space of X_C is isomorphic to the formal completion of the Hilbert scheme $\mathscr{H}ilb$ at the point $[C]$ corresponding to C, consequently that local moduli space is also non-reduced. A more precise form of the first part of Mumford's remark is the following, which is an algebraic version of a special case of [52, Thm. 6].

> *For every Artinian local ring R with residue field k, and for every flat morphism $\mathfrak{X} \to \mathrm{Spec}(R)$ whose closed fiber is X_C, there exists a closed subscheme $\mathcal{C} \subset \mathbb{P}^3_R$ which is flat over R and an R-isomorphism from $\mathfrak{X}$ to the blow-up of $\mathbb{P}^3_R$ along $\mathcal{C}$.*

Let $\mathscr{M}$ be the local moduli problem which assigns to every Artinian local ring R with residue field k the set of all equivalence classes of morphisms $f : \mathfrak{X} \to \mathbb{P}^3_R$ over R such that $\mathfrak{X}$ is flat over R and the closed fiber of f is bl_C, where two such maps f_1 and f_2 over R are declared to be equivalent if and only if there is an isomorphism $\alpha : \mathfrak{X}_1 \xrightarrow{\sim} \mathfrak{X}_2$ such that $f_2 \circ \alpha = f_1$. The above statment implies that the local moduli problem $\mathscr{M}$ is represented by the formal completion $\mathscr{H}ilb^{/[C]}$ of the Hilbert scheme $\mathscr{H}ilb$ at the point $[C] \in \mathscr{H}ilb(k)$. Note that if we consider the deformation problem $\mathscr{D}ef(X_C)$ of X_C instead of the moduli problem $\mathscr{M}$ above, then $\mathscr{D}ef(X_C)$ is isomorphic to the quotient of $\mathscr{H}ilb^{/[C]}$ by the formal group attached to $\mathrm{PGL}(4)$.

3.14 Historical Remarks

The theory of schemes was first announced in 1958 in [2], followed by a big bang, FGA exposé 182 in May 1959. Besides the definition and basic properties of schemes, the latter contains: finiteness theorems of proper morphisms, relation between formal and algebraic geometry (GFGA), deformation theory of schemes, definition of finite étale covers and the étale fundamental group of a scheme, and the computation of the prime-to-p part of étale fundamental group of a complete irreducible smooth algebraic curve.

For an interesting discussion (between Grothendieck and Mumford) about the use of the word "variety" versus "scheme", see [23, pp. 730–733].

4. The Algebraic Fundamental Group

> *… J'en ai bien entendu à une définition algébrique du groupe fondamental …*
>
> Grothendieck 22.11.56, [10, page 55]

This is an example of the clarity and depth of Grothendieck's thinking: well-known classical ideas freed from their initial restrictions give a theory, of great beauty, but also, as we will see, with many clean and direct applications. Here is a mental exercise for the reader.

On the one hand you know *Galois theory*: classify finite, separable extensions of fields by group theory (where the action of the Galois group interchanges zeros of an equation). This is algebra and a field extension $K \subset L$ does not look like a geometric object.

On the other hand you know the theory of the *topological fundamental group*: for a topological space S you want to consider "all" ("unramified") coverings $T \to S$, and classify them by group theory (where the action of the fundamental group interchanges sheets of the universal covering). Do you feel these two notions are two aspects of one and the same theory? And what is that theory?

This uniform approach is given by what we now call *the Grothendieck fundamental group*, or *the algebraic fundamental group*. This material is discussed in SGA 1. Start by thinking instead of a finite, separable extension of fields

$K \hookrightarrow L$ as a geometric "covering": $\mathrm{Spec}(K) \leftarrow \mathrm{Spec}(L)$.

Once you realize this (and if you have the daring insight of Grothendieck) you define unramified finite covering of schemes (no further restrictions), and you prove that these are all classified by a (pro-finite) group. Note that this theory enables you to study examples like $\mathrm{Spec}(\mathbb{C}(t))$, or $\mathrm{Spec}(\mathbb{Z}[1/p])$, or $\mathrm{Spec}(\mathbb{F}_p[T])$ and so on. Moreover, as in topology a continuous map (a morphism) between topological spaces (between schemes) gives a covariant map on the fundamental groups involved. There are incredible applications, e.g. the specialization of the fundamental group.

One example: for any field K, the Grothendieck fundamental group is just the Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$.

Another example: for a complete algebraic variety V over $\mathbb{C}$ the Grothendieck fundamental group is the pro-finite completion of the topological fundamental group of $V(\mathbb{C})$; this "comparison" enables us to compute this group by topological methods. One of the results along this line proved by Grothendieck is the computation of the (prime to p part of) the fundamental group of an algebraic curve in positive characteristic SGA 1, exp. X Cor. 3.10: lift coverings of the curve to characteristic zero, classify these by topological methods, and use the specialization of the fundamental group. We have a rich, powerful and beautiful technique at our disposal.

This theory was precisely described and further developed in SGA1. Grothendieck was very fond of this theory as he wrote to Serre. We have learned from Grothendieck, especially in this case, that going to the core of the problem, and deleting all unnecessary assumptions can produce (in capable hands) a beautiful revolution.

We describe one detail of this theory. Suppose you have a scheme X over a field K. Consider the fundamental group $\pi_1(X)$ (apologies: we should include base points in our notation, but we want to keep things short). This has two aspects:

the *geometric part*, $\pi_1(\overline{X})$, where $\overline{X} = X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})$,
the *arithmetic part* $\pi_1(\mathrm{Spec}(K)) = \mathrm{Gal}(K^{\mathrm{sep}}/K)$

and they fit into an exact sequence

$$1 \to \pi_1(\overline{X}) \to \pi_1(X) \to \pi_1(\mathrm{Spec}(K)) \to 1 \quad \text{(g-a)},$$

see SGA1 exp. IX Thm. 6.1. This mixture of arithmetic and geometry leads to deep applications.

5. Descent Theory and Grothendieck Topologies

J'en arrive à un petit début de compréhension, pour l'instant largement heuristique, et à la constatation que ma théorie de dualité pour faisceaux cohérents sera guide épatant pour l'édification d'une théorie générale de dualité, englobant celle-ci en même temps qu'une théorie de dualité des groupes algébriques ou schémas en groupes, et la dualité en cohomologie de Weil. Cela m'amène à élargir d'autant mon programme prévu, ces questions me semblant maintenant beaucoup plus abordables que je ne pensais. Une fois cette théorie développée, j'espère bien que les conjectures de Weil viendront toutes seules.

Grothendieck 9.8.1960, [10, p. 104]

The theory of descent appeared in FGA exp. 190, the first of the series *Technique de descent et théorèmes d'existence en géométrie algébrique* in FGA, in December 1959. The predecessors of Grothendieck's general descent theory include Hilbert's theorem 90, Weil's method of "descent of base fields" for separable field extensions, extended by Cartier to purely inseparable extensions of height one (extensions L/K with $L^p \subseteq K$). The notion of Grothendieck topology is closely related to the theory of descent.

References: Exp. VIII and IX of SGA 1 for descent; Exp. IV of SGA 3 for Grothendieck topologies; [35], [36] and [37] for algebraic stacks.

5.1 Grothendieck Topology

Grothendieck had the insight that in the definition of topology, one can

> replace *open immersions* (i.e. subsets) *by maps with suitable properties*

in a categorical setting, and arrive at a new notion which captures the essence of the concept of topology. We will illustrate the main idea using the category of schemes.

A Grothendieck pretopology on the category of scheme is an assignment which attaches to scheme X, a collection $\mathrm{Cov}'(X)$ of families of morphisms, satisfying the following properties (C0)–(C2). (Elements of $\mathrm{Cov}(X)$ are called *covers* of X for the Grothendieck pretopology.) A pretopology is a Grothendieck topology if the saturation property (C3) below is satisfied. The property (C3′) is a consequence of (C2) and (C3).

(C0) For every isomorphism $Y \xrightarrow{\sim} X$, the singleton family $\{Y \xrightarrow{\sim} X\}$ is in $\mathrm{Cov}(X)$.

(C1) A pull-back of a cover is a cover: if $\{X_i \to X \mid i \in I\}$ is in $\mathrm{Cov}'(X)$, then its base change $\{X_i \times_X Z \to Z \mid i \in I\}$ is in $\mathrm{Cov}(Z)$ for every morphism $f : Z \to X$.

(C2) A cover of a cover is a cover: if $\{f_i : X_i \to X \mid i \in I\}$ is in $\mathrm{Cov}(X)$ and $\{g_{ij} : X_{ij} \to X_i \mid j \in J_i\}$ is in $\mathrm{Cov}(X_i)$ for each $i \in I$, then the family $\{g_{ij} \circ f_i : X_{ij} \to X \mid i \in I, j \in J_i\}$ is in $\mathrm{Cov}(X)$.

(C3) A family refined by a cover is a cover: Let $\{f_j : Y_j \to X \mid j \in J\}$ be a family of morphisms and let $\{g_i : X_i \to X \mid i \in I\}$ be a family in $\mathrm{Cov}(X)$. If for every $i \in I$, there exists an element $j \in J$ and a morphism $h_{ji} : X_i \to Y_j$ such that $f_j \circ h_{ji} = g_i$, then the family $\{f_j : Y_j \to X \mid j \in J\}$ is in $\mathrm{Cov}(X)$.

(C3′) Suppose $\{Y_j \to X \mid j \in J\}$ is a family of morphisms and $\{X_i \to X \mid i \in I\}$ is a family in $\mathrm{Cov}(X)$. If the family $\{Y_j \times_X X_i \to X_i \mid j \in J\}$ is in $\mathrm{Cov}(X_i)$ for each $i \in I$, then $\{Y_j \to X \mid j \in J\} \in \mathrm{Cov}(X)$.

Remark. (i) The notion of a Grothendieck pretopology is analogous to the notion of a basis of open sets of a topological space.

(ii) There is a saturation procedure, similar to using a basis of open sets to define a topology, which slightly enhances a Grothendieck pretopology to a Grothendieck topology, so that every cover in the newly produced Grothendieck topology is refined by a cover in the pretopology.

Examples.

(0) Zariski topology. Define a pretopology by assigning to any scheme X the collection of all families $\{U_i \hookrightarrow X \mid i \in I\}$, where each $U_i \hookrightarrow X$ is an affine open subscheme of X and $\bigcup_{i\in I} U_i = X$. Every family $\{V_j \hookrightarrow X \mid j \in J\}$ such that V_j is a Zariski open subset of X for each j and $\bigcup_{j\in J} V_j = X$ is a cover of X for the associated Grothendieck topology.

(2) Étale topology. This is the Grothendieck topology generated by the pretopology which assigns to each scheme X the collection of all families $\{X_i \to X \mid i \in I\}$ where each morphism $X_i \to X$ is étale and the map $\coprod_{i\in I} X_i \longrightarrow X$ is surjective. Here the symbol "$\coprod$" means "disjoint union".

(3) Flat fpqc topology. Define a Grothendieck pretopology by taking $\mathrm{Cov}(X)$ to be the set of all families of morphisms obtained in the following manner. Take a Zariski open cover $\{U_i : i \in I\}$ of X where each U_i is an affine open subscheme of X. For each $i \in I$, let $g_i : X_i \to U_i$ be an faithfully flat morphism between affine schemes; let $f_i : X_i \to X$ be the composition of g_i with the inclusion map $U_i \hookrightarrow X$. Declare the family $\{f_i : X_i \to X \mid i \in I\}$ to be in $\mathrm{Cov}(X)$. The associated Grothendieck topology is called the fpqc topology. For every faithfully flat quasi-compact morphism $f : Y \to X$, the singleton family $\{f\}$ is a cover of X for the fpqc topology. Note that fpqc is the acronym for "fidèlement plate et quasi-compact" (faithfully flat and quasi-compact).

If in the above we replace the condition that each g_i is a faithfuly flat morphism between affine schemes by requiring that g_i is a faithfuly flat and quasi-finite morphism between affine schemes, the resulting topology is called the fppf topolgy (for "fidèlement plat et de présentation finie").

These two topologies are finer than the étale topology, which in turn is finer than the Zariski topology:

$$(\mathrm{fpqc}) \ \geq \ (\mathrm{fppf}) \ \geq \ (\mathrm{et}) \ \geq \ (\mathrm{Zar})$$

5.2 Descent

There are several things (among others) we can do using a topology.

(a) Verify a property (say about either a manifold or a map between manifolds) using local charts, if that property is local.
(b) Produce a map between two manifolds or a vector bundle on a manifold, by gluing, i.e. first produce maps or vector bundles using local charts, then verify compatibility condition for different charts.
(c) Produce a manifold by gluing smaller open pieces.
(d) Define global invariants of manifolds (such as cohomology groups and homotopy groups), and use them to study geometric properties of manifolds.

The theory of descent is mostly about (a)–(c).

Main ideas of descent.

(a) Properties of local nature. Many important properties of morphisms of schemes are local with respect to suitable Grothendieck topologies, for the target and/or for the source. For instance suppose $\{X_i \to X \mid i \in I\}$ and $\{g_j : Y_j \to Y \mid j \in J\}$ are coverings for the fpqc topology. Then a morphism $f : Y \to X$ is proper (respectively flat, smooth, or étale) if and only if $f \times_X X_i : Y \times_X X_i \to X_i$ is for each $i \in I$: being proper (respectively flat, smooth, or étale) is local for the target in the fpqc topology, therefor also in the étale and Zariski topology. Similarly being flat is local for the source in the fpqc topology (and the étale and the Zariski topology): f is flat if and only if $f \circ g_j$ is flat for each $j \in J$. However being smooth (respectively étale) is local for the source in the étale topology and the Zariski topology, but not for the fpqc topology.

Localizing with respect to a suitable Grothendieck topology is a basic tool for studying properties which are local for this topology, and is often used silently. For instance results in textbooks on algebraic geometry such as [46] are often stated for algebraic varieties over algebraically closed fields. The base change/descent method allows one to immediately draw conclusions for varieties over arbitray base fields from the case of algebraically closed base fields.

(b) Descent of morphisms or coherent sheaves. Suppose that $X \to S$ and $Y \to S$ are morphisms of schemes, and $\{S_i \to S \mid i \in I\}$ is a cover of S for the fpqc topology. Let $S_{ij} := S_i \times_S S_j$ for all $i, j \in I$, and let $\mathrm{pr}_1 : S_{ij} \to S_i$, $\mathrm{pr}_2 : S_{ij} \to S_j$ be the two projections. We have a short exact sequence of sets

$$\mathrm{Mor}_S(X,Y) \longrightarrow \coprod_{i\in I} \mathrm{Mor}_{S_i}(X \times_S S_i, Y \times_S S_i) \overset{\mathrm{pr}_1^*}{\underset{\mathrm{pr}_2^*}{\rightrightarrows}} \coprod_{i,j\in I} \mathrm{Mor}_{S_{ij}}(X \times_S S_{ij}, Y \times_S S_{ij})$$

of sets. In particular, if we have S_i-morphisms $f_i : X \times_S S_i, Y \times_S S_i$ for $i \in I$ such that the pull-backs of f_i and f_j coincide as morphisms from $X \times_S S_{ij}$ to $Y \times_S S_{ij}$ for all $i, j \in I$, then the f_i come from a unique morphism $f : X \to Y$: they glue over the flat cover $\{S_{ij} \to S\}$ and descend uniquely to an S-morphism from X to Y.

In the case when $S = \mathrm{Spec}(K)$ and the of S is $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ for a finite Galois extension L/K of fields, the above descent of morphism says something familiar: suppose X, Y are schemes over K and if $f_L : X \times_{\operatorname{Spec} K} \operatorname{Spec} L \to Y \times_{\operatorname{Spec} K} \operatorname{Spec} L$ is an L-morphism which is fixed under all conjugations by elements the Galois group $\mathrm{Gal}(L/K)$, then f_L is the base change to L of a unique K-morphism $f : X \to Y$.

The fpqc descent for coherent sheaves is similar.

(c) Descent of schemes. Let S be a scheme, let $\{U_i \to S \mid i \in I\}$ be a cover of S for the fppf topology. Let $S_0 := \coprod_{i\in I} U_i$, and let $\mathrm{pr}_0 : S_0 \to S$ be the structural morphism for S_0. Let $S_1 := S_0 \times_S S_0$, let $S_2 := S_0 \times_S S_0 \times S_0$. We have projections $\mathrm{pr}_{1,1}, \mathrm{pr}_{1,2} : S_1 \to S_0$ to the first and the second factor of S_1; similarly we have projections

$$\mathrm{pr}_{2,12}, \mathrm{pr}_{2,23}, \mathrm{pr}_{2,13} : S_2 \to S_1$$

and

$$\mathrm{pr}_{2,i} : S_3 \to S_0, \quad i = 1,2,3.$$

Suppose we have a morphism of schemes $X \to S$. Let $X_i := X \times_S S_i$ for $i = 0,1,2$. The fact that X_0 is the base change of an S-scheme gives rise to a natural S_1-isomorphism $\alpha : \mathrm{pr}_{1,2}^* X_1 \to \mathrm{pr}_{1,1}^* X_1$ which satisfies the following cocycle condition:

$$\mathrm{pr}_{2,12}^* \alpha \circ \mathrm{pr}_{2,32}^* \alpha = \mathrm{pr}_{2,13}^* \alpha$$

as S_2-morphisms from $\mathrm{pr}_{2,3}^* X_0$ to $\mathrm{pr}_{2,1}^* X_0$.

Define the category $\mathrm{DescSch}(S_0 \to S)$ of descent data for schemes to be the category whose objects are pairs $(X_0 \to S_0, \alpha : \mathrm{pr}_{1,2}^* X_1 \to \mathrm{pr}_{1,1}^* X_1$ which satisfy the above cocycle condition. A morphism in $\mathrm{DescSch}(S_0 \to S)$ from $(X_0 \to S_0, \alpha)$ to $(Y_0 \to S_0, \beta)$ is by definition a map $f : X_0 \to Y_0$ such that $\beta \circ \mathrm{pr}_{1,2}^* f = \mathrm{pr}_{1,2}^* g \circ \alpha$. In the case when S is the spectrum of a field K and S_0 is the spectrum of a finite Galois extension field L of K, one recovers the Galois descent data according to Weil.

Given a descent datum $(X_0 \to S_0, \alpha)$ in $\mathrm{DescSch}(S_0 \to S)$ relative to an fpqc (respective étale) cover $S_0 \to S$, a natural question is whether it comes from a scheme $X \to S$ by base change. If so, the descended scheme $X \to S$ is unique up to unique isomorphism, and one says that this descent datum is effective. Some general criteria for effectiveness were discussed in SGA 1, exp. VIII.

5.3 Descent as a Method of Constructing Algebraic Geometric Objects

In the series *Technique de descente et théories d'existence en géométrie algébrique I–V* in [5], exp. 190, 195, 212, 221, 232 and 236, Grothendieck explained the method of descent, formal existence theorems for deformation problems, construction of quotients for equivalence relations, the theory of Hilbert schemes, culminating with an existence theorem of Picard schemes, theorem 3.1 in exp. 232:

> *If $f : X \to S$ is a projective flat morphism of locally Noetherian schemes such that all geometric fibers of f are integral, then the Picard scheme $\mathscr{P}ic(X/S)$ exists.*

Here $\mathscr{P}ic(X/S)$ is the *relative Picard functor*, which assigns to every morphism $S' \to S$ the set

$$\mathscr{P}ic(X/S)(S') := \mathrm{H}^0(S', \mathrm{R}f^1_{S'}\mathbb{G}_m),$$

where $f_{S'} := f \times_S S' : X \times_S S' \to S'$ is the base change of f by $S' \to S$ and $\mathrm{R}f^1_{S'}$ is the first direct image functor of f for the fppf topology. Under the assumption that f is projective, Grothendieck first constructed the scheme $\mathscr{D}iv(X/S)$ as the disjoint union $\coprod_Q \mathscr{D}iv^Q(X/S)$ of effective relative Cartier divisors with a fixed Hilbert polynomial $Q(t)$ using the theory of Hilbert schemes. One has a natural/obvious morphism of functors $\mathscr{D}iv(X/S) \longrightarrow \mathscr{P}ic(X/S)$, which is relatively representable, and one obtains a descent datum on (a suitable open subscheme of) $\mathscr{D}iv(X/S)$ which covers $\mathscr{P}ic(X/S)$. The idea then is to prove that this descent datum is effective, which produces a scheme representing the relative Picard functor $\mathscr{P}ic(X/S)$.

5.4 Brauer-Severi Varieties and Descent

We use the Brauer-Severi variety as a baby example to illustrate the method of descent. See [68, Ch. X §6] for a concise account.

Let A be a finite dimensional central simple algebra of dimension d^2 over a field K. It is a basic fact that there exists a finite Galois extension L/K such that the central simple algebra $A_L := A \otimes_K L$ over L is isomorphic to the matrix algebra $\mathrm{M}_d(L)$. Moreover the group of L-linear automorphisms of $\mathrm{M}_d(L)$ is naturally isomorphic to $\mathrm{PGL}_d(L)$, which acts on $\mathrm{M}_d(L)$ through conjugation: for every element $\tilde{g} \in \mathrm{GL}_d(L)$, $M \mapsto \tilde{g}^{-1} \cdot M \cdot \tilde{g}$ gives an automorphism $\mathrm{Ad}(g)$ of $\mathrm{M}_d(L)$, where g is the image in $\mathrm{PGL}_d(L)$ of $\tilde{g}$. We pick an isomorphism $\xi : \mathrm{M}_d(L) \xrightarrow{\sim} A_L$, and obtain for each $\sigma \in \mathrm{Gal}(L/K)$ an element $g_\sigma \in \mathrm{PGL}(L)$ determined by ${}^\sigma\xi = \xi \circ \mathrm{Ad}(g_\sigma)$. An easy calculation shows that the function $\sigma \mapsto g_\sigma$ satisfies the cocycle relation

$$g_{\sigma\cdot\tau} = g_\sigma \cdot {}^\sigma g_\tau \quad \forall \sigma, \tau \in \mathrm{Gal}(L/K).$$

If instead of ξ we picked another isomorphism $\xi' = \xi \circ \mathrm{Ad}(h)$ with $h \in \mathrm{PGL}(L)$, then the 1-cocycle attached to ξ' is

$$\sigma \mapsto h^{-1} \cdot g_\sigma \cdot {}^\sigma h, \quad \sigma \in \mathrm{Gal}(L/K),$$

which is cohomologous to the 1-cocycle $\sigma \mapsto g_\sigma$. We conclude that every d^2-dimensional central simple algebra gives rise to a well-defined class $[A]$ in the non-abelian cohomology group $\mathrm{H}^1(\mathrm{Gal}(\overline{K}/K), \mathrm{PGL}_d(\overline{K}))$.

The Bauer-Severi variety attached to a central simple algebra A over K can be constructed in two ways.

- (by descent) Choose a cocycle $(g_\sigma)_{\sigma\in\mathrm{Gal}(L/K)}$ representing the cohomology class $[A]$ for some finite Galois extension L/K. Each $g_\sigma \in \mathrm{PGL}_d(L)$ defines an automorphism of $\mathbb{P}^{d-1}$ over L, and the 1-cocycle (g_σ) defines an (L/K)-decent datum on $\mathbb{P}^{d-1}$ over L. Such a descent datum is effective: it comes from a variety X_A over K, well-defined up to unique isomorphism. Note that the Galois descent above is a special case of fpqc descent via the natural isomorphism $\mathrm{Gal}(L/K)\times\mathrm{Spec}(L)\xrightarrow{\sim}\mathrm{Spec}(L\otimes_K L)$.
- (direct construction) Define X_A to be the closed subvariety of a suitable Grassmannian variety attached to the vector space underlying A, so that for every commutative K-algebra R, $X_A(R)$ is naturally identified with the set of all surjective R-linear maps of modules $q: A\otimes_K R \twoheadrightarrow Q$ such that Q is a projective R-module of rank $d(d-1)$ and $\mathrm{Ker}(q)$ is a right ideal of $A\times_K R$. The above condition is easily expressed as a system of equations in the Plüker coordinates of the Grassmannian variety.

There is a right principal homogenous space T_A for PGL_d over K attached to the cohomology class $[A]\in \mathrm{H}^1(\mathrm{Gal}(\overline{K}/K),\mathrm{PGL}_d(\overline{K}))$, such that the contraction product $T_A\times^{\mathrm{PGL}_d}\mathbb{P}^{d-1}$ is naturally isomorphic to X_A. The variety T_A can be constructed by descent, and can also be written down explicitly by equations. The scheme T_A is defined so that $T_A(R)$ is naturally identified with the set of all R-linear isomorphisms $\mathrm{M}_d(R)\xrightarrow{\sim}A\otimes_K R$ of algebras, for every commutative K-algebra R. To get explicit equations, choose a K-basis $(N_{i,j})_{1\le i,j\le n}$ of A, and we have structural constants $\left(c_{kl;ij,ab}\right)_{1\le k,l,i,j,a,b\le d}$ of A given by

$$N_{i,j}\cdot N_{a,b}=\sum_{1\le k,l\le d} c_{kl;ij,ab}N_{k,l}.$$

Let $(E_{i,j})_{1\le i,j\le d}$ be the standard basis of M_d, and write $\sum_{1\le i,j\le d} x_{ij;kl}N_{ij}$ for the image of $E_{k,l}$ under a "varying" isomorphism $\mathrm{M}_d(R)\to A\otimes_K R$. Then in the variables $\left(x_{ij;kl}\right)_{1\le i,j,k,l}$, the equations for T_A is

$$\sum_{1\le i,j,a,b\le d} c_{kl;ij;ab}\,x_{ij;rs}\cdot x_{ab;uv}=\delta_{su}\cdot x_{kl;rv}\quad \forall r,s,u,v=1,\dots,d.$$

It would be quite unmanageable to try to understand T_A from the above messy-looking system of equations. Localization in the étale topology tells us that this system of equations defines a right principal homogeneous space over K under PGL_d.

5.5

The method of descent, including taking quotients of actions of reductive groups using Mumford's geometric invariant theory, have become a basic tool in algebraic geometry. Grothendieck's approach has been extended in two related directions: algebraic spaces due to M. Artin, and algebraic stacks due to Deligne–Mumford and M. Artin. These notions fits into an ascending sequence

$$\begin{aligned}\text{schemes} &\le \text{algebraic spaces}\\ &\le \text{Deligne–Mumford stacks}\le \text{Artin stacks}\end{aligned}$$

in their level of generality; the representability condition is weakened in each step, while important geometric structures are still preserved.

5.5.1

As an illustration of their usefulness, below are two statements for the existence of Picard schemes, both due to Artin, in [37, p. 186–187] and [36, Thm 7.3, p. 67] respectively.

(i) *Let $f: X\to S$ be a proper flat morphism of algebraic spaces. The relative Picard stack $\mathscr{PIC}(X/S)$ is an Artin stack.*
By definition the relative Picard functor $\mathscr{PIC}(X/S)$ assigns to every morphism $S'\to S$ the groupoid category of all invertible $\mathscr{O}_{X\times_S S'}$-modules on $X\times_S S'$.
(ii) *If moreover f is cohomologically flat, i.e. the formation of $f_*\mathscr{O}_X$ commutes with arbitrary base change $S'\to S$, then the relative Picard functor $\mathscr{P}ic(X/S)$ is represented by an algebraic space locally of finite presentation over S.*

5.5.2

Here is an easy example of Deligne–Mumford stack that appears "in nature"; see [55, §3] for more information. Let C be a connected smooth projective algebraic curve over $\mathbb{C}$ of genus $g\ge 2$. It is known that there exists a discrete cocompact subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})^0$ such that $C(\mathbb{C})$ is isomorphic to $\Gamma\backslash\mathfrak{H}$, where $\mathfrak{H}$ is the upper half-plane Let

$$\begin{aligned}\mathrm{Comm}(\Gamma)=:\{\gamma\in\mathrm{PSL}_2(\mathbb{R})^0\mid \gamma\cdot\Gamma\cdot\gamma^{-1}\cap\Gamma &\text{ has finite index}\\ &\text{in }\Gamma\text{ and in }\gamma\cdot\Gamma\cdot\gamma^{-1}\}\end{aligned}$$

For "most curves of genus g", Γ is of finite index in $\mathrm{Comm}(\Gamma)$. The stack quotient of $\mathfrak{H}$ by $\mathrm{Comm}(\Gamma)$ is a "stacky curve" $\mathfrak{Y}$, which comes with a morphism $C\to Y$. This stacky curve $\mathfrak{Y}$, called the *core* of Y in [55], has the property that every finite étale algebraic correspondence $C\leftarrow D\to C'$ factors through $\mathfrak{Y}$. One can say that $\mathfrak{Y}$ "controls" all algebraic correspondences of Y.

6. Etale Cohomology

I started thinking on the cohomology of schemes, after reading your notes which I find quite useful. (As for comments *of detail, we will discuss about it when you are here and we are organizing the seminar). I got a few results: … Of course, the main interest of 6) is to allow computations of cohomology in characteristic $p > 0$ from transcendental results in characteristic 0, just as for the fundamental group. Besides, the main steps in the key results 1) and 5) are the analogous statements on fundamental groups. The main techniques I developed so far in algebraic geometry have to be used: the existence theorem on coherent algebraic sheaves, nonflat descent, Hilbert and Picard schemes (the latter for nice relative curves only), Lefschetz techniques. Thus it was not so silly after all to postpone Weil cohomology after all this.*

Grothendieck to M. Artin, Feb. 14, 1963, [23, pp. 675–676]

6.1

According to [20] the exposition [67], where Serre introduced the notion of *locally isotrivial* algebraic fibrations (those which Zariski locally becomes trivial after passing to a finite étale cover), inspired Grothendieck to define the notion of étale topology. In [2, p. 103–104], Grothendieck wrote

> …, it seems clear now that the Weil cohomology has to be defined by a completely different approach. Such an approach was recently suggested to me by the connections between *sheaf-theoretic cohomology and cohomology of Galois groups on the one hand, and the classification of unramified coverings of varieties on the other* (as explained quite unsystematically in Serre's tentative Mexico paper), and by Serre's idea that a "reasonable" algebraic principal fiber space with structure group G, defined on a variety V, if it is not locally trivial, should become locally trivial on some covering of V *unramified* over a given point of V. This has been the starting point of a definition of the Weil cohomology (involving "spatial" and Galois cohomology), which seems to be the right one, and which gives clear suggestion how Weil's conjecture may be attacked by the machinery of Homological Algebra.

Grothendieck was optimistic that because the étale topology gives the "correct" Weil cohomology for H^1, it should also give the "correct" Weil cohomology for the higher H^i's. In the spring 1962 Harvard seminar [34], Artin gave a precise definition of the étale topology, computed the étale cohomology of constructible torsion sheaves on algebraic curves based on Tsen's theorem, and also the étale cohomology of algebraic surfaces fibered by curves. Then between September 1962 and March 1963, Artin and Grothendieck established the basic theorems of étale cohomology with torsion coefficients. The resulting ℓ-adic étale cohomology has since become a basic tools in algebraic geometry; its basic properties are documented in SGA4, SGA$4\frac{1}{2}$ and SGA5. We refer to the wonderful article [20] for a nuanced overview and also an excellent guide to the literature.

6.2

Why does the étale the topology produce a "correct Weil cohomology"? This is a question many students and non-experts may have. As indicated in [2], the étale cohomology fuses the cohomology of sheaves and also the cohomology of profinite groups. Given a smooth algebraic variety X, one can try to produce successive fibrations $U_{i.0} \to U_{i,1} \to \cdots \to U_{i,d}$, with $d = \dim(X)$, so that each arrow is a "fibration by smooth open curves", and the U_i's form a basis of open neighborhoods of X, broadly interpreted. Suppose we know the etale cohomology of curves, then one can hope to understand the etale cohomology of the U_i's by Leray spectral sequence, and eventually get to the cohomology of X.

From complex analysis, we know that for every smooth open algebraic curve X over $\mathbb{C}$ (i.e. the complement of finitely many points of an irreducible complete smooth algebraic curve over $\mathbb{C}$), the punctured Riemann surface $X(\mathbb{C})$ is a uniformized by a discrete Fuchsian subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})^0$: $X(\mathbb{C}) \cong \Gamma\backslash\mathfrak{H}$. In particular $X(\mathbb{C})$ is an Eilenberg-MacLane space $K(\Gamma,1)$: its higher homotopy groups $\pi_i(X(\mathbb{C})) = (0)$ for all $i \geq 2$. So the cohomology groups of $K(\Gamma,1)$ are the cohomologies of the group Γ. In particular for every locally constant torsion coefficient system L on $X(\mathbb{C})$, corresponding to a finite abelian group L with action by Γ, we have

$$\mathrm{H}^i(X(\mathbb{C}),\mathcal{L}) \cong \mathrm{H}^i(\Gamma,L) \cong \varinjlim_{\Delta} \mathrm{H}^i(\Gamma/\Delta,L) \quad \forall i \geq 0,$$

where Δ runs through all normal subgroups of Γ of finite index.

What about higher dimensions? In algebraic topology we know that from the homotopy exact sequence that a fibration over a $K(\Gamma_1,1)$ with fiber a $K(\Gamma_2,1)$ is again a $K(\pi,1)$. So for a good system of fibrations $U_{i.0} \to U_{i,1} \to \cdots \to U_{i,d}$ by curves, it is reasonable to expect that the "limit" of all finite etale coverings of $U_{i,0}$ is in some sense "acyclic for torsion coefficients", and the étale topology produces "the correct Weil cohomology" for $U_{i,0}$. The general formalism of homological algebra tells us that the same should hold at least for all smooth varieties.

6.3

We illustrate the étale cohomology in an example. The Diophantine equation involved was studied by Gauss. It "is" an elliptic curve over $\mathbb{Q}$ with (potential) complex multiplication by Gaussian integers, and has good reduction outside of 2. It is also the modular curve $X_0(32)$, which classified isogenies between elliptic curves whose kernel are cyclic of order 32.

6.3.1

Here is the *last entry* in Gauss's mathematical diary, July 7, 1814.

> A most important observation made by induction which connects the theory of biquadratic residues most elegantly with the lemniscatic functions. Suppose, if $a + bi$ is a prime

number, $a-1+bi$ divisible by $2+2i$, then the number of all solutions of the congruence

$$1 = x^2+y^2+x^2y^2 \pmod{a+bi}$$

including $x=\infty, y=\pm i, x=\pm i, y=\infty$, is $=(a-1)^2+b^2$.

A facsimile reproduction and a transcript we find in [43]. Also see [51], with the Last Entry on page 33. For a brief history see [44, page 97].

For any field K of characteristic different from 2, the equation $X^2+Y^2+X^2Y^2-1=0$ defines a smooth affine curve $E_{\mathrm{aff},K}$ over K. Let E_K be the complete smooth model of $E_{\mathrm{aff},K}$. It turns out that the genus of E_K is equal to 1. The complement $E_K \smallsetminus E_{\mathrm{aff},K}$ of $E_{\mathrm{aff},K}$ in E_K is the disjoint union of two copies of $\mathrm{Spec}(K[X]/(X^2+1))$. Clearly $E_{\mathrm{aff},K}$ has at least four K-rational points, with $(x,y) = (0,1),(0,-1),(1,0),\text{or }(-1,0)$. We pick the point $P_0=(0,1)$ to give a group law on E_K with P_0 as the unity element, so that E_K becomes an elliptic curve over K. The elliptic curve $E := E_{\mathbb{Q}}$ over $\mathbb{Q}$ has good reduction over $\mathbb{Z}[\frac{1}{2}]$ in the sense that there exists a one-dimensional abelian scheme $E_{\mathbb{Z}[1/2]}$ over $\mathbb{Z}[\frac{1}{2}]$ whose generic fiber is E. For every field K of characteristic different from 2, we have $E_K \cong E_{\mathbb{Z}[1/2]} \times_{\mathrm{Spec}(\mathbb{Z}[1/2])} \mathrm{Spec}(K)$.

The prediction of Gauss is that if p is a prime number with $p\equiv 1 \pmod 4$, write p as a sum of two squares $p=a^2+b^2$ with a odd and $a-1\equiv b \pmod 4$, then

$$\mathrm{card}(E_{\mathbb{F}_p}(\mathbb{F}_p)) = (1-a-\sqrt{-1}\,b)\cdot(1-a+\sqrt{-1}\,b) = p-2a+1.$$

Note that the conditions that $a-1\equiv b \pmod 4$ and a is odd uniquely determine the integer a. The first proof of this prediction by Gauss was given by Herglotz in [49].

Define $\mathbb{Z}[\sqrt{-1}] := \mathbb{Z}[X]/(X^2+1)$, $\mathbb{Q}(\sqrt{-1}) := \mathbb{Q}(X)/(X^2+1)$, to the effect that "$\sqrt{-1}$" denotes the image of X in either $\mathbb{Z}[X]/(X^2+1)$ or $\mathbb{Q}(X)/(X^2+1)$. (We make no association between the symbol $\sqrt{-1}$ and the complex number "i".) The ring $\mathbb{Z}[\sqrt{-1}]$ of Gaussian integers shows up naturally because elements of $\mathbb{Z}[\sqrt{-1}]$ define endomorphisms of the elliptic curve $E_{\mathbb{Z}[\sqrt{-1}][1/2]}$ (respectively E_L for every field L over $\mathbb{Z}[\sqrt{-1}][1/2]$); see Section 6.3.3 (2).

6.3.2

The theory ℓ-adic étale cohomology produces, for the elliptic curve E over $\mathbb{Q}$ with structural morphism $f: E\to \mathrm{Spec}(\mathbb{Q})$, a $\mathbb{Q}_\ell$-sheaf $\mathrm{R}^1 f_*\mathbb{Q}_\ell$ on $\mathrm{Spec}(\mathbb{Q})$ under the étale topology of $\mathrm{Spec}(\mathbb{Q})$. Such a sheaf over $\mathrm{Spec}(\mathbb{Q})_{\mathrm{et}}$ is naturally identified with a continuous linear representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on a finite dimensional $\mathbb{Q}_\ell$-vector space. For the sheaf $\mathrm{R}^1 f_*\mathbb{Q}_\ell$ on $\mathrm{Spec}(\mathbb{Q})_{\mathrm{et}}$, the underlying vector space is $V_\ell = \mathrm{H}^1_{\mathrm{et}}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$, of dimension 2 over $\mathbb{Q}_\ell$. The associated Galois representation ρ_ℓ is unramified outside of 2 because the elliptic curve E has good reduction over $\mathbb{Z}[1/2]$. We will explicitly describe this Garlois representation ρ_ℓ.

For each odd prime number p different from ℓ, the characteristic polynomial $\mathrm{ch}_p(T)$ of $\rho_\ell(\mathrm{Fr}_p^{-1})$ is a monic quadratic polynomial with integer coefficients of the form

$$\mathrm{ch}_p(T) = T^2 - A_pT + p, \quad A\in\mathbb{Z}.$$

Here Fr_p^{-1} denotes an element of the decomposition group above p which induces $x\mapsto x^{1/p}$ on the residue field $\mathbb{F}_p$. Moreover the integer A_p is determined by $\mathrm{card}(E(\mathbb{F}_p)) := 1-A_p+p$. Gauss's prediction amounts to a formula for A_p when $p\equiv 1 \pmod 4$.

As will be shown in Section 6.3.3 (2) below, $A_p=0$ if $p\equiv 3 \pmod 4$. Together with Gauss's formula of A_p for $p\equiv 1 \pmod 4$, we arrive at the following description of the Galois representation ρ_ℓ.

(i) For every prime number $\ell\neq 2$, the $\mathbb{Q}_\ell$-Zariski closure G_ℓ of the image of the Galois representation ρ_ℓ has two connected components. Its neutral component G_ℓ^0 is naturally isomorphic to the algebraic torus over $\mathbb{Q}_\ell$ whose set of R-points is $(R\otimes_{\mathbb{Q}}\mathbb{Q}(\sqrt{-1}))^\times$ for every commutative ring R over $\mathbb{Q}_\ell$.

(ii) The composition of the homomorhism $\rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to G_\ell$ with the surjection $G_\ell \twoheadrightarrow G_\ell/G_\ell^0$ is the quadratic character attached to the imaginary quadratic field $\mathbb{Q}(\sqrt{-1})$.

(iii) There is a one-dimensional Galois representation

$$\rho'_\lambda : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{-1})) \longrightarrow (\mathbb{Q}(\sqrt{-1})_\lambda)^\times$$

where $\mathbb{Q}(\sqrt{-1})_\lambda$ is the λ-adic completion of a place λ above ℓ, such that

$$\rho_\ell\otimes_{\mathbb{Q}_\ell}\mathbb{Q}_\lambda \cong \mathrm{Ind}_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-1})}(\rho'_\lambda),$$

the induced representation from $\mathbb{Q}(\sqrt{-1})$ to $\mathbb{Q}$ of ρ'_λ. The one-dimensional Galois representation will be made explicit in (v), following Gauss.

(iv) This one-dimensional character ρ'_λ of the Galois group $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{-1})\right)$ is unramified outside the prime ideal $((1+\sqrt{-1})\ell)\mathbb{Z}[\sqrt{-1}]$ of $\mathbb{Z}[\sqrt{-1}]$; its conductor away from ℓ is equal to $(1+\sqrt{-1})^3$, or equivalently $(2+2\sqrt{-1})$ in Gauss's expression. The conductor of ρ_ℓ away from ℓ is 2^5. The two conductors are related by the conductor-discriminant formula because ρ_ℓ is induced from ρ'_λ.

(v) The homomorphism ρ'_λ (and hence also ρ_ℓ) can be described in a manner that is independent of ℓ:

- Let $\mathcal{I}\left(\mathbb{Z}[\sqrt{-1}][\frac{1}{2}]\right)$ be the group of all fractional ideals of $\mathbb{Q}(\sqrt{-1})$ which are prime to 2. Let

$$\psi : \mathcal{I}\left(\mathbb{Z}[\sqrt{-1}][\tfrac{1}{2}]\right) \to \mathbb{Q}(\sqrt{-1})^\times,$$

be the homomorphism such that $\psi(\wp)$ is the generator $\pi_\wp$ of $\wp$ with $\pi_\wp \equiv 1 \pmod{(1+\sqrt{-1})^3}$. More explicitly,

$$\psi(\wp) = \begin{cases} a+\sqrt{-1}\,b & \text{if } \wp = (a+\sqrt{-1}b), \\ & a^2+b^2 = p, \\ & a \text{ odd}, \ a-1 \equiv b \pmod 4 \\ -p & \text{if } \mathbb{Z}[\sqrt{-1}]/\wp \cong \mathbb{F}_{p^2} \end{cases}$$

– For every *odd* prime ideal $\wp \subset \mathbb{Z}[\sqrt{-1}]$ which is relatively prime to ℓ, we have $\rho'_\ell(\mathrm{Fr}_\wp^{-1}) = \psi(\wp)$.

We note that the above homomorphism ψ is a *grössencharacter of type* A_0 in Weil's terminology. Property (iii) above implies that the Hasse-Weil zeta function $\zeta_E(s)$ attached to the elliptic curve E is

$$\zeta_E(s) = \frac{L(\psi,s)}{\zeta(s)\cdot\zeta(s-1)}$$

where $\zeta(s)$ is the Riemann zeta function, and $L(\psi,s)$ is the Hecke L-function attached to ψ.

(vi) For every power p^n of an odd prime number p, we have

$$\mathrm{card}\,(E_{\mathbb{F}_{p^n}}(\mathbb{F}_{p^n})) = \begin{cases} (1-\psi(\wp)^n))\cdot(1-\psi(\bar{\wp})^n) \\ \quad \text{if } p \equiv 1 \pmod 4, \ (p) = \wp\cdot\bar{\wp} \\ (1-\sqrt{-p}^n)\cdot(1+\sqrt{-p}^n) \\ \quad \text{if } p \equiv 3 \pmod 4 \end{cases}$$

where $\wp$ and $\bar{\wp}$ are the two prime ideals of $\mathbb{Z}[\sqrt{-1}]$ above p when $p \equiv 1 \pmod 4$.

6.3.3 Some Information About the Above Elliptic Curve

(1) *Other equations defining the same curve.*

Recall that x,y are the image of X,Y in the fraction of the integral domain $K[X,Y]/(X^2+Y^2+X^2Y^2-1)$. For every field K of characteristic different from 2, define rational functions

$$t := \frac{1-x^2}{y}, \quad v := \frac{2x+2}{-x+1}, \quad u := \frac{t(v+2)^2}{4}$$

on E_K, so that

$$x = \frac{v-2}{v+2}, \quad y = \frac{1-x^2}{v+2}, \quad t = \frac{u}{4(v+2)^2}.$$

Using (X,T) and (V,U) as coordinates, we get two other affine equations for E_K:

$$0 = T^2-(1+X^2)(1-X^2)$$

and

$$0 = U^2-V(V^2+4).$$

The point P_0 is $(X,T) = (1,0)$ in the (X,T)-coordinates, and is the "point at ∞" for the Weierstrass equation in (U,V). More information can be found in [59]. A non-zero holomorphic differential ω on E is given below in three coordinate systems:

$$\omega = \frac{dx}{y(1+x^2)} = \frac{dx}{t} = \frac{4\,dv}{u}.$$

(2) *Relation to lamniscatic elliptic integrals.*

This curve E is closely related to the lemniscatic elliptic integral

$$\int \frac{dx}{\sqrt{1-x^4}}.$$

See [72, pp. 524–525], in particular Example 4 on p. 525, which asserts that

$$\sin\mathrm{lemn}^2\phi + \cos\mathrm{lemn}^2\phi + \sin\mathrm{lemn}^2\phi\cdot\cos\mathrm{lemn}^2\phi = 1.$$

In other words the elliptic curve $E_{\mathbb{C}}$ is uniformized by the lemniscatic functions $\sin\mathrm{lemn}\,\phi$ and $\cos\mathrm{lemn}\,\phi$.

(3) *Number of elements of* $E_{\mathbb{F}_p}(\mathbb{F}_p)$ *when* $p \equiv -1 \pmod 4$.

Using the Weierstrass equation $0 = U^2-V(V+4)$ for E_K in (1), it is easy to determine the cardinality of $E_{\mathbb{F}_p}(\mathbb{F}_p)$ if $p \equiv -1 \pmod 4$. The curve $E_{\mathbb{F}_p}$ is isomorphic to the elliptic curve $E'_{\mathbb{F}_p}$ with Weierstrass equation $0 = Y^2+X(X^2+4)$. Because -1 is not a quadratic residue modulo p, for every element $a \in \mathbb{P}^1(\mathbb{F}_p)$, there are exactly two elements in $E_{\mathbb{F}_p}(\mathbb{F}_p) \sqcup E'_{\mathbb{F}_p}(\mathbb{F}_p)$ lying above a. So

$$2\cdot\mathrm{card}(E_{\mathbb{F}_p}(\mathbb{F}_p)) = 2\cdot(1+p) \quad \text{for } p \equiv -1 \pmod 4.$$

Explanation. As p is not split in $\mathbb{Q}(\sqrt{-1})/\mathbb{Q}$ in case $p \equiv -1 \pmod 4$ we see that E_K is supersingular for $K = \mathbb{F}_p$; hence A_p is divisible by p; for $p \geq 7$ the result $A_p = 0$ follows (as $A_p^2-4p < 0$). For $p = 3$ a small extra argument gives the same result.

(4) *Complex multiplication.*

The elliptic curve E_L admits complex multiplication by $\mathbb{Z}[\sqrt{-1}]$ for any field L over $\mathbb{Z}[\sqrt{-1},\frac{1}{2}]$, i.e. a field L of characteristic different from 2 plus a specified element $e \in L$ with $e^2 = -1$: there is an automorphism ι of E over L of order 4 such that

$$\iota^*(x) = \frac{1}{x}, \quad \iota^*(y) = ey, \quad \iota^*(t) = \frac{et}{x^2}, \quad \iota^*(u) = eu, \quad \iota^*(v) = -v.$$

(5) *Rational points.*

The four obvious $\mathbb{Q}$-rational points $(X = 0, Y = \pm 1)$ and $(X = \pm 1, Y = 0)$ are the only $\mathbb{Q}$-rational solutions of the equation $X^2+Y^2+X^2Y^2-1 = 0$. Hence $\#(E(\mathbb{Q})) = 4$. The elliptic curve E has conductor 32 as already mentioned, and its minimal discriminant is -2^{12}.

(6) *E is a modular curve.*

It turns out that the curve E is isomorphic over $\mathbb{Q}$ to the modular curve $X_0(32)$. Under the uniformization of (the non-cusp part of) $X_0(32)$ by the upper-half plane, the automorphism ι of E corresponds to the map $\tau \mapsto \tau + \frac{1}{4}$ on the upper-half plane.

7. The Monodromy Theorem

Je viens m'apercevoir qu'il y a une partie de tes conjectures avec Tate que se démontre de façon quasi-triviale, savoir celle précisément qui n'a pas d'explication «transcendante» evident:

Grothendieck to Serre, 24.9.64, [10, p. 182]

We discuss (a simplified version of) a proof by Grothendieck of the monodromy theorem, outlined in the 24.9.1964 letter to Serre [10, p. 182], and written down by Serre and Tate in the appendix of [69] as "Proposition (Grothendieck)" on page 515.

Ever since the work of Gauss, we have known the problem of monodromy; that is, to determine the representation of the fundamental group of a punctured curve (or a punctured open unit disc) S^0 associated to analytically continuing functions around the missing point. We may similarly try to understand the representation associated to the cohomology of a fibration over S^0. Gauss studied the substitutions corresponding to prolonging hypergeometric functions and Jordan proved that these substitutions generate a group (now called the monodromy group.)

The eigenvalues of such a monodromy representation are roots of unity; there have been many proofs in various situations. Here we show how the action of the arithmetic part on the geometric part of the fundamental group gives access to this result. First a motivating example:

Lemma 1. *Consider $K = \mathbb{Q}(T)$. For some $n \in \mathbb{Z}_{>1}$ consider*

$$f = X^n - T \in K[X]$$

and let L/K be the splitting field,

$$K = \mathbb{Q}(T) \subset E = \mathbb{Q}(\zeta_n, T) \subset L = \mathbb{Q}(\zeta_n, \sqrt[n]{T}).$$

Then we have an exact sequence

$$\begin{aligned} 1 \to N = \mathrm{Aut}(L/E) \cong \mathbb{Z}/n \to \mathrm{Aut}(L/K) \\ = G \to H := \mathrm{Aut}(E/K) \cong (Z/n)^{\times} \to 1. \end{aligned}$$

We leave the proof of this lemma to the reader. We will give an interpretation of this result in the vein of the geometric-arithmetic exact sequence (g-a) in Section 4:

- the *geometric* part: the covering of the affine line with coordinate T by the affine line with coordinate $\sqrt[n]{T}$ is analogous to an unramified cyclic-n cover of the open unit disk,
- the *arithmetic* part is $H = (Z/n)^{\times}$;
- moreover the exact sequence is a semi-direct product given by the natural $(Z/n)^* \to \mathrm{Aut}(N)$.

This observation, generalized to the geometric-arithmetic exact sequence (g-a), is the starting idea of the wonderful proof of Grothendieck of the monodromy theorem. Let us give a simplified version of the proof in the appendix as Proposition (Grothendieck) on page 515 of [69]. (However, the result and the proof below give the essence of the more general result.)

Lemma 2. *Let $M \in \mathrm{GL}(m, \mathbb{Q})$ be a $m \times m$-matrix over $\mathbb{Q}$, with determinant not equal to zero.* (This will be the monodromy matrix.) *Suppose there is an invertible matrix $S \in \mathrm{GL}(m, \mathbb{Q})$ and a positive integer $r \in \mathbb{Z}_{\geq 2}$ such that*

$$S^{-1} \cdot M \cdot S = M^r.$$

(The matrix S comes from the action of the Galois group on the geometric part, and this equality is the essence of Lemma 1.) *Then there exists $e \in \mathbb{Z}_{>0}$ such that for any eigenvalue $\lambda \in \overline{\mathbb{Q}}$ of M we have $\lambda^e = 1$.*

Proof of Lemma 2. Let $\Lambda \subset \overline{\mathbb{Q}}$ be the set of eigenvalues of M. Note that all these eigenvalues are unequal to zero. Then Λ is also the set of eigenvalues of $S^{-1} \cdot M \cdot S$. Hence

$$\lambda \mapsto \lambda^r \text{ gives a permutation of } \Lambda.$$

Iterating this permutation $(m!)$ times we obtain the identity map on Λ. This is just saying that

$$\lambda^{r^{(m!)}} = \lambda, \quad \text{hence} \quad \lambda^e = 1 \quad \text{with} \quad e := r^{(m!)} - 1.$$

This proves Lemma 2. $\square$

We invite the reader to read the more technical result as recorded in [69], proposition on page 515, and be convinced that the short and elegant argument above is the essence of the proof by Grothendieck.

A Remark on Terminology. In 19-th century mathematics (and also at present) we find the terminology "the monodromy theorem" for the fact that analytic continuation of an analytic function over a simply connected domain gives a univalent function (after a "dromos", a path, we get "mono", univalent, functions). Nowadays the terminology "monodromy theorem" is used more generally, so that both the local monodromy theorem discussed in this section and the purity of the monodromy weight filtration are included under the same name.

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For many periods of his life in 1970–2014 it was unclear to us how to reach Grothendieck, personally and also at his changing locations. Hence it came as a surprise that one of us received a gentle and interesting letter from Grothendieck, written on February 3, 2010, where the need of contact and communication was obvious. In this we see the gentle side of Grothendieck and also the need to express his feelings for the past. We reproduce a scan of this letter, and we provide a transcript.

Lasserre Feb. 3, 2010

Dear Frans Oort,

This letter is motivated by your "coffee table" expository paper on the "algebraic fundamental group" in the book "Geometric Galois Actions, 1", published in 1997. I was informed last month only, by the editors, of the existence of this book, and of the existence of the Luminy conference 27 august – 1 september 1995, of which this book is the first volume of Proceedings. (The editors met me the very next day of the conference, on September 2 1995, without ever mentioning the conference, nor the project of this book.) When looking through this book, I was struck at once by your presence among the contributors, and by your contribution. And I was glad to find there an earnestness, a warmth of tone, highly unusual nowadays in mathematical spheres, and even elsewhere. I then thought at once of writing you, with a word of thanks for your warm implication in this coffee-table account. This was Jan. 12, and in the meanwhile some other matters have occurred to me, about which to communicate with you, if you should be interested. First I must check if this letter reaches you at your former adress (as you may be presently retired). If so, I would appreciate getting your personal adress, and will then write more extensively about matters close to my heart.

At present, I restrict to an inquiry: did you get the copy of "Récoltes et Semailles" I certainly sent you, among the very first, with a word of dedication to you, by september 1985? I do not remember that I got any echo from you in response, at that time or later. This "Reflection and Testimonial" on my life as a mathematician, unreadable as it is I admit, has much meaning for me, if not to anyone else!

With my best regards, and hoping to read you very soon

Alexander Grothendieck

Letter from Alexander Grothendieck to Frans Oort, 2010.

Transcript of this letter.

Alexander Grothendieck to Frans Oort
Lasserre February 3, 2010.

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With my best regards, and hoping to read you very soon

Alexander Grothendieck

Later comments by FO:

- The "coffee table" expository paper Grothendieck mentions is in [62]. We describe the basic idea in Section 7 above.
- The copy of "Récoltes et Semailles" mentioned in Grothendieck's letter never reached me;
- an answer to this letter was opened, resealed, and came back with the text: "retour á l'envoyeur [personal address missing]".
- a letter of admiration from an audience at UPenn was returned unopened.

Alas – further attempts for communication were in vain.