Open Problems

compiled by Kefeng Liu[*] and Hao Xu[†]

Note. The readers are welcome to propose the solutions. The authors should send their solutions to liu@math.ucla.edu and post their solutions in arXiv. The correct solutions will be announced and some souvenirs will be presented to the solvers.—*The Editors*

Problem 2016005 (Spectral Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Can one characterize those manifolds whose eigenfunctions of the Laplacian have uniform bound, independent of the eigenvalues. The same question for graphs and all generalized Laplacians. A good estimate will give means to use eigenfunctions to cut the graph in an efficient manner.

Problem 2016006 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

In my paper with Schoen [1] on the existence of black hole due to condensation of matters, we introduce a concept of radius: Let X be a closed Jorden curve in M which is homotopically trivial, it is homotopically trivial in a tubular neighborhood of radius r in X when r is small. There is the first number r that X is homotopically trivial. Such r can be called the Rad of X. Take supremum of the Rad of X for all X, we call this the Rad of M. Since this is related to closed curves, we denote it by $\mathrm{Rad}(1,M)$. We can use higher homotopy groups of M and get $\mathrm{Rad}(i,M)$. Are there relations between them? In the paper with Schoen, we observed that on a three dimensional manifold, we can give an upper estimate of the first eigenvalue of the operator $-\Delta+\frac{1}{2}R$, where Δ is the Laplacian and R is the scalar curvature of, in term of $\mathrm{Rad}(1,M)$. What about the other $\mathrm{Rad}(i,M)$?

[*] Department of Mathematics, University of California at Los Angeles
E-mail: liu@math.ucla.edu
[†] Department of Mathematics, University of Pittsburg, Pennsylvania, U.S.A.
E-mail: mathxuhao@gmail.com

[1] R. Schoen and S.-T. Yau, The existence of a black hole due to condensation of matter. Comm. Math. Phys. 90 (1983), no. 4, 575–579.

Problem 2016007 (Spectral Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

For an algebraic manifold with a Kähler metric, in my paper with Bourguignon and Peter Li [1], we gave an upper estimate of the first eigenvalue of the Laplacian in terms of its volume and the degree of its embedding into a projective space. We expect that there is an upper estimate of the i-th eigenvalue in terms of i and similar quantities. For Riemann surfaces, this was proved by N. Korevaar [3] for the case of Riemann surfaces, in answering a question I asked.

In my paper with Christodoulou [3] on the positivity of Hawking mass, we gave an upper bound of the gap of the first two eigenvalues of the operator L for Riemann surface, where L is equal to the Laplacian plus a constant multiple of the scalar curvature, in terms of the area and the Euler number of the surface. Will similar estimates hold for Kähler manifolds? Is this related to the stability of Kähler manifolds?

[1] J.-P. Bourguignon, P. Li and S.-T. Yau, Upper bound for the first eigenvalue of algebraic submanifolds. Comment. Math. Helv. 69 (1994), no. 2, 199–207.
[2] D. Christodoulou and S.-T. Yau, Some remarks on the quasi-local mass. Mathematics and general relativity (Santa Cruz, CA, 1986), 9–14, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988.
[3] N. Korevaar, Upper bounds for eigenvalues of conformal metrics. J. Differential Geom. 37 (1993), no. 1, 73–93.

Problem 2016008 (Complex Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

It was proved by Siu [1, 2] that plurigenera are deformation invariant within projective algebraic manifolds. But except in special case, it is not known for Kähler manifolds [3]. A smooth manifold may support wo complex structures whose Kodaira dimensions are different. But for a given smooth manifold that support two different two complex structures of general type, would the complex structures have the same Chern classes?

Without the general type condition, they do not even have to have the same Chern numbers. LeBrun (Pacific J. Math. 1999) found pairs of non-Kähler 3-fold examples of Kodaira dimension 0. Kotschick (J. Topol. 2008) then gave Kähler 3-fold pairs of Kodaira dimension $-\infty$. Libgober and Wood (Topology 1982) found diffeomorphic complete intersections of different multidegrees with the same Chern classes. Perhaps the first example of this type was due to Calabi in the sixties where he constructed a different complex structure over the complex 3-torus, that has nonzero first Chern class. Perhaps we can ask a slightly restrictive question. If a diffeomorphism of two complex manifolds of general type preserves the first Chern class, will the diffeomorphism also preserves other higher Chern classes?

Suppose we have two complex structures of general type over the same smooth manifold and assume they have the same Chern classes, do they have the same plurigenera? If the canonical bundle K is ample, then by Hirzebruch-Riemann-Roch and Kodaira vanishing theorem, this is true for $p_m = \dim \Gamma(mK)$, $m \geq 2$.

[1] Y.-T. Siu, Invariance of plurigenera. Invent. Math. 134 (1998), 661–673.

[2] Y.-T. Siu, Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semipositively twisted plurigenera for manifolds not necessarily of general type. In: Complex Geometry, ed. I. Bauer et al., Springer-Verlag 2002, pp. 223–277.

[3] M. Levine, Pluri-canonical divisors on Kähler manifolds, Invent. Math. 74 (1983), 293–903. II. Duke Math. J. **52** (1985), 61–65.