Open problems

compiled by Kefeng Liu* and Hao Xu†

Note. The readers are welcome to propose the solutions. The authors should send their solutions to liu@math.ucla.edu and post their solutions in arXiv. The correct solutions will be announced and some souvenirs will be presented to the solvers.—*The Editors*

Problem 2016001 (Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

In 1974, Lawson and Yau proved that for compact spin manifolds with nonzero KO-characteristic number, it does not admit effective non-abelian compact Lie group action. In general, such manifolds may admit effective circle action. Can one find conditions on these manifolds so that they do not admit n-dimensional torus action with $n > 1$? Can we find conditions on a manifold which does not support three non-vanishing vector fields whose bracket relations are the same as that of $\mathfrak{su}(2)$? Same question for two non-vanishing vector fields whose Lie bracket is identically zero. In general, one can ask whether we can find a set of non-vanishing vector fields that satisfy the relation of a finite dimensional Lie algebra. Also find a condition on the manifolds which do not support simple finite non-abelian group actions.

Problem 2016002 (Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Is there a positive lower estimate of the volume of a compact Einstein manifold with scalar curvature $+1$ that depends only on the dimension, if the manifold satisfies one of the following conditions: (a) It is 4 dimensional; (b) It is even dimensional and locally irreducible; (c) It is even dimensional and simply connected.

Wang-Ziller [1] showed this was not true of Einstein metrics on the simply connected 5-manifold $S^2 \times S^3$. Van Coevering [2] more recently constructed counter-examples on connected sums $(S^2 \times S^3)\#\cdots\#(S^2 \times S^3)$.

[1] M. Wang and W. Ziller, Einstein metrics with positive scalar curvature. Curvature and topology of Riemannian manifolds (Katata, 1985), 319–336. Lecture Notes in Math., 1201. Springer, Berlin, 1986.

[2] C. van Coevering, Sasaki-Einstein 5-manifolds associated to toric 3-Sasaki manifolds. New York J. Math. 18 (2012), 555–608.

Problem 2016003 (Complex Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Given two Calabi-Yau manifolds defined over integers which are mirror to each other in the sense of string theory, what is their relation when we define them mod p? Over characteristic zero, we can relate Kodaira-Spencer theory of one manifold to the counting of algebraic curves in the other one. Do we have similar properties over characteristic p? Candelas et al. studied this problem over p for quintic threefolds. But it is still not clear what the general cases will be.

Problem 2016004 (Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

In 2008, B. Farb and S. Weinberger [1] conjectured the following: For a compact aspherical, smoothly irreducible manifold such that it's fundamental group contains no nontrivial normal abelian subgroups, if there exists a positive constant C depending only on the fundamental group such that $[\mathrm{Isom}(\widetilde{M}) : \pi_1(M)] > C$ where $\widetilde{M}$ is the universal cover of M. Then M is a locally symmetric space. Is there any way to determine

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whether a closed aspherical manifold to be diffeomorphic to a locally symmetric space based on information on topology alone? For example, can we replace group of isometrics of the universal cover by a semisimple Lie group acting smoothly on the universal cover of M. Note that M is called smoothly irreducible if no finite cover of M is diffeomorphic to a product manifold.

[1] B. Farb and S. Weinberger, Isometries, rigidity and universal covers. Ann. of Math. (2) 168 (2008), 915–940.