Reflections on studies and outlooks

by Theodore Yao-Tsu Wu*

With long admiration for The International Congress of Chinese Mathematicians, Presided by Prof. Shing-Tung Yau of The Harvard University, it is a great pleasure to present some of my representative works by self-learning studies with reflections and outlooks in view. In high spirit, I wish to present some of my studies on characteristic subjects for mutual interest.

1. Group Self-Learning

I recall in fond memory on attending the Provincial Shanghai High School where we students were enthused conducting class-group studies aspired to pursue from excurricular books in addition to what we were already receiving epic teaching by our most famed teachers. As I recall, we studied geometry from a voluminous college-book found by Lu Ting, a classmate who is now teaching at the Courant Institute in New York. We rejoiced from resolving such problems as the *Nine-point circle set on a generic triangle*, which we pursued separately for all the individual solutions providing the group discussions at last. That was a huge step to the differential geometry, as we realized later for being benefited from earlier preparations.

2. Base-Matrices Elimination

For resolving a set of $n \geq 2$ linear equations, e.g., with a generic regular matrix $A = A_{n\times n}$, $A\boldsymbol{x}_{n\times 1} = \boldsymbol{b}_{n\times 1}$ say, the *Gaussian matrix elimination* is widely useful. It operates to reduce the matrix $A = [a_{ij}]$ $\forall |A| \neq 0$ by row- and/or column-vectors reductions in order to obtain A^{-1}, the inverse of A, $\exists A^{-1}A = AA^{-1} = I_n$, the unit matrix.

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A new approach is by mapping the unit matrix I_n backward into A (rather than conventionally mapping A into I_n) by applying three basic operators called the *base matrices*,

(1a)
$$E_I : E_{R_i \leftrightarrow R_j}, \quad E_{II} : E_{R_k \mapsto aR_k}, \quad E_{III} : E_{R_i \mapsto R_i + bR_j}, \quad (a, b \in R),$$

where they signify: i., interchanging row R_i and row R_j; ii., row $R_k \mapsto a \times R_k$; and iii., row $R_i \mapsto [R_i + bR_j]$, and with each inverse given by

(1b)
$$E_I^{-1} : E_{R_i \leftrightarrow R_j}, \quad E_{II}^{-1} : E_{R_k \mapsto a^{-1}R_k}, \quad E_{III}^{-1} : E_{R_i \mapsto R_i - bR_j}.$$

Example 1 (Base-matrices elimination). *This case first exemplifies for inverting the matrix*

(1c)
$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix},$$

By (1a), (1b), this matrix A can be mapped from I_2 by

(1d)
$$A = E_2 E_1 I_2 \equiv PI_2, \quad E_1 = E_{R_1 \mapsto R_1 + R_2}, \quad E_2 = E_{R_2 \mapsto R_2 + 2R_1},$$

hence by (1a)-(1d), the inverse A^{-1} of A is given by

(1e)
$$A^{-1} = P^{-1} I_2 = E_1^{-1} E_2^{-1} I = E_{R_1 \mapsto R_1 - R_2} E_{R_2 \mapsto R_2 - 2R_1} I_2$$
$$= \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \equiv B, \quad \longrightarrow \quad B = A^{-1},$$

for $BA = AB = I_2$, hence $B = A^{-1}$, resulted from applying solely the relevant base matrices.

Therefore, by induction to order $n \geq 2$, there affords the proclamation as:

Theorem 1 (Base-matrices elimination). *By using base matrices, $E_j's$ $(j = 1, \cdots, n,)$ if*

(1f)
$$A = A_{n\times n} = E_\ell \cdots E_2 E_1 I_n \equiv PI_n, \quad \longrightarrow$$
$$A^{-1} = P^{-1} I_n = E_1^{-1} E_2^{-1} \cdots E_\ell^{-1} I_n,$$

which has the inverse A^{-1} produced by using solely the relevant base matrices. This also shows that the base matrices can provide all regular matrices, A_n, just like the base vectors for building all vectors.

Therefore, this establishes the *base matrices elimination*, in parallel to Gaussian's matrix elimination. They differ soundly between that *any generic regular matrix A_n and the unit matrix I_n can be related all by the base matrices, whereas Gaussian's matrix elimination may differ in general.*

3. Uniformly Continuous Analytica Functions

Cauchy's integral theorem and integral formulas (1855) provide the values of the Cauchy integral, $J[f(z)]$, for an analytic function $f(z)$ of a complex variable $z = x + iy$ in the Argand z-plane, as shown by

$$\text{(2a)} \quad J[f(z)] \equiv \oint_C \frac{f(t)}{t-z} dt = 2\pi i f(z) \quad (z \in \mathcal{D}^+ - \text{ open domain inside } C),$$

$$\text{(2b)} \quad = 0 \quad (z \in \mathcal{D}^- - \text{ open domain outside } C),$$

with $f(t)$ integrated over Cauchy's kernel, $(t-z)^{-1}$, around a Jordan contour C. Here, (2a) is the *integral formula*, providing the value for $J[f(z)]$ as shown within domain $\mathcal{D}^+$ inside contour C, and the value $J[f(z)] = 0$ in domain $\mathcal{D}^-$ outside C, since the integrand of $J[f(z)]$ is then regular for $z \in D^-$. However, it is so conspicuous that (2a)–(2b) provide no value for $J[f(z)]$ when z lies exactly on contour C, since contour C is left out by *domains $\mathcal{D}^+$ and $\mathcal{D}^-$, both being open.*

The task for extending (2a)–(2b) to cover also contour C was taken by Wu [7]. Despite the continuity condition for $f(z)$ on contour C as so asserted by Cauchy, Wu extended it to a new one assuming that

$$\text{(2c)} \quad f(z) \in C^n - \text{ class } \forall (n < \infty, \ |z - z_0 \in C| < \epsilon \ll 1),$$

i.e., $f(z)$ is n $(< \infty)$-times continuously differentiable in a neighborhood striding across contour C on both sides. This then enables contour C to be indented for letting a generic point z_+ in $\mathcal{D}^+$ and a point z_- in $\mathcal{D}^-$ both tending to a generic point z_o exactly on contour C for the integral retaining its value intact, since contour C is thus being never crossed by z_+ and z_- in integration, as shown in Figure 1.

This assumption (2c) is first necessary because the $f(z)$ in $J[f(z)]$ may well also be such a function as so assumed by (2c). Thus, by apt proceedings with counting the residues at the indented contour, hence yielding

$$\text{(2d)} \quad (I): \ f_+(z_o) = f(z_o); \qquad (II): \ f_-(z_o) = 0;$$

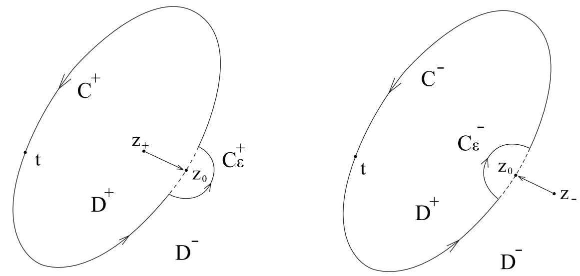

Figure 1. A Cauchy integral $\oint_C f(t)dt/(t-z)$ around a contour C enclosing domain $\mathcal{D}^+$ and excluding domain $\mathcal{D}^-$ has its contour deformed, $C \mapsto C^{\pm}$, with only a small semicircle $C_\epsilon^{\pm}$ indented onto the $\mathcal{D}^{\mp}$ side, centered at a generic point $z_0 \in C$ so as to let point $z_\pm \in \mathcal{D}^{\pm}$ tend, respectively, to $z_0 \in C$ without crossing $C^{\pm}$.

$$(III): \ f(z_o) = \frac{1}{\pi i} \mathcal{P} \oint_C \frac{f(t)}{t - z_o} dt \quad (z_o \in C),$$

where relations (I)–(II) prove that $f(z)$ is *uniformly continuous in the entire z-plane, and relation (III) for the uniform convergence of the generalized integral formula of its principal value twice that of (2a).* Thus, what was assumed by (2c) for $f(z)$ of $J[f(z)]$ behaving merely in a neighborhood striding across contour C is now proved to cover the entire z-plane, thereby lending the theory valid with the complete consistency for having multiple merits such as in comprising all the derivatives of integral $J[f(z)]$ uniformly continuous everywhere. This invaluable result bears more implications, one lending success in establishing the following new field.

4. Generalized Line-Integral Transforms

First, it may ascertain if there exists an integral analog of the Cauchy-Riemann differential relations between the conjugate functions $u(x,y)$ and $v(x,y)$ of analytic function $f(z) = u + iv$. This leads to generating the *generalized Hilbert-type transforms.* Critically, they appear to rely on the geometry of contour C whether it is inevitable for reaching the goal.

Above all, *four such geometric regions readily exist. They are (i), upper-half Argand plane; (ii), lower-half Argand plane; (iii), inside and (iv), outside of a unit disc.*

To exemplify, we take the pioneering case (i) given by David Hilbert (1862–1943), which is for an analytic function $f(z)$ regular in domain $\mathcal{D}^+$ in the upper half z-plane, bounded by the upper semicircular contour C_u, use of which renders (2d) to becoming

$$(I): \ f_+(x) = f(x),$$

(2e) $(II):\ f_-(x)=0,$

$$(III):\ f(x)=\frac{1}{\pi i}\mathcal{P}\int_{-\infty}^{\infty}\frac{f(\xi)}{\xi-x}d\xi \quad (x\in C),$$

since the integral round the upper semi-circle vanishes in the limit. Hence, substituting $f(x)=u(x)+iv(x)$, $f(\xi)=u(\xi)+iv(\xi)$ in (2e) yields:

(2f)
$$u(\xi)=H[v(x)]=\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{v(x)dx}{x-\xi},$$
$$v(x)=H^{-1}[u(\xi)]=\frac{-1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{u(\xi)d\xi}{\xi-x}.$$

Here, we note that all the four pairs of this four groups of the transforms given in (2f) are as proved, based on ((1a)-(1f))-((2a), (2b)), by invoking (2d)-(2e), also with the original pair.

In (2f), $u(x)$, $v(x)$ are said to be *conjugate to each other*, in *skew-reciprocal signs*. The reciprocity has $f(x)=u(x)+iv(x)(=u(x,0)+iv(x,0))$ along the $y=0$-axis afford a new proof that a generic analytical function $f(z)$ is determined in all the Argand plane if either $u(x)$ or $v(x)$ is given, a factor well known.

Further, the successive transforms of H and its inverse H^{-1} yield the unity operator as

(3a) $$H^{-1}H[v(x)]=v(x),\quad HH^{-1}[u(\xi)]=u(\xi),\quad \longrightarrow$$
$$H^{-1}H=HH^{-1}=I \quad \text{(unity operator)}.$$

These two formulas are shown for specific $u(x)$ or $v(x)$ by consecutive transform integrals as exemplified. In general, for arbitrary $v(x)$ (or $u(x)$), interchanging the order of integrations in (2f) yields

(3b) $$H^{-1}H[v(x)]=\frac{1}{\pi^2}\int_{-\infty}^{\infty}v(\xi)d\xi\int_{-\infty}^{\infty}\frac{dt}{(t-\xi)(t-x)}=v(x),$$

with a product of two Cauchy kernels therein. This clearly implies the principle of generalized distribution for Dirac's delta function such that

(3c) $$\int_{-\infty}^{\infty}\frac{dt}{(t-\xi)(t-x)}=\pi^2\delta(x-\xi),\quad \longleftrightarrow$$
$$\delta(x)=\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{dt}{(t-x)t}=\delta(-x),$$

as much previously marked for vital applications and further developments.

For most general cases, (3c) is still abiding by the *Poincaré-Bertrand formula:*

(3d)
$$\int_L\frac{dt}{t-x}\int_L\frac{f(t,\xi)}{t-\xi}d\xi=\pi^2 f(x,x)+\int_L d\xi\int_L\frac{f(t,\xi)dt}{(t-x)(t-\xi)}$$
$$(\text{(Poincaré-Bertrand formula)},\ \text{for } x\in L)$$

where L is a regular Jordan arc, assumed finite (or infinite) in length, with end-points at $t=a$ and $t=b$ and without double point, the integration variable t moves from a to b, and function $f(t,\xi)$ is assumed regular in a neighborhood of the entire line L, while each of the integrals assumes its own principal value ever, as understood.

If the function f depends only on one variable, $f=f(t)$, then (3d) reduces to

(3e)
$$\int_L\frac{d\xi}{\xi-x}\int_L\frac{f(t)}{\xi-t}dt=\pi^2 f(x)\quad \text{(Poincaré-Bertrand formula)},$$

which is called, as for (3d), also the Poincaré-Bertrand formula.

5. Matched Perturbation Expansion Theory

This theory is first given to rise in fluid mechanics. Intuitively speaking, consider a solid body, of typical length L, moving with velocity U through a viscous fluid (air and water being viscous, albeit weakly), the viscous effects exert concentrated in a neighborhood of the body surface $S_b(\boldsymbol{x})$, having both the body and fluid velocities kinematically equal at body surface, giving rise to the *no-slip condition*. Dynamically, swirling *vortices (or eddies)* of all sizes are generated at solid surfaces whilst being retarded and diffused sidewise by viscosity in producing a thin layer enveloping the body surface, reckoned as the *laminar boundary layer*, (well streamlined, as those seen with model aircraft, or might become turbulent due to flow instabilities), whereas outside this thin layer the flow is *inviscid, or non-viscous, or called potential, as the velocity vector can be derived from a scalar potential*. The eddies carried downstream form also a *viscous flow*. However, the eddies gradually decay off by nature, owing to the intrinsic viscous friction within the flow wake. Eventually, the wake flow reduces back into a uniform potential flow far downstream, being all inertial in nature just like it in Newton's law. So the final steady flow comprises the inner viscous flow, with a potential flow trailing.

The contrast between the two flows is characterized by a parameter called the *Reynolds number*, Re. The Reynolds number is the ratio of the inertial force to the viscous force as

(4) $$Re=\rho U^2L^2/\mu UL=\rho UL/\mu=UL/\nu,$$

where the defining ratio is that between the typical two forces, with μ being the dynamic viscosity and ν the kinematic viscosity (per unit mass) of the fluid of density ρ.

To illustrate a boundary layer being generated, set a flat plate of length L and ignorable thickness in parallel to a uniform unbounded free stream of velocity U, in an apt Re range, thereby producing a steady boundary layer, $y = \pm\delta(x)\ \forall(0 \le x < L)$ symmetrically on both sides, we may proceed evaluating this layer iteratively as follows. At first, the thin layer is set to vanish, $\delta(x)_0 \equiv 0$, it then lends the first-order outer flow velocity $u(\boldsymbol{x}) \simeq U_0(\boldsymbol{x}) = U$ in the reduced outer region. Next taking the inner free variable $\hat{\boldsymbol{x}}_1 = (x, y/\delta_1) \equiv (x, Y_1)$ for describing the inner flow inside the *expanded region* $(0 < y < Y_1)$ with $v/u = O(\delta_1/L)$, thus with $Y_1 = \delta_1$ yet undetermined so as to obtain the 1st-order inner flow velocity (e.g. by means of elementary methods) $u_1(\hat{\boldsymbol{x}}_1)$ as well as the outer flow velocity $U_1(\boldsymbol{x})$ beyond the expanded layer $S_b(\boldsymbol{x})_1 = S_b(\boldsymbol{x}) + \delta_1$. Then by matching both flow fields of $U_1(\boldsymbol{x})$ and $u_1(\hat{\boldsymbol{x}}_1)$ at their common boundary along $S_b(\boldsymbol{x})_1 = S_b(\boldsymbol{x}) + \delta_1$, which gives a scalar equation with no new unknowns appearing, hence precisely determines $\delta_1(x)$ as the sole unknown to bring forth the first-order solution to completion. Similarly, the same scheme can apply in analogy to yield solutions of higher orders by induction, thus having illustrated the perturbation theory, whenever the scheme converges. This therefore proves that the matching step is the vital step in this theory, hence aptly called the *matched perturbation theory*.

In operating this matched perturbations stepwise, a singular trend of the flow solutions is found arising at the leading edge of the plate. Such a limit of this matched perturbation scheme can also be simulated by the intimately associated differential equation given by

(5a) $$\epsilon\frac{d^2u}{dx^2} + a\frac{du}{dx} + bu = 0 \quad (u(0) = A,\ u(1) = B;\ (0 \le x \le 1; A,\ B \in R),$$

which is singular with $0 \le \epsilon \ll 1$ being the coefficient of the leading term in (5a).

In the limit equation, $(\epsilon = 0)$, (5a) reduces by one order, hence can adopt only one of the two conditions, yet aptly defines the outer equation as

(5b) $$a\frac{du}{dx} + bu = 0,$$

holding valid in the outer region, R_o say, spanning $0 < \delta\ (= O\epsilon) < x \le 1$, with the outer variable x, intact from (5a), by assuming its outer expansion in a power series of ϵ given by

(5c) $$u(x;\epsilon) \cong u_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \cdots, \quad (u_0(1) = B,\ u_j(1) = 0,\ j = 1, 2, \cdots),$$

where the various $u_j(x)$ terms satisfy the following non-homogeneous differential equations:

(5d) $$a\frac{du_0}{dx} + bu_0 = 0, \quad a\frac{du_j}{dx} + bu_j = -\frac{d^2u_{j-1}}{dx^2}, \quad (j = 1, 2, \cdots),$$

in which $u_0(x)$ has the solution given by

(5e) $$u_0(x) = Be^{(1-x)^{b/a}}, \quad \longrightarrow \quad u_0(1) = B, \quad u_0(0) = Be^{b/a} = C,$$

say, with $C \ne A$ in general, hence a boundary layer exists in the inner region, $R_i\ \forall(0 \le x < \delta = O(\epsilon) \ll 1)$ say. The suitable inner variable is Poincaré's *expanded variable* $X = x/\epsilon$, since it uniquely converts (5a) into an equation essentially with the same comparable coefficients for yielding the inner equation given as

(6a) $$\frac{d^2u}{dX^2} + a\frac{du}{dX} + \epsilon bu = 0 \quad (X = x/\epsilon,\ u(0) = A),$$

with one boundary condition at $X = 0$. Likewise, the adequate inner expansion is

(6b) $$u(x;\epsilon) \cong U_0(X) + \epsilon U_1(X) + \epsilon^2 U_2(X) + \cdots, \quad X = x/\epsilon \quad (U_0(0) = A,\ U_j(0) = 0,\ j = 1, 2, \cdots).$$

Thus, the $U_j(X)$'s satisfy their equations in the inner limit given as

(6c) $$\frac{d^2U_0}{dX^2} + a\frac{dU_0}{dX} = 0, \quad \frac{d^2U_j}{dX^2} + a\frac{dU_j}{dX} + bU_{j-1} = 0 \quad (U_0(0) = A,\ U_j(0) = 0,\ \forall j = 1, 2, \cdots).$$

Their inner solutions are readily derived to obtain as

(6d) $$U_0(X) = Ae^{-aX} + B_0(1 - e^{-aX}),$$

where the constant B_0 comes from matching $U_0(X)$ with $u_0(x)$.

It is convenient to adopt an intermediate limit for this problem given by

(6e) $$\text{Limit}(\epsilon \to 0):\ x_\eta = \frac{x}{\eta(\epsilon)}, \quad O(\epsilon) < O(\eta) < 1,$$

by means of which it can be shown that

(6f) $$u(x;\epsilon) = Be^{(1-x)^{b/a}} + (A - Be^{b/a})e^{-ax/\epsilon} + o(1),$$

is the first approximation uniformly valid in $0 \le x \le 1$. Higher approximations can be pursued in analogy, iteratively.

This theory was a primary pursuit for vital generalization taken by Prof. Paco Lagerstrom at Caltech who was a leading master on the Navier-Stokes theory in fluid mechanics and mathematics since 1940 or so with his research group. That was a remarkable group, pursuing studies in high spirit and serenity, sharing warmly closed friendship, and gathered daily for revealing day's achievements at evening coffee-shop break. Archives publications from the group on the central fields together with diversified and profound applications was so abundant that the papers so promoted to have appeared in the leading professional journals from allover were found full of the same coined terms like *inner and outer variables for* *inner and outer expansions*, *etc.* This dynamic tide surely illustrated the tremendous power of the perturbation expansion methods being so effective that numerous problems formerly hindered from resolution then readily solved, hence deserving to be attributed to the new methods.

In this regard, Wu and students also resolved problems of interest. One is concerning the classic problem for water-ski of a two-dimensional flat plate, of length L, advancing initially at an infinitesimal incident angle α, returning a very thin spray sheet to the front infinity over the quiescent surface of a deep water at rest, being a status-quo equivalent to the lower half solution-space of an unbounded uniform stream past the plate as a wing with lift, thereby yielding half or so lift to the water-skiing plate. This initial free-surface flow is next extended by adding more weight onto the plate to be more deeply submerged, hence rendering the spray-sheet thicker as well as increasing its up-shooting angle β leaving the plate. Critically, the spray sheet shoots vertically upward at $\beta = 90°$, then falling behind in forming a *cavity flow*, with the former free surface forming the inside cavity surface, leaving the plate in the new cavity flow to lose about half its wing lift owing to having the low suction pressure on the wing top-side replaced by higher cavity pressure. (Cf. [4]).

Another task is Wu's Ph.D. thesis expounding on the Navier-Stokes flow of a compressible fluid embedded with a stationary point-source of heat, a rather unique study in this respect. The diversified proceedings require analytical computations, in scrutiny, for the internal molecular motions of the fluid, in translational, rotational, and vibrational modes all required to be accounted in precision as high as possible. This is necessary for the flow so heated near the source origin prior to being diffused and convected away at subsonic speeds, or along the shock waves and flow wake at supersonic speeds. It was given an inquery from Prof. Theodore von Kármán, Wu's much revered grand teacher, the teacher of Prof. H. S. Tsien, or splendidly known later as Qian, Xue-sen in China, my post-doctoral advisor earlier at Caltech. In gentle smile, the grand teacher nodded, knowing that in supersonic speeds, the flow propagating along the shock waves slanted back from the heat-source is splendidly isentropic, holding a constant value of entropy whenever the flow analytically computed in the neighborhood surrounding the heat-source had been achieved in great precision, yet would be more and more remiss in retaining the epic isentropy had the computational results found less and less accurate in particular. Reflecting on such verifications, the grand teacher would clap in smile, saying that "This is good, for it shows the all-embracing beauty of the nature."

6. Fluid Mechanical Boundary-Layer Theory

As regards boundary layer in fluid flows, it has been explained physically in clarity. We continue to study its structure and properties. In expounding *two-dimensional fluid flows* of various kinds, the fundamental base is a streamline, merely one is for all kinds of flows, and a bunch is a flow field. For the base it is the *stream function*, $\psi(x,y)$, which provides the velocity $\boldsymbol{u}(\boldsymbol{x}) = (u,v,0)$, $\boldsymbol{x} = (x,y,0)$, and its total differential $d\psi$ given by

$$\text{(7a)}\qquad \begin{aligned} u(x,y) &= \frac{\partial \psi}{\partial y}, \quad v(x,y) = -\frac{\partial \psi}{\partial x}, \\ d\psi &= -vdx + udy = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy, \end{aligned}$$

where $d\psi = 0$ indeed signifies a streamline with $v/u = dy/dx \ \forall d\psi = 0$. The vorticity, $\omega(x,y)$, realized as $\omega \equiv \boldsymbol{e}_3 \cdot \nabla \times \boldsymbol{u}(\boldsymbol{x})$ ($\boldsymbol{x} = (x,y,0)$, $\boldsymbol{u} = (u,v,0)$, which has one component perpendicular out of the paper) given as

$$\text{(7b)}\qquad \omega(x,y) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\nabla^2 \psi,$$

a result of clarity signifying that the vorticity $\omega(x,y)$ is invariant along each abd every steamline for $\psi = const.$, or, in other words, *vorticity propagates on steamlines.*

To illustrate boundary layers, an apt example is for a uniform free stream past a flat plate in parallel, of length unity and ignorable thickness, with its boundary layer denoted by δ.

At first, take the initial $\delta_0 = 0$, for which the potential flow velocity is simply $U_0 = \nabla x$.

As regards boundary layers, it all began with Ludwig Prandtl (1875–1953), a pioneer master on fluid mechanics, who defined the boundary layer in fluid flows by a paper of a most penetrating new conception as he presented on August 12, 1904 at the Third International Congress of Mathematicians in Heidelberg, Germany. For a smooth solid body of typical length L moving with velocity U in a viscous fluid It simplifies the fluid mechanics by dividing the flow field into two areas: one inside the boundary layer dominated by viscosity with friction, and the other outside the layer where ignoring viscosity with negligible effects on the solution. This provides a closed-form solution for the flow in both areas, a significant simplification of the full Navier-Stokes equations.

At this vital point, the keystone goal is therefore to find the deftly useful equation desired. For this purpose, introducing a set of dimensionless variables given as follows.

With an innovative idea, namely, how best to differentiate the flow properties near the body and also far away, as he delivered his at the 1904 The key notion is in regard with an arbitrary boundary layer attached at body surface, characterized by a magnifying normal velocity gradient and by a vorticity varying rapidly in the layer, denoted by a typical thickness δ, outside of the boundary layer is the inviscid (non-viscous) potential flow. For two-dimensional plane flows, the complex variable $z = x + iy$ provides the basic equations for potential flows as

$$f(z) = \phi(x,y) + i\psi(x,y), \quad \frac{df}{dz} = u(x,y) - iv(x,y), \quad u = \phi_x = \psi_y, \quad v = \phi_y = -\psi_x, \tag{8a}$$

where $\phi(x,y)$ is the potential and $\psi(x,y)$ the stream functions, and u, v the velocity components in the x, y directions.

For thin boundary layers along the x-axis, the derivatives of the tangential velocity u arise with $|u_x| \ll |u_y|$, $|u_{xx}| \ll |u_{yy}|$, and by the mass conservation, $u_x + v_y = 0$, it also follows that $v/u = O(\delta/L)$, all in accord to Prandtl's hypothesis. Hence the flow with nearly uniform pressure reads

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}. \tag{8b}$$

Higher the Reynolds number, narrower becomes the viscous flow field until it reduced so narrow to become the *boundary layer*.

For moderate to higher Reynolds numbers ($Re > 10^4$ say), the layer is given by $\delta \simeq L/\sqrt{Re}$, implying the layer diminishing with increasing Re, making the normal velocity gradient at body surface more singular as the outer potential flow has its tangential velocity remaining almost intact. (The relation $\delta \simeq L/\sqrt{Re}$ is evident by the dimensionless Navier-Stokes equation with coefficient Re^{-1} for the viscous stress terms. Therefore it gives rise to the *kinematic viscosity* parameter as the proportional coefficient of the viscous stress on the right-hand side of (8b)).

Similarly, this perturbation expansion scheme can be applied to various problems.

7. Solitary Waves

The first solitary wave was encountered in 1834 by John Scott Russell, a commanding engineer, who was observing a boat rapidly drawn by two horses along a Scottish canal, when the boat suddenly stopped, whilst the water was agitated round the prow in forming a large solitary wave rolling forward in quite uniform shape and speed as observed by Russell catching it on horseback. Physically, the forward rolling wave carried with it a certain momentum, hence sending an equal and opposite momentum, by Newton's law, onto the boat and the horses that were apparently lacking the power to move on. This may be rich with implications.

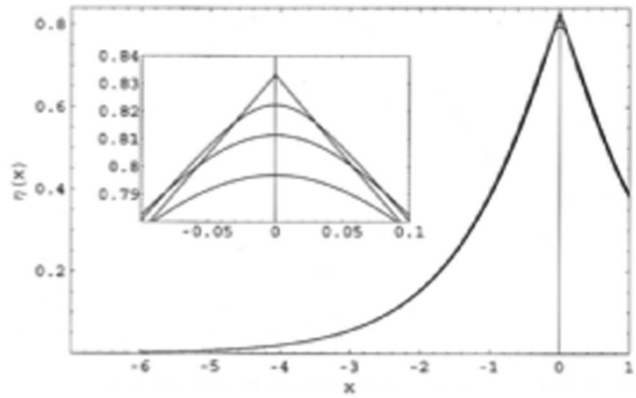

Figure 2. $\alpha = 0.8331990,\ 0.822279,\ 0.811386,\ 0.796952$; $Fr = 1.290890,\ 1.291738,\ 1.293358,\ 1.294208$.

Following such pioneers as Sir George G. Stokes (1880), a new theory called the *unified intrinsic functional expansion* (UIFE) theory was announced for evaluating fully nonlinear and fully dispersive solitary wave of height a moving in rest water of depth h with speed c under gravity constant g (i.e., [6]). This theory is founded on the wave velocity distributions in terms of the *intrinsic component functional expansion* (ICFE) representing the regional wave properties, including the exponential falling-off far away and the wave crest regime approaching the highest wave with cornered 120° crest owing to the primary and secondary singularities so that the wave can be determined accurately in these specific algorithms. The method is based on minimizing the mean-square error of the wave energy equation stepwise, by finding the steepest descent of the error with interactive optimizations. It is exemplified here to illustrate the results for wave amplitude $\alpha = a/h$, and speed Froude number $Fr = c/\sqrt{gh}$ for several high waves as shown in Fig. 2, with the fourth highest wave quite close to the fastest solitary wave specified with $\alpha_{fst} = 0.7959034$, $Fr_{fst} = 1.294211$, also in a telescopic view of these waves.

These single solitary waves are systematically closed, for the viscous friction exerted on the canal water by the horizontal river bottom, being the sole force, is negligible. However, the boat under tow by horses is an open system, producing dynamic water waves propagating away behind the boat constituting a primary part of the drag force on the boat. The water waves in response can result in fascinating phenomena of great significance, as we now expound next.

8. Periodic Forward Radiating Solitons

The open system of water waves produced by exterior critical forcing involves issues lends more

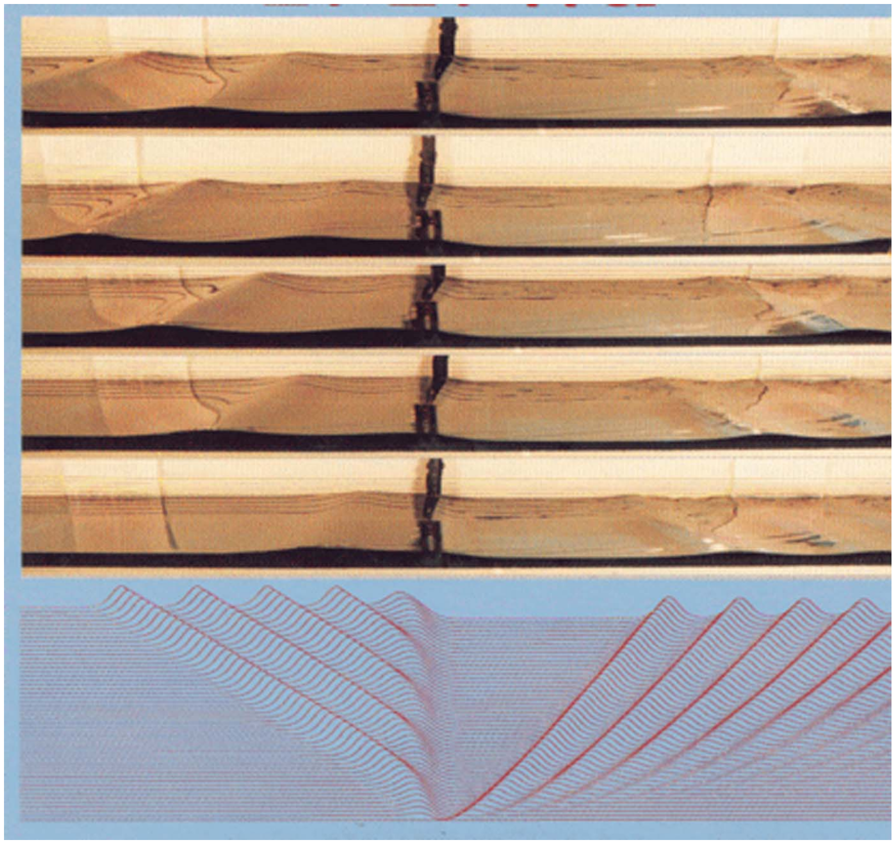

Figure 3.

facets for pursuit. (i), First, it involves acting and reacting momenta, as just mentioned. (ii), The physical progressions have to lie in a critical speed range, ($0.8 < Fr < 1.2$ or so, say), lasting long enough for the critical nonlinear coupled with critical dispersive effects jointly conceived long enough in acquiring the water mass and momentum up to maturity for radiating each solitary wave propagating forward away as a free wave. (iii), As it consumes a duration in radiating a single solitary wave, it is conceivable that if the critical forcing be long endured, periodic radiation of solitary waves appears evident to be realized.

For this purpose, the basic set of partial differential equations were derived, and used in dedicated computations by Prof. Demin Wu, visiting from Harbin, China. The vital snag to be conquered was for determining the boundary conditions at the front and rear progressing edges of the computation region in order to sustain the same stationary state as everything else. As this also implies the wave group velocity as the energy transfer rate that may well take the steepest ascend to its critical nonlinear limit, which requires determination in scrutiny with optimum iterations. As exemplified with Figure 3 in display, the two-dimensional motion is generated by a humped steel yard-stick with its image darkened at the center of the body-frame of reference, with each and every curve representing a wave ridge along the longitudinal axis, its height distribution in y, and the time t as the transverse axis. The waves propagating downstream to the right are the regular ship waves, whilst the waves to the left are the newly discovered solitary waves (or *solitons*) being radiated upstream, periodically generated by the uniformly critical forcing. The mass of the new solitary waves is drawn from the water further behind the downstream waves, pulled forward through an ever-lengthening stretch of higher flow velocity, falling in this stretch (just behind the stick) to a depressed water surface exposed to the constant air pressure. The darkened longitudinal image is rendered by the light deflected at the glass side wall of the water tank, an image which is found in perfect agreement with the computed wave profiles, a result thus verifying the theory by experiment.

Incidentally speaking, research endeavors may also be incidental. Had there been available with more horses pulling the boat, the periodic radiating solitons might have been seen by Russell. On the other hand, once discovered, the flyers received in return at all have also been cheerful. Similar phenomena have subsequently been also found in other disciplines, such as in plasma, nonlinear critical optics and nonlinear sound wave media, density-jump layers existing for inner waves arising in atmosphere and in the oceans, even in fundamental particles, as disseminated in seminar lectures by Wu, e.g. for commemorating Sir Geoffrey I. Taylor in Cambridge, U.K. [5].

The group research interests have expanded to diversified types of animal self-locomotion, the first being on fish swimming.

9. Fish Swimming

The joy in watching fish swimming has the classic tale (circa 300 B.C.) of Zhuangfucius and Huifucius strolling across a bridge, seeing a good cluster of whitish fish swimming below. With a rejoicing sigh, Zhuang: "What a joy to fish swimming effortless around! Hui queried, "You are not fish, how do you know of their joy? Zhuang, rebutting: "You are not me, how do you know I don't know of their joy? On the other hand, this academic era began in 1936 with Sir James Gray's *paradox on fish swim*: 'Either the drag of cruising fish is only tenth of that from the laboratory model tests, or the fish muscle power can be ten folds of the equivalent warm-blood animals. This furnished at once a brisk stimulus to mechanics and biology professions for resolution. Cf. e.g., [2, 3, 8].

In this regard, Wu stressed on a principle that resolving this paradox of great significance might benefit from a strongly linked endeavors between both mechanics and biology, just as it appeared to Gray. Following this principle, Wu utilized Brett's experimental data on the oxygen consumption by salmon specimens of five size groups working in five distinct levels of activities were analyzed to carry out the scaling energetic effects on metabolic conservation in biology in parallel with Prof. von Kármán's *specific energy cost* in mechanics to have attained experimental proof showing that fish swim all streamlined with laminar boundary-layers rather than any turbulent friction at all for all fishes. This provides the solid and sound data base for a complete resolution to Gray's paradox. Hence all the species of fishes may enjoy their effortless cruising. More details can be referred to the review: [10]. Summing up, the same observations across thousands of years on the joy of watching fish, joy by fish, and all assays for fish are now bound together by Yuk Yung and Wu at Caltech and Jiachuan Li of The Chinese Academy of Sciences in a poem, signifying that the truth is invariant timelessly as seen by ageless views of curiosity, singing swirling around the pillars in the palace:

> Behold, the white fish cruising so effortlessly, how magnificently they exhibit their delight, observed the witty Zhuangfucius, all resolved, with no turbulence, fish swim all streamlined.

10. Bird/Insect Flight

For oscillating wing flight, the pioneering linear theory by Prof. von Kármán and W. R. Sears brought forth the keystone proposition for the distribution of vortices. It lends $\gamma_0 + \gamma_1$ spanning the wing surface S_b jointly with γ_w covering the wake surface S_w, with γ_0 standing for the instantaneous steady vortex on S_b whilst γ_1 interacting with γ_w in producing $\gamma_0 + \gamma_1$ just for satisfying the time-varying velocity of the oscillating wing surface S_b for time-stepwise computations, under the Kutta condition at the wing trailing edge. This principle further renders the algorithm to establish a theory published in [9]. This theory is fully nonlinear and fully unsteady, entirely general, even with swift wing deforming, hence capable of describing such wings flashing with such wings bending as that of a humming-bird fleeting away from a flower.

Of the various types of animal self-propulsion, one limiting group comprises the single celled flagellates and ciliates of microscopic scales. For research studies in this field, it requiems a special discipline called the *low-Reynolds number fluid mechanics*, $Re \ll 1$, where Re, the Reynolds number, stands for the ratio between the *inertial force* and the *viscous force*, or $Re = \rho U^2 L^2/\mu UL = \rho UL/\mu = UL/\nu$, where ρ denotes the density of the fluid medium supporting the relevant body motion within it, U, L are typical body velocity and length, and μ is the dynamic viscosity coefficient of the fluid, and $\nu = \mu/\rho$ the kinematic viscosity coefficient. For microscopic cellular self-locomotion, $Re < 10^{-4}$, down to $Re \simeq 10^{-6}$ or so. In this range, the general Navier-Stokes equations reduce, with all the inertial factors omitted, to the linear Stokes equations. The fully nonlinear theory for swiftly deforming wing surface $S_b(t)$ is given by formulas in [9]. The relevant fluid dynamics did advance to furnishing fully nonlinear and fully unsteady theory, even for deformable bodies. For references, see "Fish Swimming and Bird/Insect Flight" [10].

11. Resolving the Spirochete Paradox

This paradox is well known in biology for lack of expounding for the helical motion of its single-celled body swirling at uniform speed of several score body-lengths per second, yet with self-induced force and torque vanishing by Newton's law. The final resolution was taken by Allen T. Chwang for his Ph.D. thesis. The basic concept stems from assuming the cell membrane being capable of spinning opposite in sense to the coil rotation, thereby resulting in zero torque, with results all verified with our microscope. This thesis was instantly accepted by Sir James Lighthill, the Secretariat of the Royal Society [1]. Two decades later, it was indeed discovered inside each self-propelling

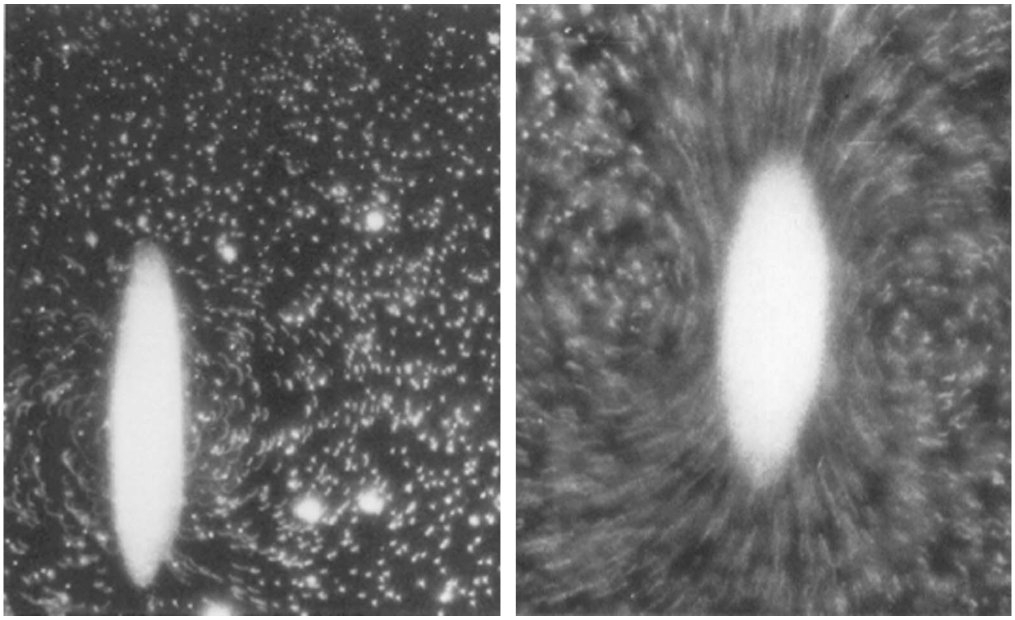

Figure 4.

flagellate that there are interior flagellar rooted on a platform driven by proton motor swirling on bearings to whip the cell membrane opposite to the coil rotation, all reported from advanced hi-tech on such phenomena. This is of course the same phenomenon as postulated in the proposition advanced in Chwang's 1971 Thesis.

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