On special zeta values in positive characteristic

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Introduction

In mathematics, we are interested in special numbers, particularly "constants" which arise from various mathematical contexts and structures. For example π, e. These special values are frequently transcendental numbers, meaning that they are not algebraic numbers, substituting such a number into any nonzero polynomial with integral coefficients, you will never get 0. However the transcendence of a specific number needs to be proved. Hermite did such a proof for e in 1873, and Lindemann did such a proof for π in 1882. Whenever you encounter a number for which there is no apparent reason to be algebraic, then it is most natural to suspect that this number is transcendental. Ironically, more often than not, mathematicians are unable to confirm the transcendence nature of specific numbers.

When we are confronted with several transcendental numbers, a natural question is asking for their relationships, if they are related or not. This is a more subtle challenge. Ideally, we would like to know whether, say e and π are algebraically dependent, i.e. whether there are two variables polynomials $P(X,Y)$ with coefficients from $\mathbf{Z}$ not all zero so that $P(e,\pi)=0$. We see no reason that e and π should be algebraically dependent. Hence we conjecture that they are algebraically independent. However till today, no one can prove such a statement about this pair of very classical constants! Whenever a set of transcendental special values is given to us, we would dream to find out and to explain all the possible algebraic relations among them. Only after that, can we say that we really "understand" these special values.

A very classical family of special values comes from the following series, for integer $m > 1$,

$$\zeta(m) = \sum_{n=1}^{\infty} \frac{1}{n^m}.$$

These are the values of the Riemann zeta function taken at positive integers greater than 1. As is well-known to all students, this family goes back to Euler and Bernoulli who proved for even $m = 2s$ the beautiful formula:

$$\zeta(2s) = \frac{-B_{2s}\left(2\pi\sqrt{-1}\right)^{2s}}{2(2s)!},$$

where $B_{2s} \in \mathbf{Q}$ are the Bernoulli numbers. When m is odd, very little is known about these special values, the irrationality of $\zeta(3)$ is confirmed by R. Apéry (1976) while the transcendence of $\zeta(3)$ is still open. Nevertheless we would believe that all these values $\zeta(m)$ are transcendental and the Bernoulli-Euler identity above is the only source of algebraic relations among the very special values from this classical family. This means, in particular, that all $\zeta(m)$ with odd $m > 1$ should be algebraically independent from π, and they are also algebraically independent from each other.

Euler went on to introduce multizeta values (henceforth abbreviated MZV's) which are defined by the reciprocal power sums:

$$\zeta(s_1, \cdots, s_r) := \sum_{n_1 > \cdots > n_r \geq 1} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}},$$

where $s_1, \ldots, s_r$ are positive integers with $s_1 \geq 2$. Here r is called the depth and $w := \sum_{i=1}^{r} s_i$ is called the

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weight of the presentation $\zeta(s_1,\ldots,s_r)$. The following are some known evaluations of weight $2w$ multiple zeta values:

$$\zeta(2w,2w) = \frac{(\zeta(2w))^2 - \zeta(4w)}{2},$$
$$\zeta(2,2,\cdots,2) = \frac{\pi^{2w}}{(2w+1)!},$$
$$\zeta(3,1,\ldots,3,1) = \frac{2\pi^{2w}}{(2w+2)!}.$$

A given multizeta value MZV $\zeta(\mathbf{s}) = \zeta(s_1,\ldots,s_r)$ of weight $w = s_1 + \ldots + s_r$ and depth r is said to be Eulerian if the ratio $\zeta(\mathbf{s})/(2\pi\sqrt{-1})^w$ is rational. In the case of depth 1 Riemann zeta value at positive integer $s > 1$, $\zeta(s)$ is Eulerian if and only if s is even. Depth > 1 Eulerian zeta values should be rare, to date there is no explicit rule completely describing these rather exceptional MZV's.

We conjecture that all multizeta values are transcendental numbers. However, once depth $r > 1$ are taken into considerations, Euler already knew that there are many $\mathbf{Q}$-linear relations among monomials of MZV's with the same weight, e.g. Euler's stuffle relation:

$$\zeta(s_1,s_2) + \zeta(s_2,s_2) + \zeta(s_1+s_2) = \zeta(s_1)\zeta(s_2).$$

Moreover for monomials of MZV's having different weights, the conjecture is that they should be linearly independent over $\mathbf{Q}$. Let $\mathcal{Z}$ be the $\mathbf{Q}$-algebra inside $\mathbf{R}$ generated by all the MZV's, and let $\mathcal{Z}_w$ be the $\mathbf{Q}$-span of all weight w MZV's. Then $\mathcal{Z}_{w_1}\mathcal{Z}_{w_2} \subset \mathcal{Z}_{w_1+w_2}$. It has been conjectured (Goncharov, Zagier) that:

$$\mathcal{Z} = \mathbf{Q} \oplus_{w\geq 2} \mathcal{Z}_w.$$

Naively we would regard weight w monomials of MZV's as kind of degree w polylogarithms. For nonzero logarithms of algebraic numbers, there is the phenomenon described by a fundamental theorem of Alan Baker [Bak] asserting that these logarithms are in fact linearly independent over $\overline{\mathbf{Q}}$ (the field of all algebraic numbers) if they are known to be linearly independent over $\mathbf{Q}$. We suspect that there is also such a phenomenon for monomials of MZV's with equal weight. Let $\overline{\mathcal{Z}}$ be the $\overline{\mathbf{Q}}$-algebra inside $\mathbf{C}$ generated by all the MZV's, and let $\overline{\mathcal{Z}_w}$ be the $\overline{\mathbf{Q}}$-span of all weight w MZV's. Then we conjecture:

$$\overline{\mathcal{Z}} = \overline{\mathbf{Q}} \oplus_{w\geq 2} \overline{\mathcal{Z}_w} = \mathcal{Z} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}.$$

This implies that for any given set of MZV's of the same weight w, if they are linearly independent over $\mathbf{Q}$ then they are actually linearly independent over all algebraic numbers.

A consequence of the above extremely bold conjecture answers the following enquiry. We say that a given set of nonzero numbers has the Euler dichotomy property if taking any two numbers from this set either their ratio is rational or else they are algebraically independent. We are particularly interested in the following problem: consider the set S_w of weight w MZV's together with the period $(2\pi\sqrt{-1})^w$, is this set of special values satisfying the Euler dichotomy?

Multizeta values have been widely studied in recent years. The wealth of $\mathbf{Q}$-rational algebraic relations is a mathematical treasure. Breakthroughs include the 2012 proof ([B12]) by F. Brown of Hofmann's conjecture that all MZV's can be written as $\mathbf{Q}$-linear combinations of $\zeta(s_1,\cdots,s_r)$, with $s_i \in \{2,3\}$. Also there is the beautiful dimension formula conjectured by D. Zagier which describes the dimension of $\mathbf{Q}$-vector space of weight w double zeta values (depth two MZV's) via the dimension of the space of weight w cusp modular forms for $\mathrm{SL}_2(\mathbf{Z})$.

Special Zeta Values for $\mathbf{F}_q[\theta]$

We now move to the special values in a world of positive characteristic. Fix a finite field $\mathbf{F}_q$ with q elements. Let $A = \mathbf{F}_q[\theta]$ be the polynomial ring in the variable θ over $\mathbf{F}_q$ with quotient field k. We embed k into its completion $k_\infty = \mathbf{F}_q((\frac{1}{\theta}))$ with respect to the infinite place ∞ of k. Let A_+ be the set of monic polynomials in A and consider the series, for $n \in \mathbf{N}$,

$$\zeta_A(n) := \sum_{a\in A_+} \frac{1}{a^n} \in \mathbf{F}_q((\frac{1}{\theta})).$$

These values, called Carlitz zeta values [Car], are analogues of classical Riemann zeta values. We note that in this "non-archimedean" situation the series for $\zeta_A(1)$ does converge in $\mathbf{F}_q((\frac{1}{\theta}))$. Also if p is the characteristic of the base finite field $\mathbf{F}_q$, there are the obvious Frobenius relations among these zeta values:

$$\zeta_A(p^m n) = (\zeta_A(n))^{p^m},$$

for any positive integer m.

In the function field setting the role of the multiplicative group $\mathbf{G}_m$ is played by the so-called Carlitz module, and the role of $2\pi\sqrt{-1}$ is played by a fundamental period $\tilde{\pi}$ for the Carlitz module:

$$\frac{1}{\tilde{\pi}} := (-\theta)^{\frac{-q}{q-1}} \prod_{i=1}^{\infty}\left(1 - \frac{\theta}{\theta^{q^i}}\right) \in \mathbf{C}_\infty,$$

where $(-\theta)^{\frac{1}{q-1}}$ can be any choice of $(q-1)$-st root of $-\theta$, and $\mathbf{C}_\infty$ is the completion of a fixed algebraic closure of k_∞. The period $\tilde{\pi}$ is transcendental over the rational function field k (Wade 1941). There is the $\mathbf{F}_q$-linear Carlitz exponential function $\exp_C(z)$ linearizing the Carlitz module:

$$0 \to \tilde{\pi}A \to \mathbf{G}_a(\mathbf{C}_\infty) \xrightarrow{\exp_C} \mathbf{G}_a(\mathbf{C}_\infty) \to 0$$

$$\begin{array}{ccc}\mathbf{C}_\infty & \xrightarrow{\exp_C} & \mathbf{G}_a(\mathbf{C}_\infty)=\mathbf{C}_\infty \\ {\scriptstyle\theta(\cdot)}\downarrow & & \downarrow{\scriptstyle x\mapsto\theta x+x^q} \\ \mathbf{C}_\infty & \xrightarrow{\exp_C} & \mathbf{G}_a(\mathbf{C}_\infty)=\mathbf{C}_\infty\end{array}$$

The multi-valued inverse map to the Carlitz expoential is called the Carlitz logarithm function which has the following simple series expansion near the origin:

$$\mathrm{Log}_C(z) := \sum_{i=0}^{\infty}\frac{z^{q^i}}{L_i},$$

where the denominator of the coefficients are given by $L_0 := 1$, and $L_i := (\theta-\theta^q)\cdots(\theta-\theta^{q^i})$ for $i\in\mathbf{N}$.

Set $D_0 = 1$ and $D_i := \prod_{j=0}^{i-1}(\theta^{q^i}-\theta^{q^j})$ for $i\in\mathbf{N}$. For a non-negative integer n, we express n as

$$n=\sum_{i=0}^{\infty} n_i q^i \quad (0\le n_i\le q-1, n_i=0 \text{ for } i\gg 0),$$

and the Carlitz factorials are defined by

$$\Gamma_{n+1} := \prod_{i=0}^{\infty} D_i^{n_i}\in A.$$

On the other hand the Carlitz exponential function has entire expansion $\exp_C(z)=\sum_{i\ge 0} z^{q^i}/D_i$. This leads to the Bernoulli-Carlitz numbers $\mathrm{BC}(n)\in k$:

$$\frac{z}{\exp_C(z)}=\sum_{n\ge 0}\frac{\mathrm{BC}(n)}{\Gamma_{n+1}}z^n.$$

In [Car], Carlitz derived an analogue of Euler's formula if n is divisible by $q-1$ (called these integer n q-even):

$$\zeta_A(n)=\frac{\mathrm{BC}(n)}{\Gamma_{n+1}}\tilde{\pi}^n.$$

We note that $\tilde{\pi}^n\in\mathbf{F}_q((\frac{1}{\theta}))$ if and only if n is divisible by $q-1$, and so Carlitz's result implies that $\zeta_A(n)/\tilde{\pi}^n\in k$ if and only if n is *q-even*. When $q=2$, every integer is q-even, it follows from the above formula and the transcendence of $\tilde{\pi}$ that all $\zeta_A(n)$ are transcendental over k (hereafter we will just say that these values are transcendental).

The story of all Carlitz zeta values was unfold more than a quarter of a century ago in two papers [AT90], [Yu91]. In particular all the positive characteristic zeta values $\zeta_A(n)$ for n "q-odd" are proved to be transcendental. When positive integers n are not divisible by $q-1$, the ratio $\zeta_A(n)/\tilde{\pi}^n$ are also shown to be transcendental by [Yu91], as a highly non-trivial consequence of the k-linear independence of $\zeta_A(n)$ with $\tilde{\pi}^n$. An analogue of Baker's theory on linear forms of logarithms is hereby established in the positive characteristic world, for Carlitz logarithms of algebraic *functions* in $\bar{k}$, as well as for degree n last-coordinate logarithms of algebraic points on the n-th tensor powers of the Carlitz module. For those special values in question, we find again the phenomenon that linear independence over the rationals *forces* linear independence over the algebraic closure.

Since a decade ago it has also been discovered that analogue of Baker's theory in positive characteristic can be further refined to deal with the algebraic independence of special values. Continuing the efforts of Anderson, Brownawell and Yu, Papanikolas in [P08] solved the problem of algebraic independence of Carlitz logarithms of algebraic "numbers", assuming only that these logarithms are linearly independent over k. Chang-Yu [CY07] goes on replacing Carlitz logarithms by higher degree Carlitz polylogarithms. Applying this powerful tool to the Carlitz zeta values, they are able to prove that all the polynomial relations among the values $\zeta_A(n)$ come from the Bernoulli-Carlitz identity, together with the obvious Frobenius relations. This confirms the speculation that every $\zeta_A(n)$ with n not divisible by $q-1$ should be algebraically independent from the fundamental period $\tilde{\pi}$ over k, and these *odd* zeta values $\zeta_A(n_1),\zeta_A(n_2)$ are algebraically independent over k if n_1/n_2 is not a power of the characteristic p.

Replacing $\mathbf{Z}$ by the polynomial ring $A=\mathbf{F}_q[\theta]$, D. Thakur 2004 [T04] introduced multizeta values $\zeta_A(s_1,s_2,\cdots,s_r)$ over A, generalizing the depth one Carlitz zeta value at positive integers. For $\mathfrak{s}=(s_1,\ldots,s_r)\in\mathbf{N}^r$, these characteristic p multizeta values are:

$$\zeta_A(s_1,\ldots,s_r) := \sum\frac{1}{a_1^{s_1}\cdots a_r^{s_r}}\in\mathbf{F}_q((\frac{1}{\theta})),$$

where the sum is taken over r-tuples of monic polynomials $a_1,\ldots,a_r$ with $\deg a_1>\cdots>\deg a_r$. Not surprisingly, there are interesting k-linear relations as well as $\mathbf{F}_p$-linear relations for these positive characteritic multizeta values. Euler's stuffle relation is replaced by the following more subtle [Chen]:

$$\begin{aligned}\zeta_A(s_1)\zeta_A(s_2) &= \zeta_A(s_1,s_2)+\zeta_A(s_2,s_1)+\zeta_A(s_1+s_2)\\ &\quad+\sum_{i+j=s_1+s_2,\, q-1|j}\Big[(-1)^{s_1-1}\binom{j-1}{s_1-1}\\ &\quad+(-1)^{s_2-1}\binom{j-1}{s_2-1}\Big]\zeta_A(i,j).\end{aligned}$$

As samples of k-linear relations, we cite [T09] (here $[i]$ stands for $\theta^{q^i}-\theta$):

$$\zeta_A(q-1,(q-1)^2)=(\frac{-1}{[1]})^{q-1}\zeta_A(q^2-q),$$

$$\begin{aligned}&\zeta_A(q-1,(q-1)q,\cdots,(q-1)q^{r-1})\\ &\quad=\frac{[r-1]\cdots[1]}{[1]^{q^{r-1}}\cdots[r-1]^q}\zeta_A(q^r-1),\end{aligned}$$

$$\zeta_A(1,(q-1)q,q^3-q^2+q-1)=\frac{[3]-1}{[3][2][1]^{q^2-q+1}}\zeta_A(1)^{q^3}.$$

Chang [C14] succeeds in proving that all these positive characteristic multizata values $\zeta_A(\mathfrak{s})$ are transcendental (over the rational field k). Following Thakur we say that $\zeta_A(\mathfrak{s})$ is *Eulerian* if the ratio $\zeta_A(s_1,\ldots,s_r)/\tilde{\pi}^{s_1+\cdots+s_r}$ is in k. Contrast to our knowledge about classical MZV's, such quotients are proved to be either rational or transcendental over k. Indeed, we already know that either $\zeta_A(s_1,\ldots,s_r)/\tilde{\pi}^{s_1+\cdots+s_r}$ is in k or $\zeta_A(s_1,\ldots,s_r)$ and $\tilde{\pi}$ are algebraically independent over k [C14]. Moreover a much strengthened analogue of Goncharov's conjecture is proved, and the whole theory of algebraic relations for multizeta values over A (henceforth also abbreviated as MZV's) is reduced to studying the k-inear relations among these positive characteristic special values. Let $\mathcal{Z}_A$ be the k-algebra inside k_∞ generated by all the MZV's, and let $\mathcal{Z}_w$ be the k-span of all weight w MZV's. Then again $\mathcal{Z}_{w_1}\mathcal{Z}_{w_2}\subset\mathcal{Z}_{w_1+w_2}$ ([T09]). Let $\overline{\mathcal{Z}_w}$ be the $\bar{k}$-span of all weight w MZV's. Then Chang has verified:

$$\overline{\mathcal{Z}_A}=\bar{k}\oplus_{w\geq 1}\overline{\mathcal{Z}_w}=\mathcal{Z}_A\otimes_k\bar{k}.$$

Given any set of multizeta values over A of the same weight w, if they are linearly independent over k then they are in fact linearly independent over $\bar{k}$. These special values of weight k in positive characteristic are indeed the "ideal" polylogarithms of degree k. In particular, such a set of multizeta values always has the Euler dichotomy property, meaning that two such special values either has their ratio in k or are algebraically independent over k. In fact, as Chang shows, any set of fixed weight multipolylogarithms of algebraic elements in $\bar{k}$ satisfies Euler dichotomy.

Recently Chang, Papanikolas and Yu ([CPY], [KL]) have found an effective criterion (algorithm) determining whether given MZV over A is Eulerian. Data from implementing this algorithm lead to conjectural description of all the Eulerian MZV's in the positive characteristic world: Fix prime power q, and call the sequence of r-tuples below Eulerian r-tuples with respect to $\mathbf{F}_q$:

$$\mathrm{Eu}_1:=(q-1)\text{ and }\mathrm{Eu}_{r+1}:=(q-1,q\mathrm{Eu}_r)\in\mathbf{N}^{r+1}.$$

For each depth r, we introduce a sequence of r-tuples in $\mathbf{N}^r$ as follows:

$$\mathrm{Eu}_r(\ell):=(q^\ell-1,q^\ell\mathrm{Eu}_{r-1}),\text{ for } r>1,\ \ell\geq 1,$$

and $\mathrm{Eu}_1(\ell):=(q^\ell-1)$. Call this the canonical sequence of depth r with respect to $\mathbf{F}_q$. The corresponding MZV $\zeta_A(\mathrm{Eu}_r(\ell))$ are all Eulerian. This follows from the Euler-Carlitz identity and the following inductive formula of Chen, for all $r\geq 2$ and $\ell\geq 1$:

$$\zeta_A(\mathrm{Eu}_r(\ell))=\zeta_A(q^\ell-1)\zeta_A(\mathrm{Eu}_{r-1})^{q^\ell}-\zeta_A(\mathrm{Eu}_{r-1}(\ell+1)).$$

When $q>2$ and depth $r>2$, these seem account for all the Eulerian multizeta values (there is an extra family of Eulerian MZV's and also an exceptional one when $r=2$, and more exceptions when $q=2$).

Motivic Transcendence Theory

We interpret number theory in a broad sense. Both the algebraic number fields and the algebraic function fields (in one variable over finite constant fields) are included as our global fields. We are interested in arithmetic objects defined over these global fields, and we study closely the parallels between phenomena in characteristic zero and in positive characteristic p. Opportunities arise to examine some of the most difficult open problems in classical number theory, not in the original context but in the positive characteristic world.

Let $t\neq\theta$ be another variable and we consider the rational function field $\bar{k}(t)$, where $\bar{k}$ is a fixed algebraic closure of the rational field $k=\mathbf{F}_q(\theta)$. The (inverse) Frobenius automorphism on $\bar{k}$ gives rise the following natural "conjugation" on $\bar{k}[t]$ (hence also on $\bar{k}(t)$) by taking $1/q$-th root on their coefficients in $\bar{k}$:

$$f=\sum_i a_it^i\mapsto f^{(-1)}:=\sum_i(a_i)^{\frac{1}{q}}t^i.$$

Let V be a finite-dimensional $\bar{k}(t)$-vector space, together with a conjugate linear bijective operator σ on V, satisfying, for $v\in V$:

$$\sigma(fv)=f^{(-1)}\sigma(v).$$

We call such pair (V,σ) a Frobenius module of rank $r=\dim_{\bar{k}(t)}V$. Choose a basis of V, then the operator σ is represented by a matrix $\phi\in\mathrm{Mat}_r(\bar{k}(t))$. If $B\in\mathrm{GL}_r(\bar{k}(t))$ changes the original basis to another basis, then the matrix representation of σ with respect to the new basis becomes $B^{(-1)}\phi B^{-1}$, where $B^{(-1)}$ is obtained from B via replacing all entries by their $1/q$-th root. We regard these Frobenius modules as pre-*t-motives* ([A86], [P08]).

The t-motivic study of special zeta values in positive characteristic was initiated by Anderson-Thakur [AT90]. They introduces a sequence of "Bernoulli type elements" H_n inside $\mathbf{F}_q[t,\theta]$, now called the Anderson-Thakur polynomials. Take y to be yet another variable, and define polynomials $G_n(y)\in\mathbf{F}_q[t,y]$ by $G_0(y):=1$ and $G_n(y)$ for $n\in\mathbf{N}$ such that

$$G_n(y):=\prod_{i=1}^n\left(t^{q^n}-y^{q^i}\right).$$

Note that $G_{n+1}(y^q)=(t-y^q)^{q^{n+1}}G_n(y)^q$. For $n=0,1,2,\ldots$ the sequence of Anderson-Thakur polynomials $H_n\in$

$\mathbf{F}_q[\theta][t]$ is given from the generating function identity:

$$\left(1-\sum_{i=0}^{\infty}\frac{G_i(\theta)}{D_i|_{\theta=t}}x^{q^i}\right)^{-1}=\sum_{n=0}^{\infty}\frac{H_n}{\Gamma_{n+1}|_{\theta=t}}x^n.$$

We note that for $0\le n\le q-1$ simply $H_n=1$.

It turns out that the Carlitz zeta values $\zeta_A(n)$ can be recognized as "period" of a dimension two Frobenius module with its σ operator represented by the following matrix

$$\Phi_n:=\begin{pmatrix}(t-\theta)^n & 0\\ H_{n-1}^{(-1)} & 1\end{pmatrix}.$$

Given a Frobenius module with its operator σ represented by matrix $\Phi\in\mathrm{Mat}_r(\bar{k}[t])$, we consider the following Frobenius difference equation:

$$\Psi^{(-1)}=\Phi\Psi,$$

with Ψ to be solved in $\mathrm{Mat}_{r\times 1}(\mathcal{E})$, where $\mathcal{E}$ is the ring of entire power series $\sum_{n=0}^{\infty}a_nt^n\in\bar{k}[[t]]$ such that

$$\lim_{n\to\infty}\sqrt[n]{|a_n|_\infty}=0,\ [k_\infty(a_0,a_1,a_2,\dots):k_\infty]<\infty.$$

Here the Frobenius twisting $f\mapsto f^{(-1)}$ is extended from to $\bar{k}[t]$ to $\mathcal{E}$, and then to matrices with entries in $\mathcal{E}$ by twisting entry-wise. These Frobenius difference equations are analogues of systems of classical linear differential equations.

Explicitly solving the Frobenius equation of Φ_n, we introduce the entire power series

$$\Omega(t):=(-\theta)^{\frac{-q}{q-1}}\prod_{i=1}^{\infty}\left(1-\frac{t}{\theta^{q^i}}\right).$$

Then $\frac{1}{\Omega(\theta)}=\tilde{\pi}$ and Ω satisfies the functional equation $\Omega^{(-1)}=(t-\theta)\Omega$. Also define the following family of entire power series $\mathcal{L}_n$:

$$\mathcal{L}_n(t):=\sum_{i\ge 0}(\Omega^nH_{n-1})^{(i)}.$$

Then

$$\Psi_n:=\begin{pmatrix}\Omega^n\\ \mathcal{L}_n\end{pmatrix}$$

gives the desired solution of our equation. Specializing at $t=\theta$, we arrived at ([AT09])

$$\Psi_n(\theta)=\begin{pmatrix}\frac{1}{\tilde{\pi}^n}\\ \frac{\Gamma_n\zeta_A(n)}{\tilde{\pi}^n}\end{pmatrix}.$$

To study multizeta values $\zeta_A(s_1,\cdots,s_r)$ of depth r in positive characteristic, we consider Frobenius modules of rank $r+1$ with its operator σ represented by

$$\Phi_{\mathbf{s}}:=\begin{pmatrix}(t-\theta)^{s_1+\cdots+s_r} & 0 & 0 & \cdots & 0\\ H_{s_1-1}^{(-1)}(t-\theta)^{s_1+\cdots+s_r} & (t-\theta)^{s_2+\cdots+s_r} & 0 & \cdots & 0\\ 0 & H_{s_2-1}^{(-1)}(t-\theta)^{s_2+\cdots+s_r} & \ddots & & \vdots\\ \vdots & & \ddots & (t-\theta)^{s_r} & 0\\ 0 & \cdots & 0 & H_{s_r}^{(-1)}(t-\theta)^{s_r} & 1\end{pmatrix}.$$

A solution to the corresponding Frobenius difference equation is of the form:

$$\Psi_{\mathbf{s}}:=\begin{pmatrix}\Omega^{s_1+\cdots+s_r}\\ \Omega^{s_2+\cdots+s_r}\mathcal{L}_{21}\\ \vdots\\ \vdots\\ \Omega^{s_r}\mathcal{L}_{r1}\\ \mathcal{L}_{(r+1),1}\end{pmatrix},$$

where the last coordinate entire power series $\mathcal{L}_{(r+1,1)}$ is

$$\mathcal{L}_{(r+1,1)}=\mathcal{L}_{\mathbf{s}}:=\sum_{i_1>\cdots>i_r\ge 0}(\Omega^{s_r}H_{s_r-1})^{(i_r)}\cdots(\Omega^{s_1}H_{s_1-1})^{(i_1)}$$

which specializes at $t=\theta$ to the value ([AT09])

$$\mathcal{L}_{\mathbf{s}}(\theta)=\psi_{\mathbf{s}}:=\Gamma_{s_1}\cdots\Gamma_{s_r}\zeta_A(s_1,\dots,s_r)/\tilde{\pi}^{s_1+\cdots+s_r}.$$

Given any finite set of multizeta values with the same weight, $\zeta_A(\mathbf{s}_1),\cdots,\zeta_A(\mathbf{s}_m)$, a crucial theorem of Chang [C14] asserts that linear independence over k of the corresponding last coordinate periods $\psi_{\mathbf{s}_1},\cdots,\psi_{\mathbf{s}_m}$ forces their linear independence over $\bar{k}$. In other words, these "periods" behave as if they are the usual logarithms of algebraic numbers. However, proof of this theorem is based on principles quite different from the theory laid down by Baker half century ago.

Frobenius modules are just finite-dimensional vector spaces with a "twisting" operator σ. The matrices $\Phi_{\mathbf{s}}=\Phi_{(s_1,\cdots,s_r)}$ describing the Frobenius modules in question possess a key property discovered by Anderson-Brownawell-Papanikolas [ABP] which makes our transcendence dream coming true: all entries of $\Phi_{\mathbf{s}}$ are integral, i.e. inside $\bar{k}[t]$, and furthermore $\det\Phi_{\mathbf{s}}$ equals $c(t-\theta)^{\ell}$ for some nonzero $c\in\bar{k}$. Under such a condition on $r\times r$ matrix Φ describing a given Frobenius module, if Ψ is analytic solution of the Frobenius difference equation $\Psi^{(-1)}=\Phi\Psi$ in $\mathcal{E}$, any $\bar{k}$-linear relation among the coordinates of $\Psi(\theta)$ can be "lifted" to a $\bar{k}[t]$-linear relation among the coordinates of the vector Ψ. In other words if $\rho\in\mathrm{Mat}_{1\times r}(\bar{k})$ and $\rho\Psi(\theta)=0$, then there exists vector $P\in\mathrm{Mat}_{1\times r}(\bar{k}[t])$ such that $P\Psi=0$ and $P(\theta)=\rho$ holds.

The above ABP criterion formulated (2004) for Frobenius modules is extremely powerful analytic tool for transcendence theory in positive characteristic. It explains linear relations among special values by way of linear relations among analytic functions which come from solutions of Frobenius difference equations relating to these special values. This ABP criterion can be viewed as a motivic reformulation of the theory of logarithmic vectors of algebraic points on abelian t-modules developed by Yu in the 1990's ([Yu97]). Further traced back, Yu's theory was inspired by G. Wüstholz's theory in the 1980's in the context of logarithms on commutative algebraic groups. If one considers the specific group which is product of copies of $\mathbf{G}_m$, it goes back to the celebrated work of A. Baker in 1960's.

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