Open Problems

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Note. The readers are welcome to propose the solutions. The authors should send their solutions to liu@math.ucla.edu and post their solutions in arXiv. The correct solutions will be announced and some souvenirs will be presented to the solvers.—*The Editors*

Problem 2015002 (Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

If G is a closed subgroup of the group of isometries of a complete Riemannian manifold, then G acts properly on the manifold. If a Lie group G acts properly on a manifold M, then there always exists an invariant metric on M, so G is a subgroup of the group of isometries of M. Does there exist an invariant metric on M such that G is the full group of isometries of M? Note that every compact Lie group can be realized as the (full) group of isometries of a compact Riemannian manifold [2]. Whether this is true for general Lie groups?

[1] S.-T. Yau, Remarks on the groups of isometries of a Riemannian manifold, Topology **16** (1977), 239–247.

[2] R. Saerens and W. Zame, The isometry groups of manifolds and the automorphism groups of domains, Trans. Amer. Math. Soc. **301** (1987), 413–429.

Problem 2015003 (Geometry). *Proposed by Shing-Tung Yau, Harvard University; Steve Zelditch, Northwestern University and John Toth, McGill University.*

Let G be a compact group acting by isometries on a compact manifold. Fix an irreducible representation of G, it is natural to ask: How often does this representation appear in the representation of G on the eigenspaces of the Laplacian? Which one appears infinite number of times? How does it grow when the eigenvalue grows? These questions have been answered in [1,2,3]. In particular, see Corollary 1.6 of [1]. Recently, Jakobson, Strohmaier and Zelditch [4] did the same kind of problem for Kähler manifolds, where one looks at the $sl_2(\mathbb{C})$ action by the Leftschetz operater etc. in eigenspaces (and a larger symmetry algebra introduced by physicists). An interesting question is that for finite groups (or even general Lie groups) G and for generic invariant metrics, are the eigenspaces of G irreducible? E. Wigner [6] termed this "no accidental degeneracies". It is unknown if generic G-invariant Laplacians or Schrödinger operators satisfy this. The only result was proved in [5] when the irreducible representations all have low degrees.

In the boundaryless case, the references [1] and [2] answer the question only at the level of leading asymptotics in the G-equivariant Weyl laws. Recently, in [7], G-equivariant Weyl laws with sharp asymptotic remainders were proved for general compact, closed manifold with effective, isometric G-actions with G a compact and connected Lie group. That answers the question in detail for closed manifolds.

For compact manifolds with boundary, much less seems to be known. Recently, in [8], an G-equivariant Weyl law was proved for bounded Euclidean domains invariant under an isometric G-action where G is a compact Lie group acting isometrically on $\mathbb{R}^n$. However, for smooth boundaries, the Weyl error term in [8] (see Theorem 8) seems far from sharp. So, proving an G-equivariant Weyl-law with sharp remainder for

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compact manifolds with boundary invariant under an isometric G-action seems like an interesting and open problem.

[1] J. Brüning and E. Heintze, Representations of compact Lie groups and elliptic operators. Invent. Math. **50** (1978/79), 169–203.

[2] H. Donnelly, G-spaces, the asymptotic splitting of $L^2(M)$ into irreducibles, Math. Ann. **237** (1978), 23–40.

[3] B. Helffer and D. Robert, Étude du spectre pour un opérateur globalement elliptique dont le symbole de Weyl présente des symétries. II. Action des groupes de Lie compacts. Amer. J. Math. **108** (1986), 973–1000.

[4] D. Jakobson, A. Strohmaier and S. Zelditch, On the spectrum of geometric operators on Kähler manifolds, J. Mod. Dyn. **2** (2008), 701–718.

[5] S. Zelditch, On the generic spectrum of a riemannian cover, Ann. Inst. Fourier (Grenoble) **40** (1990), 407–442.

[6] E. Wigner, Group Theory and its Applications to the Quantum Mechanics of Atomic Spectra, Academic Press, 1959.

[7] P. Ramacher, Singular equivariant asymptotics and Weyl's law, arXiv:1001.1515, 2010.

[8] R. Cassanas and P. Ramacher, Reduced Weyl asymptotics for pseudodifferential operators on bounded domains. II: The compact group case, J. Funct. Anal. **256** (2009), 91–128.

Problem 2015004 (Sympletic Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Given a closed contact three-dimensional oriented manifold with a canonical Reeb vector field R, the dynamics of this flow appears in Hamiltonian mechanics, geodesic flows, and Gromov-Witten theory. The Weinstein conjecture says that R has at least one closed orbit. This was proved by Cliff Taubes in 2007 in dimension three. The Weinstein conjecture is still open in dimension greater than three, although it has been established for compact hypersurfaces of contact type in $\mathbb{R}^{2n}$. The techniques used in Taubes's proof (e.g., Seiberg-Witten Floer homology and embedded contact homology) are special to three dimension.

Can we find a refinement of the conjecture? It is natural to ask how many such orbits in dimension three? Hutchings and Taubes proved that any nondegenerate contact form on any three-manifold has at least 2 embedded Reeb orbits and has at least 3 if the manifold is not a lens space. Cristofaro-Gardiner and Hutchings proved that there are at least two embedded Reeb orbits for any contact form on any three-manifold.

An important question is: If the three manifold is not Lens space, are there infinite number of closed Reeb orbits? Is there always a "short" orbit, i.e., given an orbit, one can define the action to be the period of the contact form along this orbit. Hutchings and Cristofaro-Gardiner ask whether a closed orbit exists such as the action is less than square root of the volume? Recently they [1] proved that for a closed contact three-manifold (Y,λ), either λ has at least three embedded Reeb orbits or λ has exactly two embedded Reeb orbits, and their symplectic actions T, T' satisfy $TT' \leq \mathrm{vol}(Y,\lambda)$.

[1] D. Cristofaro-Gardiner and M. Hutchings, From one Reeb orbit to two, arXiv:1202.4839.

Problem 2015005 (Complex Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Given a holomorphic vector bundle of rank n over a compact complex manifold with dimension m. Here we assume $m \geq n$. Suppose that it admits a Hermitian metric whose curvature, suitably defined, is nonnegative. Prove that the n-th Chern class of the bundle is numerically nonnegative and that it is numerically zero if and only if the bundle is flat.

When the bundle is ample and over an algebraic manifold, this is due to Bloch and Gieseker [1].

[1] S. Bloch and D. Gieseker, The positivity of the Chern classes of an ample vector bundle. Invent. Math. **12** (1971), 112–117.