The Story of Riemann's Moduli Space

by Lizhen Ji*

Abstract. Compact Riemann surfaces and projective algebraic curves over $\mathbb{C}$ are two realizations of the same class of objects. Their moduli space was first introduced by Riemann and is one of the central objects of the contemporary mathematics and mathematical physics. In this note, we explain the birth and evolution of Riemann's moduli space by emphasizing several points which might not be so well-known to the general mathematicians and possibly even to some mathematicians who are interested in either complex analytic or algebraic geometric theories of moduli spaces. For example, who was the first to formulate the moduli problem precisely? What is the meaning of moduli spaces? Who was the first to give complex analytic and algebraic variety structures to Riemann's moduli space? Why did they study these problems? How was Riemann's moduli space first used? And why did Riemann choose and use the word moduli? Why did Teichmüller study Teichmüller space? What were Teichmüller's contributions to Riemann's moduli space? What were Teichmüller's contributions to algebraic geometry?

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1. Introduction

Mathematics is interesting and complicated. It also has interesting and complicated stories, which are often intriguing. Heroes can be mathematicians, or mathematical concepts and results. But interaction between them is probably more interesting.

In this article, we will tell the story of Riemann's moduli space. There are several reasons for choosing this topic. First, it had an honorable origin in the masterpiece "The theory of Abelian functions" by Riemann in 1857 [65, pp. 79–134]. Second, Riemann's moduli space of Riemann surfaces has been attracting more and more of people's attention, and research on it will probably continue for a long time into the future. As it is often said, history is a mirror to the future, and it is important to understand the history of such an important subject as Riemann's moduli space. Third, the most relevant reason is that this is really a good story full of twists and surprises.

One puzzling question to me is how ideas are transmitted or communicated in mathematics. This issue often leads to some fights for priority and credit in mathematics, ranging from the famous dispute between Newton and Leibniz, to the less known fight between two good friends Klein and Lie [39], and many other contemporary ones. The story of Riemann's moduli space will show that the processes are complicated.

Even for fairly recent topics and results which are published and available to the mathematics community in principle, conclusions cannot be clear cut or definite either. It is perhaps appropriate to quote from a textbook for an undergraduate course on mathematics history by the distinguished mathematics historian Jeremy Gray. In Chapter 13 of [32, p. 131], which is titled *On Writing the History of Geometry*, he wrote:

> The opinion that good historians simply tell us what happened in the past is hopelessly naive. Arguments, opinions and judgements play an inescapably large role in the writing of history; one cannot look at the different views of historians and pretend that somehow one will transcend these and reach the true and correct opinion.

Therefore, there could be many different stories of Riemann's moduli space. There should be many stories for such a central space (or concept) which involves so many diverse fields in contemporary mathematics and has been evolving for a long time.

Though I am interested in the story of moduli spaces, I do not know many things about subjects related to Riemann surfaces, complex analysis, moduli spaces and algebraic geometry etc. Therefore, to avoid my mistakes and misunderstanding in paraphrasing what others have written, I will quote extensively from heroes of this story. Their words speak for themselves.

2. Definitions

Consider the set of all compact Riemann surfaces Σ_g of genus g, or equivalently, the set of smooth projective curves over $\mathbb{C}$ of genus g. Two Riemann surfaces Σ_g, Σ'_g are defined to be *equivalent* if there exists a biholomorphic map $h:\Sigma_g \to \Sigma'_g$. Then the set of equivalence classes $\{\Sigma_g\}/\sim$ of such compact Riemann surfaces Σ_g is *Riemann's moduli space of Riemann surfaces*, usually denoted by $\mathcal{M}_g$. When compact Riemann surfaces are identified with smooth projective algebraic curves over $\mathbb{C}$, $\mathcal{M}_g$ is also the equivalence classes of smooth projective algebraic curves of genus g over $\mathbb{C}$. As given, $\mathcal{M}_g$ is only a set so far. Usually people say that *Riemann's moduli problem*, or *the moduli problem*, is to give a complex structure to $\mathcal{M}_g$. In [6, p. 4], Ahlfors wrote:

> Riemann's classical problem of moduli is not a problem with a single aim, but a program to obtain maximum information about a whole complex of questions which can be viewed from several different angles.

In [7, p. 152], Ahlfors wrote more specifically: "Actually, the classical problem calls for a complex structure".

In [15], Bers wrote:

> This address is a progress report on recent work, partly not yet published, on the classical problem of moduli. Much of this work consists in clarifying and verifying assertions of Teichmüller whose bold ideas, though sometimes stated awkwardly and without complete proofs, influenced all recent investigators, as well as the work of Kodaira and Spencer on the higher dimensional case.

In a preprint of a book on moduli spaces [48] posted on his homepage, Kollar wrote on page 7:

> For elliptic curves we get $\mathcal{M}_1 = \mathcal{A}_1$ and the moduli map is given by the j-invariant, as was known to Euler and Lagrange. They also knew that there is no universal family over $\mathcal{M}_1$. The theory of Abelian integrals due to Abel, Jacobi and Riemann does essentially the same for all curves, though in this case a clear moduli theoretic interpretation seems to have been done only later. For smooth plane curves, and more generally for smooth hypersurfaces in any dimension, the invariant theory of Hilbert produces coarse moduli spaces. Still, a precise definition and proof of existence of $\mathcal{M}_g$ appeared only in [Tei44] in the analytic case and in [Mum65] in the algebraic case.

One purpose of this note is to explain or clarify the meaning of the last sentence and also to justify it, since Teichmüller is best and mainly known for his work on Teichmüller space, which is not the moduli space $\mathcal{M}_g$, but rather an infinite covering of it.

Usually people expect that moduli spaces of complex spaces be complex spaces, and moduli spaces of algebraic spaces be algebraic spaces. This implies that $\mathcal{M}_g$ should be a complex space. When compact Riemann surfaces Σ_g are considered as algebraic curves over $\mathbb{C}$, then $\mathcal{M}_g$ should be an algebraic variety.

As it is known, one space (or a point set, a topological space, or a smooth manifold) can admit many different complex and algebraic structures (the moduli space $\mathcal{M}_g$ is precisely introduced to deal with this problem for surfaces). One basic question is what are *canonical* complex and algebraic structures on $\mathcal{M}_g$ and whether they exist or not.

What is a desirable complex structure on $\mathcal{M}_g$? What is a desirable algebraic variety structure? What are other structures and properties people want to know about $\mathcal{M}_g$? The following discussion will show that the most "natural" or "canonical" complex and algebraic structures on $\mathcal{M}_g$ can be explained in terms of *proper definitions of moduli spaces*.

There are two related notions of moduli spaces: *coarse moduli spaces* and *fine moduli spaces*. The book [58] mainly deals with moduli spaces of algebraic objects such as algebraic curves, abelian varieties, algebraic bundles. On [58, p. 96], it says:

> This algebraic setting dictates the meaning to be given to the word "classify". Obviously, one will want to classify whatever selected objects one is dealing with by another algebraic object: perhaps a scheme, perhaps a variety, or perhaps a functor of more general type such as a Q-variety. But a choice appears at this point:
>
> (1) if one stays close to the classical intuition, the problem posed is: find a scheme M whose geometric points over an algebraically closed field k are in a natural one-one correspondence with the set of selected objects defined over k. The word natural can be interpreted to mean, e.g., that given any family of the selected objects defined over a scheme S, there is a morphism from S to M mapping every geometric point s of S to that point of M which corresponds to the object in the family over the point s.
>
> (2) if one is tempted by the algebra, one tries to classify also the set of selected objects rational over an arbitrary field, or defined over a ring, or over a general scheme (i.e., a family of the objects over that scheme). Then the problem becomes: find a universal family of the selected objects defined over a scheme M, such that every other family, say defined over S, is induced from the universal one by a unique morphism from S to M.
>
> The first will be called the *coarse moduli problem*; the second will be called the *fine moduli problem.*

By replacing algebraic varieties by complex analytic spaces and morphisms by holomorphic maps, we get coarse and fine moduli spaces in the category of complex analytic spaces. In our particular case of $\mathcal{M}_g$, if it admits the structure of a complex analytic space and is a *coarse moduli space* for the moduli problem of compact Riemann surfaces, then it must satisfy the following condition: *for every holomorphic family of compact Riemann surfaces of genus g over a complex analytic space B, $\pi : X \to B$, there exists a holomorphic map $i : B \to \mathcal{M}_g$ such that for every $z \in B$, $i(z) \in \mathcal{M}_g$ is the equivalence class which contains the Riemann surface $\pi^{-1}(z)$.*

This condition alone does not necessarily determine the complex structure on $\mathcal{M}_g$ uniquely. Denote the complex analytic structure on $\mathcal{M}_g$ by J. If there exists another complex analytic structure J' on $\mathcal{M}_g$ such that the identity map on $\mathcal{M}_g$ is a holomorphic map $I : (\mathcal{M}_g, J) \to (\mathcal{M}_g, J')$, then $\mathcal{M}_g$ with the new complex analytic structure J' satisfies the same condition above.

We note that if both $(\mathcal{M}_g, J)$ and $(\mathcal{M}_g, J')$ complex manifolds, then it follows from a standard result in complex analysis that I must be biholomorphic, and hence $\mathcal{M}_g$ at most admits one complex analytic structure to make it a coarse moduli space and a complex manifold. See for example [38, Proposition 1.1.13, p. 13]. Otherwise, there is no reason why $(\mathcal{M}_g, J)$ and $(\mathcal{M}_g, J')$ should be biholomorphic to each other.

To overcome this nonuniqueness problem, usually we require that $(\mathcal{M}_g, J)$ be the *initial* (or *universal*) object among all complex analytic structures on $\mathcal{M}_g$ satisfying the above condition. Then we have a *unique coarse moduli space structure on $\mathcal{M}_g$ if it exists*. See [37, pp. 3–4] and Propositions 5.1 and 5.4 below.

For $\mathcal{M}_g$ to be qualified as a *fine moduli space*, we need the existence of a holomorphic family of Riemann surfaces U_g over $\mathcal{M}_g$ satisfying the *universal property that every family $X \to B$ is the pull-back from U_g by a unique holomorphic map $B \to \mathcal{M}_g$.*

The complex structure of a fine moduli space and the universal family over it *are unique if they exist.* On the other hand, it is known that whenever objects under classification have nontrivial automorphisms, fine moduli spaces cannot exist. The reason is that if nontrivial automorphisms exist, then one can construct nontrivial families whose fibers are isomorphic and hence cannot be a pull-back from the universal family over a fine moduli space. Since Riemann surfaces Σ_g admit nontrivial holomorphic automorphisms, $\mathcal{M}_g$ *does not admit the structure of a fine moduli space.* See [37, Chap. 2, Section A] for more detail.

Remark 2.1. Many moduli problems do not admit fine moduli spaces. This happens if the objects under classification admit nontrivial automorphisms. As we will see below, one crucial idea is to *add additional structures* to the objects to reduce the automorphism group to be trivial. Suppose that we can construct a fine moduli space for the enriched objects. In order to construct a coarse moduli space of the original objects, we need to remove the additional structures, which often amounts to take a quotient of the fine moduli space under an equivalence relation, usually given by a group action. If the group is a discrete and its action on the fine moduli space is proper, then getting a coarse moduli space is relatively easy in the category of complex analytic spaces (see Proposition 5.1

and Remarks 5.2, 5.3), but it is not so easy in the category of algebraic varieties (see Remark 7.1). On the other hand, if the group is neither discrete or compact, this procedure of passing to quotients can be subtle and complicated. Geometric invariant theory in [58] is one method to solve such problems when the group is a noncompact reductive algebraic group.

In the modern literature, moduli spaces are usually defined in terms of *moduli functors* for *moduli problems*. Existence of fine moduli spaces for moduli problems amounts to the *representability* of the moduli functors, and coarse moduli spaces are defined by a *natural transformation from the moduli functors to the functor of points*. More precisely, in this formulation, there are two important ingredients:

1. Specify a class of objects to be classified and a notion of *families of objects over allowable spaces*, which could be complex analytic spaces, varieties or schemes.
2. Choose isomorphisms between (or an equivalence relation on) families of objects under consideration.[1]

If a moduli functor is representable by a space $\mathcal{M}$, then $\mathcal{M}$ is a *fine moduli space* for the moduli problem. Similarly, we can define a *coarse moduli space* for a moduli problem. See Propositions 5.1 and 5.4 below for some concrete examples and what are the precise conditions for fine moduli and coarse moduli spaces.

[1] According to Deligne [25]: Morphisms play no role. Isomorphisms do, but for me [Deligne] any notion of object carries automatically its notion of isomorphisms. [Compare this with [37, pp. 1–2]: The equivalence relations we will wish to consider will vary considerably even for a fixed class of objects: in the second case cited above [subschemes C in $\mathbb{P}^r \times B$, flat over B, whose fibers over B are cubes of fixed genus g and degree d], we might wish to consider two families equivalent if

(a) the two subschemes of $\mathbb{P}^r \times B$ are equal,

(b) the two subcurves are projectively equivalent over B, or

(c) the two curves are (biregularly) isomorphic over B.]

A crucial property of "families" is that it gives rise to pull back, so that

$$S \to \{\text{isomorphism classes of families over } S\}$$

is a (contravariant) functor.
The story is sometimes more subtle. For $\mathrm{Pic}_{X/S}$, one starts with the functor

$$T/S \to \text{isomorphism classes of line bundles on } X \times_S T$$

but one takes an associated sheaf to this presheaf (for the fppt topology). This causes the problem that line bundles have automorphisms (but one has to assume $f_*\mathcal{O}_X = \mathcal{O}_S$, hence $f_*\mathcal{O}_X^* = \mathcal{O}_S^*$ universally: no jumps in the group of automorphisms). The map $\mathrm{Pic}_{X/S} \to \mathrm{Pic}(X \times_S T)$ will in general be neither injective (the pull back of $\mathrm{Pic}(S)$ is killed) nor surjective (for a conic with no points on $\mathbb{R}$, it is, for $T = S = \mathrm{Spec}(\mathbb{R})$: $2\mathbb{Z} \hookrightarrow \mathbb{Z}$).

See also [58, pp. 98–99] [37, pp. 1–4] and [43, Chap. 4] for the general definitions.

After the above general discussions on moduli spaces, we are going to take a look at some original papers and books to see how the story of Riemann's moduli has unfolded, for example, when definitions of the moduli space $\mathcal{M}_g$ were first given, and when the existence of complex analytic and algebraic structures on it was first proved.

3. Riemann's Count of Moduli for Riemann Surfaces

Riemann surfaces were first introduced by Riemann in 1851 in his thesis *Foundations for a general theory of functions of a complex variable* [65, pp. 1–41] as *natural domains* to study holomorphic functions of one complex variable. Later in 1857 in his paper *The theory of Abelian functions* [65, pp. 79–134], Riemann made the notion of Riemann surfaces more concrete and applied it to represent multivalued functions arising from algebraic equations and functions arising from integrals such as $\log z$. He visualized Riemann surfaces as branched coverings over domains of $\mathbb{C}P^1$. See [40] for some discussion and detail.

In this 1857 paper, Riemann introduced the problem of birational classification of plane algebraic curves and counted the number of moduli for Riemann surfaces via branching points. Specifically, on [65, p. 111], he wrote:

> We shall now consider *all irreducible algebraic equations between two complex variables, which can be transformed into one another by rational transformations, as belonging to the same class*. Thus $F(s,z) = 0$ and $F_1(s_1,z_1) = 0$ belong to the same class, if rational functions of s_1 and z_1 can be found which, when substituted for s and z respectively, transform the equation $F(s,z) = 0$ into the equation $F_1(s_1,z_1) = 0$; while equally s_1 and z_1 are rational functions of s and z.
>
> The rational functions of s and z regarded as functions of any one of them, say ζ, constitute a system of similarly branching algebraic functions. In this way, every equation clearly gives rise to a class of systems of similarly branching algebraic functions, which by introducing one of them as an independent variable, can be rationally transformed into each other. Moreover, all the equations of one class lead to the same class of systems of algebraic functions, and conversely (Section 11) each class of such systems leads to one class of equations.
>
> If the (s,z) domain is $2p+1$ times connected and the function ζ becomes infinite of first order at μ

points of this domain, the number of branch points of equivalently branching functions of ζ, which can be formed by the other rational functions of s and z, is $2(\mu+p-1)$, and the number of arbitrary constants in the function ζ is therefore $2\mu-p+1$ (Section 5). These constants can always be chosen so that $2\mu-p+1$ branch points take arbitrarily assigned values, when these branch points are mutually independent functions of the constants. This can be done in only a finite number of different ways because the conditions are algebraic. In each class of similarly branching functions with connectivity $2p+1$, there is consequently only a finite number of μ-valued functions such that $2\mu+p-1$ branch points have prescribed values. If, on the other hand, the $2(\mu+p-1)$ branch points of a surface with connectivity $2p+1$, covering the whole ζ-plane μ-times, are arbitrarily prescribed, then (Sections 3–5) there is always a system of algebraic functions of ζ branching like the surface. The remaining $3p-3$ branch points in these systems of similarly branching μ-valued functions can therefore be assigned any given values; and thus a class of systems of similarly branching functions with connectivity $2p+1$, and the corresponding class of algebraic equations, depends on $3p-3$ continuous variables, which we shall call the moduli of the class.

He then explained that this count only worked under the assumption that the genus $p \geq 2$ and worked out the case $p=1$. See an modern presentation of the above count of moduli, see [34, pp. 255–256].

From the above quote of Riemann's paper, the precise meaning of moduli is not clear. But it seems reasonable to say that *the moduli for an equivalence class of Riemann surfaces is a collection of non-redundant continuous local complex parameters needed to determine a* general *equivalence class of Riemann surfaces*.[2] This is one common, non-rigorous way for some (or many) people to understand the original meaning of moduli in Riemann's paper. For example, in the paper [46, p. 331], Kodaira and Spencer wrote:

[2] According to Deligne [25]: I think at that time there was a consensus that any reasonable space, set of objects, ..., had a dimension, which for general objects obj of the kind considered was the number of parameters needed to fix an object over obj (while for more special object less would suffices). This is the intuition now captured by "stratified space, strata manifolds of dimension $\leq d$, open strata of $\dim d$". It is because of this fact that Peano's square filling curves was met with such dismay, but I guess most mathematicians felt (rightly) that this "pathology" did not occur for the "reasonable" objects they considered. At least the Italian continued to speak of "∞^d-objects of some kind". The analysts also spoke of space of solutions (of a PDE) depending on so many functions of so many variables, and here it is less clear we can justify the intuition.

Deformation of the complex structure of a Riemann surface is an idea which goes back to Riemann who, in his famous memoir on abelian functions published in 1857, calculated the number of independent parameters on which the deformation depends and called these parameters "moduli".

This is the only place in Riemann's publications [65] where he discussed moduli of Riemann surfaces, and Riemann did not explicitly raise any question about the moduli space $\mathcal{M}_g$. It might be helpful towards understanding the above notion of moduli to note that Riemann introduced Riemannian metric and Riemannian curvature, and gave a characterization of flat spaces (i.e., locally isometric to $\mathbb{R}^n$) in terms of Riemannian curvature in his two papers: (1) *The hypotheses on which geometry is based*, (2) *A mathematical work that seeks to answer the question posed by the most distinguished academy of Paris* [65, pp. 257–272, pp. 365–381].

Remark 3.1. This 1857 paper of Riemann, *The theory of Abelian functions*, has had a huge impact on mathematics and is considered as a masterpiece by many people. Besides introducing the problem of birational classification of algebraic curves and Riemann surfaces, the notion of their moduli and the count, it also contains global topological properties of surfaces such as Betti numbers and Riemann-Hurwitz formula, function theory on Riemann surfaces including Riemann's inequality in the Riemann-Roch Theorem, a solution of general inversion problems for abelian integrals which had been solved for elliptic integrals by Abel and Jacobi and which first made Riemann famous.

4. Klein's Booklet on Riemann's Work and the Method of Continuity to Prove the Uniformization Theorem by Klein and Poincaré

Though the importance of Riemann's work in general was recognized by many people right from the beginning, Felix Klein was often considered as a champion to promote the geometric function theory of Riemann. According to Courant [64, p. 178], "If today we are able to build on the work of Riemann, it is thanks to Klein."[3]

[3] It should be emphasized that, independent of Klein, Poincaré made enormous contributions to Riemann surfaces and moduli spaces of hyperbolic surfaces. The following summary seems to be a fair assessment of Poincaré work on these topics:

1. Poincaré's work on constructing Riemann surfaces as quotients of the upper half plane by Fuchsian groups, in particular via the Poincaré polygon theorem.

In the book [45], Klein tried to explain some work of Riemann in ways he thought that Riemann did it. For example, the title of Chapter 19 of Part III of this book is *On the moduli of Algebraic Equations.* He made some statements about the moduli spaces of Riemann surfaces by essentially repeating Riemann. He did not define things precisely or give rigorous arguments to support them. On [45, pp. 80–81], he wrote:

> In one important point, Riemann's theory of algebraic functions surpasses in results as well as in methods the usual presentations of this theory. It tells us that, *given graphically a many-sheeted surface over the z plane, it is possible to construct associated algebraic functions,* where it must be observed that these functions if they exist at all are of a highly arbitrary character, $R(w,z)$ having in general the same branchings as w. This theorem is the more remarkable, in that it implies a statement about an interesting equation of higher order. For if the branch-points of an m-sheeted surface are given, there is a finite number of essentially different possible ways of arranging these among the sheets; this number can be found by considerations belonging entirely to pure analysis situs...
>
> Let us with Riemann speak of all algebraic functions of z as belonging to the same class when by means of z they can be rationally expressed in terms of one another. *Then the number in question is the number of different classes of algebraic functions which, with respect to z, have the given branch-values.*
>
> In the present and following sections various consequences are drawn from this preliminary proposition and among these we may consider in the first place the question of the *moduli* of the algebraic functions, i.e. of those constants which play the part of the invariants in a uniform transformation of the equation $f(w,z) = 0$.
>
> For this purpose let ρ be a number initially unknown, expressing the number of degrees of freedom in any one-one transformation of a surface into itself, i.e. in a conformal representation of the surface upon itself. Then let us recall the number of available constants in uniform functions on given surfaces (§13). We found that there were in general ∞^{2m-p+1} uniform functions with m infinities and that this, as we stated without proof, is the exact number when $m > 2p-2$. Now each of these functions maps the given surface by a uniform transformation upon an m-sheeted surface over the plane. *Hence the totality of the m-sheeted surfaces upon which a given surface can be conformally mapped by a uniform transformation, and therefore also the number of m-sheeted surfaces with which an equation $f(w,z) = 0$ can be associated, is $\infty^{2m-p+1-\rho}$;* for ∞^{ρ} representations give the same m-sheeted surface, by hypothesis.
>
> But there are in all ∞^{w} m-sheeted surfaces, where w is the number of branch-points, i.e. $2m+2p-2$. For, as we observed above, the surface is given by the branch-points to within a finite number of degrees of freedom, and branch-points of higher multiplicity arise from coalescence of simple branch-points as we have already explained in connection with the corresponding cross-points in §1 (cf. Figs. 2, 3). With each of these surfaces there are, as we know, algebraic functions associated. The number of moduli is therefore
>
> $$w - (2m+1-p-\rho) = 3p-3+\rho.$$
>
> It should be noticed here that the totality of m-sheeted surfaces with w branch-points form a *continuum*, corresponding to the same fact, pointed out in §13 with respect to uniform functions with m infinities on a given surface. Hence we conclude *that all algebraical equations with a given p form a single continuous manifoldness*, in which all equations derivable from one another by a uniform transformation constitute an individual element. Thus, for the first time, a precise meaning attaches itself to the number of the moduli; *it determines the dimensions of this continuous manifoldness.*
>
> The number ρ has still to be determined and this is done by means of the following propositions.

2. Poincaré's proofs of the uniformization of Riemann surfaces.

3. Though Poincaré did not contribute to the moduli space of Riemann surfaces (or complex structures), the hyperbolic metric of Riemann surfaces of negative Euler characteristic and use of Fuchsian groups (or hyperbolic metric) via the uniformization theorem has been crucial to the study of various properties of Teichmüller and moduli spaces of Riemann surfaces. For example, in the definition of Weil-Petersson metric, the hyperbolic metric of Riemann surfaces is essentially used.

4. Poincaré just used what Riemann said about the moduli space of Riemann surfaces (or complex structures) when he applied the method of continuity to prove the uniformization theorem, but he did not contribute to this moduli space of complex structures.

5. In trying to prove the uniformization theorem using the method of continuity, Poincaré had also introduced an early vague version of Teichmüller space of hyperbolic surfaces, or rather spaces of hyperbolic surfaces with additional data such as spaces of polygonal groups, i.e., Fuchsian groups generated via the Poincaré polygon theorem.

6. As for topology or algebraic topology, Poincaré was a better successor of Riemann than Klein since he started the subject of algebraic topology.

1. *Every equation for which* $p = 0$ *can by means of a one-one relation be transformed into itself* ∞^3 *times...*

2. *Every equation for which* $p = 1$ *can be transformed into itself in a singly infinite number of ways...*

3. *Equations for which* $p > 1$ *cannot be changed into themselves in an infinite number of ways...*

According to the convention at that time, a continuum meant an arcwise connected open subset of the complex plane or complex spaces $\mathbb{C}^n$ (see [2, §3] for more discussion). Since we know that $\mathcal{M}_g$ cannot be realized biholomorphically as a domain in $\mathbb{C}^n$ for any n, this definition does not apply. So its meaning in Klein's book is not exactly clear.[4] A more modern definition of continuum is that it is a compact, connected metric space, which was motivated by a definition of Cantor in 1883 as a perfect, connected set of the real line [20, p. 194]. This definition does not apply either.[5]

The German version of [45] was published in 1881. Around the same time, other major papers and books which contributed to or made substantial use of the moduli space $\mathcal{M}_g$ are the papers of Poincaré and Klein which tried to prove the uniformization theorem for Riemann surfaces of finite type (i.e., Riemann surfaces which are obtained from compact Riemann surfaces by removing finitely many points, in particular those obtained from plane algebraic curves over $\mathbb{C}$) by *using the method of continuity.* The basic idea is to use the canonical map from the *space of Fuchsian groups* to the *space of Riemann surfaces*, via quotients of the upper halfplane by Fuchsian groups, and show that this injective map is bijective. Then every Riemann surface of negative Euler characteristic is given by the quotient of the upper half plane by a Fuchsian group. For this purpose, they needed various assumptions on the moduli space $\mathcal{M}_g$, and they also essentially took or repeated what Riemann said. The second volume of books by Fricke and Klein [29] was published a few decades later and gave a summary of their proofs, but no new results on the moduli space $\mathcal{M}_g$ of Riemann surfaces were obtained there.

See [2] for more detailed comments on the applications to the method of continuity and references. The paper [2] also contains some discussions on related work of Brouwer on invariance of domains, which he applied to salvage the above proof of the uniformization of Riemann surfaces by the method of continuity. Brouwer is one hero who will appear once more below in discussing the work of Teichmüller which motivated him to study Riemann's moduli space in the early 1940s.

The above summary was basically the status of results on Riemann's moduli space $\mathcal{M}_g$ when Teichmüller entered the subject.

5. Teichmüller's Work on Moduli Space $\mathcal{M}_g$

Teichmüller is famous for the Teichmüller space, which is an infinite covering space of Riemann's moduli space. These two spaces are closely related. We will first explain his work on both spaces from the point of view of moduli theories. In the second subsection, we will explain how his work on extremal quasiconformal maps led him to study these moduli spaces of Riemann surfaces.

5.1 Teichmüller's Rigorous Formulation of and Results on Riemann's Moduli Space

Teichmüller wrote three major papers on Teichmüller spaces and quasiconformal maps between Riemann surfaces [71] [72] [73] together with several others which dealt with some special cases.[6] The long paper [71] and its sequel or supplement [72] are well-known. In his MathSciNet review of the collected works of Teichmüller [74], Strebel called the paper [71] a masterpiece and wrote:

> Teichmüller's Habilitationsschrift: "Untersuchungen über konforme und quasikonforme Abbildungen", which appeared in 1938, and the next paper: "Ungleichungen zwischen den Koeffizienten schlichter Funktionen" can be considered as the beginning of his great contributions to function theory, which culminated in his masterpiece: "Extremale quasikonforme Abbildungen und quadratische Differentiale", 1939. In this paper and its complement: "Bestimmung der extremalen quasikonformen Abbildungen bei geschlossenen orientierten Riemannschen Flächen" (1943), Teichmüller laid the basis of what is now known as the theory of Teichmüller spaces. He further developed the theme in one of his last papers: "Veränderliche Riemannsche Flächen" (1944).

[4] According to Deligne [25]: "continuum" may have had this specialized meaning in some cases, but Klein works within the consensus explained above in Footnote 2.

[5] The meaning of the word "manifoldness" is not precise or clear either, and it should also be understood in the consensus explained by Deligne in Footnote 2 above.

[6] In this note, we are only interested in the mathematics of Teichmüller. As L. Ahlfors and F. Gehring said in the preface to Teichmüller's Collected Works [74]: "Oswald Teichmüller deserves our respect and admiration for his mathematics. His life is another matter. The charitable explanation is that he was a politically naive victim of the disease that was rampant in his country. A redeeming feature is that he did not stoop to racial slurs in his scientific papers, which shows that his regard for mathematics was stronger than his prejudices."

But his paper [73] was not so well-known, though probably most relevant to our story. The following description of [73] will show that this is the first paper *which defines precisely the moduli problem for Riemann surfaces and constructs a fine moduli space for marked Riemann surfaces,* which is the Teichmüller space $\mathcal{T}_g$ with a complex structure, consequently the moduli space *giving* $\mathcal{M}_g$ *the structure of a coarse moduli space as a complex analytic space for compact Riemann surfaces of genus* g (Proposition 5.1, Remark 5.2).

In the following, we will use the page numbers of the English translation of [73] in the *Handbook of Teichmüller theory*. The beginning of the paper [73, pp. 787–788] explained well the status on the moduli problem of Riemann surfaces at that time and the motivation for Teichmüller's work. Specifically, Teichmüller wrote:

> It has been known for a long time that the classes of conformally equivalent closed Riemann surfaces of genus g depend on τ complex constants, where
>
> $$\tau = \begin{cases} 0 & \text{if } g = 0, \\ 1 & \text{if } g = 1, \\ 3(g-1) & \text{if } g > 1. \end{cases}$$
>
> This number τ has been obtained using different heuristic arguments, and the result is being passed on in the literature without thinking too much about the meaning of this statement. The set $\mathfrak{R}$ of all classes of conformally equivalent surfaces of a genus $g > 0$ certainly has the cardinality of the continuum; if in addition to that one wants to calculate the dimension of $\mathfrak{R}$ then at first one has to turn $\mathfrak{R}$ into a space in some way.
>
> Initially it has been tried to describe the elements of $\mathfrak{R}$ with τ coordinates or also with $\nu > \tau$ coordinates satisfying $\nu - \tau$ relations. But the different calculations of τ on these grounds stood in no relationship to each other, and it was not a priori clear but rather a miracle that the calculations always lead to the same value of τ. I think that one should not primarily ask for an explicit representation of the points of $\mathfrak{R}$ via numbers in a coordinate system, but study the inner structure of the "space" $\mathfrak{R}$.
>
> One ought to start introducing a notion of neighbourhood in $\mathfrak{R}$. But up to now there is but one approach, and that is not appropriate as a foundation. But as long as $\mathfrak{R}$ is not a space with a notion of neighbourhood, it does not have a dimension in the sense of analysis or set theory.[7]
>
> Then v.d. Waerden[a] has proved with the framework of algebraic geometry that an algebraic function field depends on τ parameters if certein pathologies are excluded. I will not go into this since in this approach the problem is replaced by a purely algebraic one whose solution does not provide a function theorist with the insight he is seeking.
>
> One does not only need $\mathfrak{R}$ to have the structure of a *neighbourhood space* or an *algebraic variety* but also of an *analytic manifold*, i.e. in every neighbourhood of a point of $\mathfrak{R}$, a coordinate system with τ coordinates is wanted, and all the coordinate transformations should be analytic. Then the notion of neighbourhood is provided automatically, and in our case it is not difficult to choose a class of coordinate systems transforming algebraically into one another.
>
> But since the set $\mathfrak{R}$ can be made into an analytic manifold in different ways, we have to make sure that the choice is uniquely determined by certain properties. And these determining properties must ensure the applicability in function theory. It turns out that $\mathfrak{R}$ contains certain singular manifolds. But we will construct a covering space $\underline{\mathfrak{R}}$ without singularities.
>
> [a] B. L. van der Waerden, Zur algebraischen Geometrie. XI. Projektive Äquivalenz und Moduln von ebenen Kurven. Math. Ann. 114.

In the above quote, $\mathfrak{R}$ is the moduli space $\mathcal{M}_g$. It was clear that there was still no definition of $\mathcal{M}_g$ beyond a set in literature up to that point, in spite of a lot of intuition by various people and expected properties of $\mathcal{M}_g$.

Note that Teichmüller died at the Eastern Front in September of 1943, and probably wrote this paper [73] shortly before he went there.[8] He continued [73, p. 788]:

[7] According to Deligne [25]: In telling that one needs a topology to be allowed to speak of dimension, Teichmüller was very modern.

[8] According to wikipedia: The battles on the Eastern Front constituted the largest military confrontation in history. They were characterized by unprecedented ferocity, wholesale destruction, mass deportations, and immense loss of life variously due to combat, starvation, exposure, disease, and massacres. The Eastern Front, as the site of nearly all extermination camps, death marches, ghettos, and the majority of pogroms, was central to the Holocaust. Of the estimated 70 million deaths attributed to World War II, over 30 million, many of them civilian, occurred on the Eastern Front. The Eastern Front was decisive in determining the outcome of World War II, eventually serving as the main reason for Germany's defeat. It resulted in the destruction of the Third Reich, the partition of Germany for nearly half a century and the rise of the Soviet Union as a military and industrial superpower....
According to a biography by O'Connor and Robertson [59]: Teichmüller was drafted on 18 July 1939 as Germany prepared for World War II. He was originally supposed to do eight weeks training, but before the eight weeks were up, World War II began on 1 September 1939. He remained

In the near future I will not be able to publish all my extensive considerations in detail. Therefore at this point I just give a short overview of the methods and results.

As I am realizing now, my solution of the problem is based mainly on three newly introduced notions, namely

topological determination,
analytic family of Riemann surfaces,
winding piece coordinates.

This paper [73] is very rich in ideas. For more detailed commentaries on it, see [1] and [2]. We only want to point out the following:

1. Teichmüller was aware that nontrivial automorphisms of Riemann surfaces caused difficulty in constructing $\mathcal{M}_g$ and singularities of $\mathcal{M}_g$, and he introduced *the idea of marking to rigidify Riemann surfaces* and hence to kill nontrivial automorphisms of Riemann surfaces. This is the meaning of *topological determination*.
2. He introduced the notion of *analytic family of Riemann surfaces* in order to define moduli spaces rigorously, similar to the modern formulation in terms of categories of spaces over spaces, functors and natural transformations. As pointed out in §2, this is a crucial and beginning step in order to formulate rigorously moduli problems for Riemann surfaces.
3. He constructed the universal curve over the Teichmüller space $\mathcal{T}_g$ with a complex manifold structure so that the Teichmüller space $\mathcal{T}_g$ is a fine moduli space of marked Riemann surfaces and has a natural complex manifold structure. Consequnetly, it resolves the singularities of $\mathcal{M}_g$.
4. He undid the marking on Riemann surfaces by taking the moduli space $\mathcal{M}_g$ as the quotient of Teichmüller space $\mathcal{T}_g$ under the action of the mapping class group of a surface Σ_g.

For (1), his definition of markings on Riemann surfaces [73, p. 789] is the current one used the Teichmüller theory community:

> The **topological determination** of closed surfaces of genus g is done as follows: Let $\mathfrak{H}_0$ be a fixed and $\mathfrak{H}$ an arbitrary closed Riemann surface of genus g. Let H be a topological map from $\mathfrak{H}_0$ onto $\mathfrak{H}$.... We will have to deal with *pairs* $(\mathfrak{H}, H)$ of a surface $\mathfrak{H}$ and a map of the fixed $\mathfrak{H}_0$ onto $\mathfrak{H}$. Two such pairs are called *equal* $(\mathfrak{H}, H) = (\mathfrak{H}', H')$, if firstly $\mathfrak{H} = \mathfrak{H}'$ and secondly the map $H'^{-1}H$ from $\mathfrak{H}_0$ onto itself can be deformed into the identity....
>
> We denote a class of conformally equivalent topologically determined surfaces $\mathfrak{H}$ of genus g by $\mathfrak{h}$. These classes are the points of the "space" $\underline{\mathfrak{R}}$. In contrast to this the classes of conformally equivalent surfaces $\mathfrak{H}$ of genus g denoted by $\mathfrak{h}$, form the "space" $\mathfrak{R}$.
>
> The *moduli problem* consists of asking for the properties of the space $\mathfrak{R}$. But it turns out that it is better to study the space $\underline{\mathfrak{R}}$ at first.

For (2), we should keep in mind that the notion of high dimensional complex manifolds were not so well-known in the 1930s. Consequently, Teichmüller wrote [73, pp. 789–790]:

> We now come to the most important notion that we have to introduce here, namely the notion of an **analytic family of Riemann surfaces.** – As a start, let us recall the notion of an *analytic manifold*...
>
> Let $\mathfrak{P}$ be such an r-dimensional complex analytic manifold... Suppose that to every point $\mathfrak{p}$ in $\mathfrak{P}$ is associated a closed Riemann surface $\mathfrak{H} = \mathfrak{H}(\mathfrak{p})$ of fixed genus g... A family $\mathfrak{H}(\mathfrak{p})$ that has been made into an $(r+1)$-dimensional manifold M in the way just described, is called an *analytic family of Riemann surfaces*, and $\mathfrak{P}$ is called its *parameter manifold.*

In other words, he defined carefully a holomorphic family of compact Riemann surfaces over a complex manifold $\mathfrak{P}$.

---

in the army and, in April 1940, took part in the German invasion of Norway. After this he was transferred to Berlin to undertake cryptographic work. Bieberbach requested that Teichmüller be released from military duties to lecture at the university and indeed Teichmüller was able to teach at the university beginning at the start of session 1942–43 while continuing his cryptographic work. He still found time to continue his mathematical research with five papers being published in 1944: *Über die partielle Differentiation algebraischer Funktionen nach einem Parameter und die Invarianz einer gewissen Hauptteilsystemklasse; Beweis der analytischen Abhängigkeit des konformen Moduls einer analytischen Ringflächenschar von den Parametern; Ein Verschiebungssatz der quasikonformen Abbildung; Veränderliche Riemannsche Flächen; and Einfache Beispiele zur Wertverteilungslehre.* The first of these appeared in Crelle's *Journal*, the other four in *Deutsche Mathematik*.
The Battle of Stalingrad raged between July 1942 and February 1943. The Germans attempted to take the city but there was stubborn resistance from the Russians. Eventually the German 6th Army was tricked and, after becoming trapped, was largely destroyed. It was the first major military defeat for the German armies and a new call to arms was made across Germany. Teichmüller answered this call and, giving up his cryptographic position in Berlin, joined the forces attempting to recover from the Stalingrad defeat. The German aim was to shorten their eastern line by taking the area around Kursk where the Russian forces held positions. Teichmüller's unit took part in the offensive which began on 5 July 1943. A Russian counterattack in early August saw the Germans forced to fall back. Teichmüller was given leave of absence to return home. His unit was, at that stage, in the vicinity of Kharkov and, after a battle lasting from 3 August to 23 August, Kharkov was recaptured by the Russians. Most of Teichmüller's unit was wiped out but in early September he tried to rejoin them. He seems to have reached Poltava, southwest of Kharkov, but was killed in the confused situation as the German forces retreated in disarray before the Russian advance. He died on September 11, 1943.

For (3), we quote the main result of [73, pp. 793–794]:

For every g there exists a globally analytic family of topologically determined Riemann surfaces of genus g: $\underline{\mathfrak{H}}[\mathfrak{c}]$ where $\mathfrak{c}$ runs through a τ-dimensional complex analytic manifold $\mathfrak{C}$, with the following properties:

• *For every topologically determined surface $\mathfrak{H}$ of genus g there exists one and only one conformally equivalent $\underline{\mathfrak{H}}[\mathfrak{c}]$.*

• *If $\underline{\mathfrak{H}}(\mathfrak{p})$ is any globally analytic family of topologically determined surfaces of genus g described by the parameters $p_1, ..., p_r$ and the permanent local parameter t, there exists a map from the family $\underline{\mathfrak{H}}(\mathfrak{p})$ to the family $\underline{\mathfrak{H}}[\mathfrak{c}]$ with parameters $c_1, \cdots, c_\tau$ and permanent local parameter T, such that $c_1, \cdots, c_\tau$ become analytic functions of $p_1, \cdots, p_r$ and T becomes an analytic function of $p_1, \cdots, p_r$, t with $\frac{\partial T}{\partial t} \neq 0$ and such that the topologically determined surface $\underline{\mathfrak{H}}(\mathfrak{p})$ is mapped conformally onto $\underline{\mathfrak{H}}[\mathfrak{c}]$.*

The family $\underline{\mathfrak{H}}[\mathfrak{c}]$ is essentially uniquely determined by these properties. The (complex) dimension τ of $\mathfrak{C}$ equals ...

It is easy to see that the family $\underline{\mathfrak{H}}[\mathfrak{c}]$ is uniquely determined by the stated properties: Let $\underline{\mathfrak{H}}'[\mathfrak{c}']$ be a second family with the same properties, ...

With this theorem the **moduli problem** is solved. Namely, we obtain a one-to-one correspondence between the space $\underline{\mathfrak{R}}$ of all classes of conformally analytic topologically determined surfaces of genus g and the τ-dimensional complex analytic manifold $\mathfrak{C}$, by associating to every class $\mathfrak{h}$ in $\underline{\mathfrak{R}}$ the element $\mathfrak{c}$ in $\mathfrak{C}$ such that $\mathfrak{H}[\mathfrak{c}]$ corresponds to $\mathfrak{h}$. *Hereby, we also provide $\underline{\mathfrak{R}}$ in a unique way with the structure of a τ-dimensional complex analytic manifold.*

The fine moduli space $\underline{\mathfrak{R}}$ is the Teichmüller space $\mathfrak{C}$ with the universal curve $\underline{\mathfrak{H}}[\mathfrak{c}]$, $\mathfrak{c} \in \mathfrak{C}$, on it. It is stated exactly in the modern language which are familiar to algebraic geometers and quoted in §2 above.

For (4), he removed the topological markings of Riemann surfaces to recover the moduli space $\mathfrak{R}$ (i.e., $\mathcal{M}_g$), i.e., $\mathcal{M}_g$ is the quotient of the Teichmüller space by the mapping class group of a compact oriented surface of genus g. Teichmüller wrote [73, pp. 800–802]:

At last I want to mention briefly what happens when abolishing the topological determination.

Let $\underline{\mathfrak{H}}$ be a topologically determined Riemann surface of genus g. Let $\mathfrak{G}$ be the group of all topological maps of the underlying surface $\underline{\mathfrak{H}}$ onto itself and let $\mathfrak{A}$ be the normalizer of all elements of $\mathfrak{G}$ that can be deformed into the identity. The factor group

$$\mathfrak{F} = \mathfrak{G}/\mathfrak{A}$$

is the *mapping class group* of $\underline{\mathfrak{H}}$....

It is shown that the representation of $\mathfrak{F}$ by maps $\underline{\mathfrak{h}} \to F\underline{\mathfrak{h}}$ from $\underline{\mathfrak{R}}$ onto itself is *properly discontinuous....*

$\underline{\mathfrak{R}}$ is an analytic manifold. In all points of $\underline{\mathfrak{R}}$ that are only invariant under those elements of $\mathfrak{F}$ fixing all points of $\underline{\mathfrak{R}}$, one can directly transfer a coordinate system from $\underline{\mathfrak{R}}$ to $\mathfrak{R}$ and obtains also in $\mathfrak{R}$ an analytic coordinate system. This is because the group is properly discontinuous. Here the "general" point of $\underline{\mathfrak{R}}$ is already taken care of because the exceptional points lie on certain analytic manifolds in $\underline{\mathfrak{R}}$, as will result from the following.

Now we switch to the modern standard notation and derive some corollaries. Let $\mathcal{T}_g$ be the *Teichmüller space* of marked compact Riemann surfaces of genus g, and Mod_g be the mapping class group of a compact oriented surface S_g of genus g,

$$\mathrm{Mod}_g = \mathrm{Homeo}^+(S_g)/\mathrm{Homeo}^0(S_g),$$

where $\mathrm{Homeo}^+(S_g)$ is the group of orientation preserving homeomorphisms of S_g, and $\mathrm{Homeo}^0(S_g)$ is the identity component of $\mathrm{Homeo}^+(S_g)$. (We can also define the mapping class group Mod_g using diffeomorphisms of S_g, $\mathrm{Mod}_g = \mathrm{Diff}^+(S_g)/\mathrm{Diff}^0(S_g)$, where $\mathrm{Diff}^+(S_g)$ is the group of orientation preserving diffeomorphisms of S_g, and $\mathrm{Diff}^0(S_g)$ is the identity component. This is not true in general for mapping class groups of manifolds in high dimension).

As mentioned above, Teichmüller showed that $\mathcal{T}_g$ is the fine moduli space for marked Riemann surfaces and has a canonical complex structure. The mapping class group Mod_g acts properly and holomorphically on $\mathcal{T}_g$. Therefore, the quotient $\mathcal{M}_g = \mathrm{Mod}_g\backslash\mathcal{T}_g$ also has a canonical complex analytic structure, or rather a complex orbifold structure.

Teichmüller did not define coarse moduli spaces. To deal the space of equivalence classes of unmarked Riemann surfaces, he wrote [73, p. 788]: "But since the set $\mathfrak{R}$ can be made into an analytic manifold in different ways, we have to make sure that the choice is uniquely determined by certain properties", [73, p. 880]: "At last I want to mention briefly what happens when abolishing the topological determination", and [73, p. 802]: "The space $\mathfrak{R}$ of all classes $\mathfrak{h}$ of conformally equivalent Riemann surfaces of genus g arises from $\underline{\mathfrak{R}}$ by identifying equivalent points via $\mathfrak{F}$ ($\underline{\mathfrak{h}}_1$ and $\underline{\mathfrak{h}}_2$ are identified if and only if there is an F in $\mathfrak{F}$ with $F\underline{\mathfrak{h}}_1 = \underline{\mathfrak{h}}_2$)."

Since our emphasis is on moduli spaces, we observe the following result:

Proposition 5.1. *The space $\mathcal{M}_g$ with the complex structure induced from $\mathcal{T}_g$ is a coarse moduli space for compact Riemann surfaces of genus g.*

Proof. By putting the conditions for a coarse moduli space (see [37, pp. 3–4]) into concrete terms, we need to check two things:

1. For every holomorphic family of compact Riemann surfaces of genus g, $\pi : X \to B$, over a complex space B, there is a unique holomorphic map $f : B \to \mathcal{M}_g$ such that for every $b \in B$, $f(b)$ is the equivalence class of Riemann surfaces which contains the fiber $\pi^{-1}(b)$.
2. For any complex space $\mathcal{M}'$ whose points are in 1-1 correspondence with equivalence classes of compact Riemann surfaces of genus g and which satisfies the previous condition (1), there exists a holomorphic map $\varphi : \mathcal{M}_g \to \mathcal{M}'$ such that for any holomorphic family of Riemann surfaces $\pi : X \to B$ above, the composed map $\varphi \circ f : B \to \mathcal{M}_g \to \mathcal{M}'$ is the unique holomorphic map $B \to \mathcal{M}'$ required by Condition (1).

To prove (1), we note that the family $\pi : X \to B$ lifts to a holomorphic family $\tilde{\pi} : X' \to \tilde{B}$, where $\tilde{B}$ is the universal covering of B. Since $\mathcal{T}_g$ is a fine moduli space, we get a holomorphic map $\tilde{f} : \tilde{B} \to \mathcal{T}_g$ which is equivariant with respect to a homomorphism $\pi_1(B) \to \mathrm{Mod}_g$. Hence we get the desired holomorphic map $f : B \to \mathcal{M}_g$.

Suppose $\mathcal{M}'$ is a complex space satisfying Condition (1). Since $\mathcal{T}_g$ has the universal family of Riemann surfaces over it, we get a holomorphic map $\Phi : \mathcal{T}_g \to \mathcal{M}'$. Since $\mathcal{M}'$ is in 1-1 correspondence with equivalence classes of Riemann surfaces, this factors through the space $\mathcal{M}_g = \mathrm{Mod}_g \backslash \mathcal{T}_g$ and gives a holomorphic map $\varphi : \mathcal{M}_g \to \mathcal{M}'$. We need to show that for any family above $X \to B$, $\varphi \circ f$ is the unique map $B \to \mathcal{M}'$. Since the composed map $\tilde{B} \to \mathcal{T}_g \to \mathcal{M}'$ is the unique map for $\tilde{B} \to \mathcal{M}'$ for the holomorphic family $\tilde{\pi} : X' \to \tilde{B}$, it is clear that the composed map $\varphi \circ f : B \to \mathcal{M}_g \to \mathcal{M}'$ is also the unique desired holomorphic map for the family $X \to B$. □

Remark 5.2. This is one simple and successful instance of the general discussion in Remark 2.1 of passing from a fine moduli space of enhanced objects to a coarse moduli space of the original objects. Since the mapping class group acts properly discontinuously and holomorphically on the fine moduli space $\mathcal{T}_g$ of marked Riemann surfaces, we can easily get a coarse moduli space $\mathcal{M}_g$, as a complex analytic space, of Riemann surfaces as in Proposition 5.1.

Remark 5.3. The above discussions show that Teichmüller gave a precise and modern definition definition of fine moduli spaces in [73], but he did not give a precise definition of coarse moduli spaces. As the quotes before Proposition 5.1 show, Teichmüller was aware of taking quotients of fine moduli spaces to remove the additional marking data to get quotients. Similar ideas of taking quotients of fine moduli spaces were also present in papers of Grothendieck [35, I, pp. 7–30, 7–31, 7–32]. But the first formal definition of coarse moduli spaces seems to appear in [58, Definition 5.6, p. 99], where the first condition is usually stated and expected by almost everyone, but the second condition is often not (for example, see the quote from [12, **3. Some unsolved problems]** above Remark 8.1 in §7, and [42, p. 67]).

One corollary of the above proof of Proposition 5.1 is the following expected result.

Proposition 5.4. *If a moduli problem has a fine moduli space $\mathcal{M}$, then $\mathcal{M}$ is also a coarse moduli space.*

Proof. We take the example of $\mathcal{T}_g$. Suppose that $\mathcal{T}'$ is a coarse moduli space for marked Riemann surfaces, i.e., it is a complex space whose points are in 1-1 correspondence with equivalence classes of marked Riemann surfaces and satisfies a condition similar to Condition (1) in the proof of the previous proposition. Then the universal family on $\mathcal{T}_g$ gives the desired unique holomorphic map $\mathcal{T}_g \to \mathcal{T}'$ which satisfies a condition similar to Condition (2) above. □

Since Teichmüller died in 1943, he could not provide any detail for his paper [73], which was his last major paper. On the other hand, it might be safe to assume that he had understood enough to feel sure that writing complete proofs for all the results he claimed in this paper [73] would for him be routine. For this purpose, it might be helpful to point out that the paper [72] supplemented the earlier paper [71] and provided proofs for various conjectures he made in [71]. At the beginning of [72], he wrote:

> In 1939, it was a risk to publish a lengthy article entirely built on conjectures. I had studied the topic thoroughly, was convinced of the truth of my conjectures and I did not want to keep back from the public the beautiful connections and perspectives that I had arrived at. Moreover, I wanted to encourage attempts for proofs. I acknowledge the reproaches, that have been made to me from various sides, even today as justifiable but only in the sense that an unscrupulous imitation of my procedure would certainly lead to a proliferation of our mathematical literature. But I never had any doubts about the correctness of my article, and I am glad now to be able to actually prove the main part of it.

As it will be described in the next section, this paper [73] should have had a huge impact on the development of modern theories of moduli spaces, which are central objects on algebraic geometry. On the other hand, it seems to be not so well-known in literature. As far as I have checked, no published book on moduli spaces in algebraic geometry has cited this paper, except for a preprint of a book by Kollar on moduli spaces [48] which was mentioned above in §2. The paper [73] was not mentioned in the survey [60] on the history of $\mathcal{M}_g$, though the papers [71] [72] were.

How about in the community of complex analytic theory of Teichmüller space? It, or rather the riches in it, is probably even less understood and known. For example, in the preface to the Collected Works of Teichmüller [74], L. Ahlfors and F. Gehring wrote around 1982:

> His paper *Veränderliche Riemannsche Flächen* contains some rather vague but promising ideas that should be probably be further analyzed.

Maybe one reason is that a leading authority of the field, Ahlfors, wrote a review of [73] in AMS Math Review around 1944, the first paragraph of which is basically incorrect, and the second paragraph does not convey (clearly) basic and essential ideas at all about the moduli space of Riemann surfaces contained in this paper.

The review by Ahlfors is short and is quoted in its entirety here for the convenience of the reader so that the reader can compare it with what Teichmüller himself said as quoted from [73] above in this section:

> The conformal type of a closed Riemann surface of genus g depends on $\tau = 0, 1$ or $3(g-1)$ complex parameters, the two first cases occurring for $g = 0$ and $g = 1$. If stated without reference to a definite topological space this has no meaning. The author wishes to prove that the statement can be given a definite sense by showing that the space of conformal types can be represented as a τ-dimensional analytic manifold. The problem is easier to handle if the equivalence is restricted to conformal mappings of a given topological class. More precisely, let H_0 be a fixed surface, H a variable surface and T a topological map of H_0 onto H. Then the couple (H,T) is an element of the space $\mathfrak{R}$ if and only if H can be mapped onto H_0 by a conformal transformation S such that TS^{-1} can be deformed into the identity.
>
> Let P be an r-dimensional complex analytic manifold. To each point $\mathfrak{p} \in P$ with the local coordinates $p_1, \cdots, p_r$ is assigned a Riemann surface $H(\mathfrak{p})$ of fixed genus g; a point t of this surface is determined by a local parameter t. The couples $(\mathfrak{p}, t)$ shall again constitute an $(r+1)$-dimensional complex analytic manifold M with distinguished coordinate systems $(p_1, \cdots, p_r, t)$ any two of which are connected by relations $p_i = f_i(p_1, \cdots, p_r)$, $t = g(p_1, \cdots, p_r, t)$, with $|\partial p_i / \partial p_j| \neq 0$, $\partial t / \partial t \neq 0$. It is possible to assign topological transformations $T(\mathfrak{p})$ to the surfaces $H(\mathfrak{p})$ so that they depend continuously on p; in the large, it may be necessary to step up to a covering manifold of P. As his main result the author proves the existence of a manifold M of dimension $\tau + 1$ such that one and only one $H(\mathfrak{p})$ is equivalent to a given topologically fixed surface of genus g, i.e., to an element of $\mathfrak{R}$. The proof depends on concrete realization of the Riemann surfaces and variations of the branch-points.

Therefore, it seems reasonable to conjecture that this review had basically killed the paper [73], given that it was also published in a shorted lived Nazi journal *Deutsche Mathematik*, which was not easily available outside of Germany. (Now the paper [73] is now relatively easily available through his collected works [74].)

Remark 5.5. Teichmüller mentioned three newly introduced notions. The first two were explained above. For the last one on winding piece coordinates, he meant some complex numbers which determine how winding pieces of Riemann surfaces are glued together [73, p. 797]. They are basic ingredients to put a complex structure on the Teichmüller space and the universal Riemann surface over it. On [73, p. 796], he defined winding pieces:

> Let $\mathfrak{C}$ be a closed Jordan curve dividing the z-plane into an inner domain $\mathfrak{J}$ and an outer domain $\mathfrak{A}$. If one cuts a closed Riemann surface $\mathfrak{Z}$ over the z-plane over $\mathfrak{C}$, then the part of $\mathfrak{Z}$ lying above $\mathfrak{J}$ decomposes into finitely many pieces. The multisheeted but simply connected among those pieces are the winding pieces.

It seems that this notion of winding piece coordinates might be related to the later Kodaira-Spencer deformation theory in [46]. In his paper *On the moduli of Riemann surfaces* [77, p. 383], Weil tried to introduce a natural complex structure on the Teichmüller space and wrote:

> In order to justify the statements that have been made so far, we shall make use of the Kodaira-Spencer technique of variation of complex structures. This can be introduced in an elementary manner in the case of complex dimension 1, which alone concerns us here; this, in fact, had already been done by Teichmüller; but he had so mixed it up with his ideas concerning quasiconformal mappings that much of its intrinsic simplicity got lost.

5.2 Why Did Teichmüller Study Teichmüller Space and Riemann's Moduli Spaces?

Though Teichmüller had broad interests, one natural question was why he started to study moduli spaces. One important result or conjecture which Teichmüller wanted to prove in [71] concerns the existence of extremal quasiconformal maps between Riemann surfaces of the same topological type. (Note that his Habilitationsschrift is titled *Untersuchungen über konforme und quasikonforme Abbildungen* [70] and studies conformal and quasiconformal mappings). In doing this, he developed the Teichmüller space as the proper framework and also as an essential tool to prove this main result in [72] by using the method of continuity. In the process, he defined the moduli problem precisely for Riemann surfaces and solved it in [73].

About quasiconformal maps and Teichmüller's contributions, Ahlfors wrote [8, pp. 72–73]:

> The theory of quasiconformal mappings is almost exactly fifty years old. They were introduced in 1928 by Herbert Grötzsch in order to formulate and prove a generalization of Picard's theorem....
>
> Grötzsch's papers remained practically unnoticed for a long time. In 1935 essentially the same class of mappings was introduced by M. A. Lavrentiev in the Soviet Union.... In any case, the theory of quasiconformal mappings, which at that time had also acquired its name, slowly gained recognition, originally as a useful and flexible tool, but inevitably also as an interesting piece of mathematics in its own right.
>
> Nevertheless, quasiconformal mappings might have remained a rather obscure and peripheral object of study if it had not been for Oswald Teichmüller, an exceptionally gifted and intense young mathematician and political fanatic, who suddenly made a fascinating and unexpected discovery. At that time, many special extremal problems in quasiconformal mapping had already been solved, but these were isolated results without a connecting general idea. In 1939 he presented to the Prussian Academy a now famous paper which marks the rebirth of quasiconformal mappings as a new discipline which completely overshadows the rather modest beginnings of the theory. With remarkable intuition he made a synthesis of what was known and proceeded to announce a bold outline of a new program which he presents, rather dramatically, as the result of a sudden revelation that occurred to him at night. His main discovery was that the extremal problem of quasiconformal mapping, when applied to Riemann surfaces, leads automatically to an intimate connection with the holomorphic quadratic differentials on the surface. With this connection the whole theory takes on a completely different complexion: A problem concerned with non-conformal mappings turns out to have a solution which is expressed in terms of holomorphic differentials, so that in reality the problem belongs to classical function theory. Even if some of the proofs were only heuristic, it was clear from the start that this paper would have a tremendous impact, although actually its influence was delayed due to the poor communications during the war. In the same paper Teichmüller lays the foundations for what later has become known as the theory of Teichmüller spaces.

Ahlfors continued on [8, pp. 75]:

> Teichmüller considers topological maps $f : S_0 \to S$ from one compact Riemann surface to another. In addition he requires f to belong to a prescribed homotopy class, and he wishes to solve the extremal problem separately for each such class. Teichmüller asserted that there is always an extremal mapping, and that it is unique. Moreover, either there is a unique conformal mapping in the given homotopy class, or there is a constant k, $0 < k < 1$, and a holomorphic quadratic differential $\varphi(z)dz^2$ on S_0 such that the Beltrami coefficient of the extremal mapping is $\mu_f = k\bar{\varphi}/|\varphi|$. It is thus a mapping with constant dilatation $K = (1+k)/(1-k)$. The inverse f^{-1} is simultaneously extremal for the mappings $S \to S_0$, and it determines an associated quadratic differential $\psi(w)dw^2$ on S. In local coordinates the mapping can be expressed through
>
> $$\sqrt{\psi(w)}dw = \sqrt{\varphi(z)}dz + k\sqrt{\bar{\varphi}(z)}d\bar{z}.$$
>
> Naturally, there are singularities at the zeros of φ, which are mapped on zeros of ψ of the same order, but these singularities are of a simple explicit nature. The integral curves along which $\sqrt{\varphi}dz$ is respectively real or purely imaginary are called horizontal and vertical trajectories, and the extremal mapping maps the horizontal and vertical trajectories on S_0 on corresponding trajectories on S. At each point the stretching is maximal in the direction of the horizontal trajectory and minimal along the vertical trajectory.
>
> This is a beautiful and absolutely fundamental result which, as I have already tried to emphasize, throws a completely new light on the theory of q.c. mappings. In his 1939 paper Teichmüller gives a complete proof of the uniqueness part of his theorem, and it is still essentially the only known proof. His existence proof, which appeared later, is not so transparent,

The existence proof of extremal quasiconformal maps was given in [72]. In both papers [71] [72], marked Riemann surfaces and the Teichmüller space consisting of equivalence classes of marked Riemann surfaces were defined. Teichmüller also announced in [71] that Teichmüller space $\mathcal{T}_g$ is homeomorphic to $\mathbb{R}^{6g-6}$ and proved it in [72]. In both papers, he only treated them as topological spaces. During the period between these two papers, in order to prove his result on extremal quasiconformal maps between Riemann surfaces as mentioned by Ahlfors above, he solved the moduli problem for Riemann surfaces. On [72, p. 637], Teichmüller wrote:

> At the time I was missing an exact theory of *modules*, the conformal invariants of closed Riemann surfaces and similar "principal domains". In the meantime, particularly with regard to the intended application to quasiconformal maps, I have developed such a theory. I will have to briefly report on it elsewhere. The present proof does not depend on this new theory, and instead works with the notion of *uniformization*. However, I think one will have to combine both to bring the full content of my article **Q** in mathematically exact form.

We explain briefly why he needed marked Riemann surfaces and the space of marked Riemann surfaces. To study extremal quasiconformal maps between two Riemann surfaces, we need to fix a homotopy classes of quasiconformal maps between them. If we consider marked Riemann surfaces, then for each pair of marked Riemann surfaces, there is a *canonical* homotopy class of quasiconformal maps between them which is compatible with or rather uniquely determined by the markings. As Teichmüller wrote in item 49 of §15 of [71] titled "Topological determination of principal regions":

> Of course, mathematical problems cannot be addressed the way they have just been carried out by having a guess at the solution. We are now looking for a way – heuristic, though – by which we will be able to find the extremal quasiconformal mappings systematically.

Remark 5.6. As Teichmüller explained above, he did not set out to study Riemann's moduli space $\mathcal{M}_g$ by using quasiconformal maps and introducing the Teichmüller space $\mathcal{T}_g$. Instead, he wanted to study conformal invariants of regions and surfaces and how they change under quasiconformal maps. In item 1 of [71], he wrote

> In the present study, *the behaviour of conformal invariants under quasicon-formal mappings* shall be examined. This will lead to the problem of *finding the mappings that deviate as little as possible from conformality under certain additional conditions.* We shall give the solution of this problem, without however being able to give a rigorous proof. The solution relies on the notion of quadratic differentials (function times the square of a differential) from the theory of algebraic functions. As I have done it before, I shall here also examine quasiconformal mappings not exclusively for their own sake, but chiefly because of their connections with notions and questions that interest function theorists (see **164**ff.).

After examining examples of conformal invariants of simple regions, he wanted to study conformal invariants as functions on spaces of regions and Riemann surfaces. Therefore, he needed the moduli space of Riemann surfaces. In item 13 of [71], he wrote:

> In the simple examples treated above, the principal region was always conformally mapped onto some normalized principal region of the same type which only depended on finitely many parameters; they then provided the conformal invariants.
>
> *We shall admit the following statement without proof: when conformally equivalent principal regions are identified, principal regions of a fixed topological type form a topological manifold which is locally homeomorphic to the ($\sigma \geq 0$)-dimensional Euclidean space and shall be, for this reason, denoted as space $\mathbb{R}^\sigma$. The conformal invariants of a principal region are then precisely the functions on $\mathbb{R}^\sigma$. Hence, locally, there are precisely σ independent conformal invariants.*
>
> According to which notion of neighbourhoods conformal classes of principal regions form such a space, we do not specify. We shall naturally work with mappings onto normalized regions.

Since he wanted to understand change of conformal invariants under quasiconformal maps, it was natural for him to introduce the Teichmüller space as explained above.[9] Though the Teichmüller space was not the starting point of [71], he did propose many questions and results about it. For example, he defined Teichmüller metric on it using quasiconformal maps and explained that his results on extremal quasiconformal maps (as mentioned below) imply that the Teichmüller space is homeomorphic to an Euclidean space.

For every nonzero holomorphic quadratic forms on the domain Riemann surface, Teichmüller could construct extremal quasiconformal maps to target

[9] Incidentally, many people say and will agree that Teichmüller space was introduced to get a smooth cover of the moduli space of Riemann surfaces which is singular due to nontrivial automorphisms of Riemann surfaces. This is true and said explicitly by Teichmüller in [72] as we explained in §5.1 above. Since the Teichmüller space was first introduced in the paper [71] four years earlier, it might be surprising to find that the above quote from item 13 of [71] shows that Teichmüller at first thought that the moduli space of Riemann surfaces was a topological manifold. A natural question is whether other people at that time thought vaguely in the way about the moduli space of Riemann surfaces.

Riemann surfaces by stretching and shrinking horizontal and vertical trajectories (or coordinates) respectively determined by the holomorphic quadratic form. The question is *whether all marked Riemann surfaces can be constructed this way from any fixed domain Riemann surface* Σ_g. To do this, he considered the space $\mathfrak{C}$ of marked compact oriented hyperbolic surfaces of genus g, often called *Fricke space* now. The shrinking/stretching via holomorphic quadratic forms on Σ_g gives a map from $\mathbb{R}^{6g-6}$ to $\mathfrak{C}$. Then he applied the method of continuity and Brouwer's Theorem on invariance of domains to show that this injective map is a bijective map, and hence there exists a unique extremal quasiconformal map between every pair of marked Riemann surfaces. This application of the method of continuity is similar to its application by Poincaré and Klein to prove the uniformization of Riemann surfaces as mentioned above. The uniformization theorem was used by Teichmüller to identify compact Riemann surfaces of negative Euler characteristic with hyperbolic surfaces.

Bers [17, p. 1093] explained an application of Teichmüller's result on extremal quasiconformal maps:

> An application of Teichmüller's theorem going back to Teichmüller himself is a new proof of the difficult part of a classical theorem by Fricke. To explain this theorem we must define the *Teichmüller space* $\mathcal{T}_{p,n}$ (which, by the way, could have been called Fricke space)....
>
> Fricke's theorem reads: *The Teichmüller space* $\mathcal{T}_{p,n}$ *is homeomorphic to* $\mathbb{R}^{6p-6+2n}$; *the moduiar group* $\mathrm{Mod}_{p,n}$ *acts properly discontinuously.*
>
> The difficult statement is the first one (see Fricke-Klein [30, pp. 284–394]; [31, pp. 285–310]; ...).

The references in Bers' quote are two volumes of the book [29] cited in this note.

Remark 5.7. There seems to be a lot of confusion in the Teichmüller theory community about who first introduced Teichmüller space, or rather the space of marked hyperbolic surfaces. Was it Fricke or Teichmüller?

The most appropriate answer is probably Fricke & Klein. But then why is it called Teichmüller space by the most people, and Fricke space by some others, but not Fricke-Klein space by anyone? Here is what happened. We will also give some possible explanations of the complicated situation.

In their books [29], Fricke and Klein wanted to prove the uniformization theorem of Riemann surfaces of finite type, called the fundamental theorem in their book, by the method of continuity as mentioned in §4. The books [29] built on the works or attempts of Klein and Poincaré to prove the uniformization theorem, and were written as a research monograph to put into a more organized text of the most recent results in research papers and as well as their own new results. To apply the method of continuity, they needed two spaces and an injective continuous map between them and then showed that this map is in fact bijective (a homeomorphism). The target space is *Riemann's moduli space* $\mathcal{M}_g$, and the domain space is the *moduli space* $\mathcal{M}_g^{hyp}$ *of compact hyperbolic surfaces of genus* g, $g \geq 2$. The map $\mathcal{M}_g^{hyp} \to \mathcal{M}_g$ is the natural one by taking the conformal structure of each hyperbolic surface. The basic way to construct hyperbolic surfaces is to take discrete subgroups Γ of $\mathrm{SL}(2,\mathbb{R})$, the so-called Fuchsian groups, which act on the Poincaré upper halfplane $\mathbb{H}^2$ and form the quotient spaces $\Gamma\backslash\mathbb{H}^2$. The theory of automorphic forms gives each quotient $\Gamma\backslash\mathbb{H}^2$ the structure of an algebraic curves (or we can see directly that it is a Riemann surface since Γ acts holomorphically on $\mathbb{H}^2$). And the most effective way to construct Fuchsian subgroups is to use the Poincaré polygon theorem which specifies generators and relations of groups by matching of the geometric sides of the hyperbolic polygons. Therefore, Fricke and Klein developed spaces of polygon groups. Strictly speaking, in the papers of Klein and Poincaré from 1882–1883, spaces of polygons appeared and were used to almost prove the uniformization theorem (see [2] for more discussion and references about this), but Fricke and Klein did much more systematic work on these spaces and tried to understand them via matrices of generators of Fuchsian groups. Since the sides of the hyperbolic polygons essentially amount to markings of hyperbolic surfaces, they were constructing moduli spaces of marked hyperbolic surfaces. Due to various reasons, their construction and exposition were not so clean and it was and still is very difficult for most people to read and understand what is contained in [29]. As Bers pointed out, they had a difficult proof that the space of marked compact hyperbolic surfaces of genus g, $g \geq 2$, is homeomorphic to $\mathbb{R}^{6g-6}$, though markings and spaces of marked hyperbolic surfaces were not precisely defined in [29]. This result was important for them in order to apply the method of continuity and Brouwer's Theorem on invariance of domains.

Of course, as it is well-known, the proof of the uniformization of Riemann surfaces was not completed in [29], and the latter works of Koebe (1907) and Poincaré (1908) on uniformization of general Riemann surfaces vastly extended the result and superseded both the result and the method presented in [29].

The books [29] and even the prefaces were written by Fricke alone. On the other hand, Fricke made it clear in the prefaces that many ideas in the books came from Klein, and the name of Klein did not just appear on the cover for decoration or to make the book look better.

In order to apply the method of continuity to prove the existence of extremal quasiconformal maps between Riemann surfaces, Teichmüller [72] introduced and gave, in the section titled *Topological Determination and Uniformization of the surface* $\mathfrak{W}$, a systematical discussion of the space of marked compact hyperbolic surfaces through the matrices of the standard generators and one relation for Fuchsian groups which are representations of the fundamental group of a compact surface $\pi_1(\Sigma_g)$. For example, he showed that it is a manifold of dimension $6g-6$. As Bers pointed out above, Teichmüller's result on extremal quasiconformal maps gave an elegant and transparent proof of the fact that the space of marked compact hyperbolic surfaces is homeomorphic to $\mathbb{R}^{6g-6}$. It is more than a proof and explained why the result is true in some sense. Teichmüller also defined precisely markings on surfaces and a natural metric and hence a topology on the Teichmüller space. These are reasons why the Teichmüller space is now better known.

Another probably more important reason is that Teichmüller made several connections of Teichmüller space with Riemann surfaces, Riemann's moduli spaces and quasiconformal maps, and made quasiconformal maps into essential tools and major objects of study in geometric function theory, as Ahlfors pointed out above. These connections are important. For example, there does not seem to be any way to put a complex structure on the Teichmüller space $\mathcal{T}_g$ by using only hyperbolic geometry of surfaces such as Poincaré upper half plane, and Fuchsian groups (or representations of surface groups).

In both the works of Fricke-Klein [29] and of Teichmüller [71] [72], moduli spaces of marked Riemann surfaces or hyperbolic surfaces were just step stones. To recover moduli spaces of Riemann surfaces or hyperbolic surfaces, we should divide out the former spaces by the action of mapping class groups. The concept of mapping class groups was introduced and used in [29], but it was not clearly defined or precisely described. On the other hand, it was defined precisely in modern language in Teichmüller papers [71] [73] as we quoted above. Of course, later works showed that this action of mapping class group on the Teichmüller space is essential to the study of the mapping class group. The analysis of this note shows that Teichmüller's contribution to Riemann's moduli space $\mathcal{M}_g$ proper (not its infinite covering, i.e., Teichmüller space) is *very substantial* but *not known to many people in both the complex analytic and algebraic communities of moduli spaces.*

To conclude this section, we quote an announcement by Teichmüller [72, p. 764] of his upcoming work on the moduli problem on Riemann surfaces in [73]:

> Hereby the proof is complete. I now want to report briefly on the connection of our result with my solution of the *problem of moduli* that I have outlined elsewhere.[a]
>
> In my opinion the problem of moduli consists in turning the set $\underline{\mathfrak{R}}$ of all classes of conformally equivalent topologically determined closed oriented Riemann surfaces of genus g into an *analytic manifold* by introducing suitable "local coordinate systems", namely in such a way that one can apply them for example for *continuity arguments.* In particular, during my research I have always kept in mind the aim to give a continuity proof for the existence of the extremal quasiconformal mappings of the analytic form $E(K; a_1, \cdots, a_\sigma)$ similarly to the case of a pentagon.
>
> Elsewhere, I have briefly explained how I succeeded to introduce notions of neighbourhood and coordinates in the set $\underline{\mathfrak{R}}$ in such a way that $\underline{\mathfrak{R}}$ becomes a purely $(6g-6)$-dimensional real or a purely $(3g-3)$-dimensional complex manifold, and that this method to turn $\underline{\mathfrak{R}}$ into such a manifold is distinguished from all other possibilities by certain properties, connected with the notion of an "analytic family".
>
> Even though the proof given above has nothing to do with these new results directly, and instead is based on uniformisation, there are close connections between the theory of conformal modules and the theory of extremal quasiconformal mappings.

[a] O. Teichmüller, Veränderliche Riemannsche Flächen. Erscheint in der Deutsche Mathematik.

6. Siegel Upper Half Space and Weil's Work on Weil-Petersson Metric

There are two people whose works have had profound impact on moduli spaces of curves, but their names may not come to people's mind when people think of moduli spaces. They are Siegel and Weil.

Siegel is probably best known for his work on quadratic forms. In [69], he developed a theory of quadratic forms in n variables, by generalizing the classical study of quadratic forms in two variables. As an application of his theory, he introduced the Siegel upper half space $\mathbb{H}_g$, the Torelli space consisting of Riemann surfaces with marked bases of the first homology group of compact Riemann surfaces, and a period map from the Torelli space to the Siegel upper halfspace $\mathbb{H}_g$, in the last section of this paper, i.e., §13 titled *Modulfunctionen n^{ten} Grades*. Since the Torelli space is a quotient of the Teichmüller space $\mathcal{T}_g$, this gives a map from the Teichmüller space to the Siegel upper space, $\mathcal{T}_g \to \mathbb{H}_g$, and hence the period map of Riemann surfaces

$$\Pi : \mathcal{M}_g \to \mathcal{A}_g.$$

The Siegel upper space $\mathbb{H}_g$ and the period map Π have been crucial to the study of $\mathcal{M}_g$. For example, as mentioned above, the Torelli theorem was used in [52] to show that $\mathcal{M}_g$ is a coarse moduli space. This approach is one of the two methods in the book [58] to construct the moduli space $\mathcal{M}_g$.

In the commentaries of his collected works [80, p. 545], Weil wrote:

> La théorie des modules des courbes, inaugurée par Riemann, a fait à noter époque deux pas en avant décisifs, d'abord en 1935 du fait de Siegel (*Ges. Abh.* n^0 20, §13, vol. I, pp. 394–405), puis par les remarquables travaux de Teichmüller; sur ceux-ci, it est vrai, il continua quelque temps à planer des doutes qui ne furent levés définitivement que par Ahlfors en 1953 (*J. d' An. Math.* 3, pp. 1–58). D'ailleurs on finit par se rendre compte que la découverte par Siegel (*loc. cit.*) des fonctions automorphes appartenant au groupe symplectique atteignait en premier lieu les modules des variétés (cf. p. ex. [1957c], pp. 376) et par ricochet seulement ceux courbes, par l'intermédiare de leurs jacobiennes et du théorème de Torelli.
>
> Grâce à Siegel, on disposait ainsi d'un premier exemple de théorie des modules pour des variétés de dimension > 1. On doit à Kodaira et Spencer d'avoir découvert (*Ann. of Math.* 67 (1958), pp. 328–466) que les progrès de la cohomologie permettaient, non seulement d'aborder un nouvel aspect du même problème, mais encore, du point de vue local tout au moins, de s'attaquer au général cas des variétés à structure complexe.

Weil only briefly worked on Teichmüller space $\mathcal{T}_g$. But his name is attached to the important Weil-Petterson metric of $\mathcal{T}_g$ and $\mathcal{M}_g$. It is not clear what was the motivation for Weil to study the moduli space $\mathcal{M}_g$,[10] but his quick departure from it might be explained. Weil wrote three short notes on Teichmüller space $\mathcal{T}_g$ [76] [77] [78]. In the note [76], a Bourbaki seminar talk, he introduced the Weil-Petersson metric and announced that it is Kähler by "a stupid computation". In the note [77, p. 389], he posed it as an open problem: "This raises the most interesting problems of the whole theory: is this a Kähler metric", and in the note [78, p. 392], he claimed it again: "which turns out to be a Kähler metric." The papers [77] and [78] were published in Weil's collected works for the first time, but the paper [76] was not put into the collected works.[11] Weil's commentaries in his Collected Works tried to explain why he did not include the paper [76] and left this topic on moduli spaces. See [2] for more detail about Weil's work on moduli spaces.

7. Grothendieck's Papers at Cartan Seminar

Though many of works of Grothendieck are well-known, it is probably not so well-known that he gave ten talks at the Cartan seminar on the Teichmüller space and other moduli spaces in 1960–1961, and wrote up ten papers correspondingly. For example, these papers [35] are not listed in Math Review of AMS.

Some summaries and commentaries on these paper are given in [2] and [3]. We only want to mention the following points from the papers [35] which are most relevant to our story:

1. Grothendieck formulated precisely moduli problems in terms of categories and functors. In particular, he introduced spaces over spaces and morphisms between them. Then the existence of a fine moduli space amounts to the representability of a moduli functor.
2. He overcame the problem of nontrivial automorphisms of objects in the moduli problems by adding additional data to rigidify the objects, hence rigidifying the moduli problems and moduli functors.
3. He applied these methods to marked Riemann surfaces to construct the Teichmüller space $\mathcal{T}_g$ and the universal curve over it, hence showing that $\mathcal{T}_g$ is a fine moduli space for marked Riemann surfaces and thus admits a natural complex structure. Consequently, $\mathcal{M}_g$ also admits a natural complex structure.
4. He introduced the notion of algebraic deformations and constructed some finite covers of $\mathcal{M}_g$ as algebraic varieties, but not $\mathcal{M}_g$ itself.

---

[10] According to Deligne [26]: Weil was interested in curves (the aim of his book *Foundations of Algebraic Geometry* was to justify what he needed for the Riemann Hypothesis for curves over finite fields). He had just written about Kähler geometry. With this, I do not find it surprising that he was happy to find a nice example of a Kähler metric. [The book [79] on Kähler manifolds was finished before May 31, 1957 (the date of the book's preface) and published in 1958.]

[11] These three papers of Weil were all written in 1958. The exact order in which there were written is not clear. The paper [76] was the published lecture note for a Bourbaki seminar in May 1958, the paper [77] was for the sixtieth birthday of Emil Artin who was born on March 3, 1898, and the paper [78] was a final report for a project submitted to AFOSR, which Weil put behind [77] in his Collected Works.

The general setups, methods and concepts such as Hilbert schemes introduced by Grothendieck in these papers have had a huge impact on theories of moduli spaces in algebraic geometry. They were used by Mumford in his geometric invariant theory described in the book [58].

To define algebraic deformations, nilpotent elements and nonreduced schemes are essential. The following letter from Deligne [24] gives a very good explanation of Grothendieck's approach to moduli spaces:

> To view a moduli problem as the search for a universal family requires a good notion of "family" (flatness), and a good notion of pull-back. Here also nilpotents are also needed: in the category of reduced schemes, fiber products (pull-back) exist, but taking a pull-back in that category does not preserve flatness.
>
> To me, the Grothendieck approach to moduli problem via representable functors culminates in the work of M. Artin, who defines a slight generalization of schemes (algebraic spaces) and gives a criterion, easy to check in practice, to check whether a functor is represented by an algebraic space.

In the above list of some highlights of the papers [35], (1), (2) and (3) seem to be parallel to results in Teichmüller's paper [73]. Since the paper [73] was not cited by Grothendieck, one natural question is whether Teichmüller's work influenced Grothendieck. The answer seems to be a likely yes. The most naive and direct reason is that the subtitles of the first and the last papers in [35] contain the phrase "Teichmüller space."

A more convincing reason is that adding additional data to eliminate nontrivial automorphisms of objects under classification is crucial to the approach of Grothendieck, for example, *rigidifying moduli functors*, but it was a key and deep insight of Teichmüller [73] as pointed out in the previous section. More specifically, Grothendieck defined rigidifying functors $\mathcal{P}$ in the first paper in [35, I, pp. 7–05, 7–08], which is not necessarily adding markings on surfaces as Teichmüller did but adding any additional data to remove nontrivial automorphisms, and stated the *fundamental existence theorem: Teichmüller spaces and its variants* on [35, I, pp. 7–05, 7–08]:

> **Theorem 3.1.** There exists an analytic space $\mathcal{T}$ and a $\mathcal{P}$-algebraic curve V above $\mathcal{T}$ which are universal in the following sense: For every $\mathcal{P}$-algebraic curve X above an analytic space S, there exists a unique analytic morphism g from S to $\mathcal{T}$ such that X (together with its $\mathcal{P}$-structure) is isomorphic to the pull-back of $V/\mathcal{T}$ by g.

In a letter to Serre on November 5, 1959, Grothendieck wrote [36, p. 94]:

> The "Grand existence theorem" is progressing little by little; many technical difficulties remain, but I am more and more convinced that there is an absolutely marvelous technique at the end of all this. I have already come to the practical conclusion that every time that my criteria show that no moduli variety (or rather moduli scheme) for the classification of (global or infinitesimal) variations of certain structures (complete non-singular varieties, vector bundles etc.) can exist, despite good hypotheses of flatness, properness, and if necessary non-singularity, the only reason is the existence of automorphisms of the structure which prevent the descent from working... The remedy in moduli theory seems to me to be to eliminate bothersome automorphisms by introducing additional structures on the objects being studied: points or differential forms etc. on the varying varieties (a process which is already used for curves), trivializations at sufficiently many points of the vector bundles one wants to vary, etc.

When he wrote "a process which is already used for curves", he probably meant Teichmüller's work.[12] Teichmüller's work was quite known to the mathematics community at that time already, and this idea might have become known to experts who were interested in moduli problems. Even though papers of Teichmüller were not cited by Grothendieck, it is still consistent with his style of often asking experts for information, especially several people such as Serre.

The idea of adding markings to Riemann surfaces to construct the Teichmüller space and to study Riemann's moduli space in connection with the Teichmüller space was announced in an earlier paper [72, p. 637, pp. 764–676], as discussed in the previous section. He also said specifically [72, p. 674] that an outline of his solution to the moduli problem was to be published in *Deutsche Mathematik*. It is probably important to point out that the paper [72] was published in the well-known and easily accessible Proceedings of the Prussian Academy of Sciences, and marked Riemann surfaces also appeared in the famous paper [71] which was published in the same journal.

One expert on moduli and Teichmüller spaces who was active in the Bourkaki group was Andre Weil. Weil worked on moduli spaces and wrote three papers about them (see §6 above). Even though the paper [73] was published in an obscure Nazi journal *Deutsche*

[12] According to an email of Serre [68]: You ask whether Grothendieck knew about Teichmüller. I am pretty sure that he did not know Teichmüller's work, but that he relied on what Weil had explained one year or so before, at the Bourbaki seminar, which he certainly attended.

Mathematik and were almost impossible to be found outside Germany, Weil spent some time in University of Strasbourg, and the mathematics library of Strasbourg has a complete set of this journal which we found when we worked on the paper [2]. Weil gave a talk on Teichmüller space and Teichmüller's work at Bourbaki seminar [76, p. 413] and wrote:

> Par la combinaison des idées (récents) de KODAIRA et SPENCER sur la variation des structures complexes avec les idées (anciennes) de TEICHMÜLLER sur le probléme des modules, la théorie a fait dernièrement quelques progrès qu'on se propose d'exposer ici.

This might suggest that Weil was familiar with the work of Teichmüller, in particular all three main papers [71] [72] [73].

One of the goals of Grothendieck was to construct $\mathcal{M}_g$ as an algebraic variety, but he could not do it due to some difficulties in dealing with taking quotients of varieties. In the last of this series of papers, he wrote [35, X, p. 17–01]:

> The method indicated in the text bumps, in the context of schemes, on a difficulty related to taking quotients, which does not exist in the transcendental case. By a similar method, using in a more systematic way Picard schemes and their points of finite order, the lecturer was able to construct the Jacobi modular schemes $\mathbf{M}_n$ *of high enough level*, but, for lack of knowledge of whether $\mathbf{M}_n$ is quasi-projective, it was not possible to take a quotient by finite groups in order to obtain the moduli spaces of arbitrary level, and in particular the classical moduli space $\mathbf{M}_1$. These difficulties have just been overcome by Mumford, using a new theorem of passage to the quotient which can be applied to polarized Abelian schemes, and from there to curves.

As mentioned in the letter of Deligne above, Grothendieck's approach of algebraic deformation and the difficulty in dealing with quotients under group actions lead to algebraic spaces and algebraic deformation theory by Michael Artin.

Remark 7.1. We note that $\mathbf{M}_n$ in the above quote from Grothendieck is not the moduli space of compact Riemann surfaces of genus n. Instead, it is the Jacobi modular schemes $\mathbf{M}_n$ of curves enhanced with n level structures. When $n \gg 1$, $\mathbf{M}_n$ is a fine moduli space of curves with n level structures, and we need to pass to quotients by finite groups to obtain a coarse moduli space $\mathbf{M}_1$. This is one important example of the general principle mentioned in Remark 2.1. The difficulty to put the structure of an algebraic variety on $\mathbf{M}_1$ is also an example of the general discussion in Remark 2.1.

Besides Mumford, in the last paper of the Cartan seminar [35, X, p. 17-01], Grothendieck also mentioned Baily and cited the paper [12]. There seems to be some confusion about Baily's work on the moduli space $\mathcal{M}_g$. We will explain it in the next section.

8. Baily Compactification, Algebraic Structure on $\mathcal{M}_g$, and Torelli Morphism

In a commentary on Mumford's work on geometric invariant theory and the moduli of curves in the Selected Papers of Mumford [54, p. 1], Gieseker wrote:

> Prior to Mumford's work on the existence of $\mathcal{M}_g$, Ahlfors and Bers had shown that $\mathcal{M}_g$ existed as an analytic space and Bailey had shown that $\mathcal{M}_g$ was a quasi-projective variety over $\mathbb{C}$ [2].

The reference "[2]" cited above is the paper [11] by Baily cited in this note. The corollary on [11, p. 313] states:

> **Corollary.** *Letting $\mathcal{E}$ denote the set of points of V_n corresponding to the period matrices of Jacobian varieties, the closure of $\mathcal{E}$ in V_n^* is an algebraic subvariety I of V_n^*, and $\mathcal{E}$ is a Zariski open subset of I.*
>
> We may now, by Torelli's theorem [20], refer to $\mathcal{E}$ as the variety of moduli of curves of genus n.

We note that V_n is the quotient of the Siegel upper half space

$$\mathbb{H}_n = \{Z = X + iY \mid X, Y \text{ symmetric } n \times n \text{ matrices}, Y > 0\}$$

by the Siegel modular group $\mathrm{Sp}(2n, \mathbb{Z})$ and can be identified with the *moduli space* $\mathcal{A}_n$ of *principally polarized abelian varieties* of dimension n over $\mathbb{C}$. By [13], the Satake compactification of V_n is a normal projective variety, denoted by V_n^* above. This is the precursor of the famous Baily-Borel compactification of arithmetic locally Hermitian symmetric spaces [14].

For each compact Riemann surface Σ_g, choose a symplectic basis of $H_1(\Sigma_g, \mathbb{Z})$, and a normalized basis of holomorphic 1-forms on Σ_g. Then the periods of these 1-forms give a point in $\mathbb{H}_g$. Different choices of symplectic bases lead to points in $\mathbb{H}_g$ which are equivalent under the action of $\mathrm{Sp}(2g, \mathbb{Z})$. This leads to the *period map*:[13]

$$\Pi : \mathcal{M}_g \to \mathcal{A}_g.$$

[13] There many other names for this map $\mathcal{M}_g \to A_g$. Instead of periods of normalized holomorphic 1-forms, we can associate to each Riemann surface its Jacobian variety with the canonical principal polarization, and Π is called the *Jacobian map*. In view of the importance of the Torelli Theorem on the injecticity of the map Π, the map Π is called the *Torelli mapping* in [61]. Once $\mathcal{M}_g$ is shown to be an algebraic variety and Π is a morphism, then Π is often called the *Torelli morphism*, for example in [51].

The above result of Baily says that *the closure of* $\Pi(\mathcal{M}_g)$ *in* V_g^* *is an algebraic subvariety.* Therefore, the image $\Pi(\mathcal{M}_g)$ is a quasi-projective variety. Since the Torelli theorem says that Π is an injective map, $\mathcal{M}_g$ and $\Pi(\mathcal{M}_g)$ *can be identified as sets*, and this gives an *algebraic variety structure* to $\mathcal{M}_g$. This seems to be what [11, p. 313, Corollary] implies, but is not equivalent to what Gieseker meant in the above quote: $\mathcal{M}_g$ *with this algebraic variety structure is a coarse moduli space in the category of algebraic varieties over* $\mathbb{C}$. When we wrote the paper [2], some people also told us that Baily proved that the moduli space $\mathcal{M}_g$ is an algebraic variety $\mathbb{C}$.[14]

On the other hand, to many mathematicians who are only users of $\mathcal{M}_g$, this work of Baily on the moduli space $\mathcal{M}_g$ is probably not known. Usually they only think of the foundational work of Mumford [58] on moduli spaces. It is confirmed by the quote from Kollar [48, p. 7] in §2 above.

How to explain this? We start by noting that the paper [11] was cited by Gieseker above but was not cited in the books [58] [37]. The paper [11] was cited in the book [10], but the paper [12] was not cited. The paper [12] contains some information which is *explicitly* missing from [11].

On [12, pp. 59–60, **3. Some unsolved problems**], Baily wrote:

> A second type of problem has to do with the analytic subvariety J of V_n^*. Is it a normally imbedded subvariety of V_n^*? What, if any, are its singularities?
>
> Thirdly, in order to call $\mathcal{E}$ the variety of moduli of Riemann surfaces of genus n, one should be able to state that it is unique and in some sense universal among normal parameter varieties of algebraic systems of curves of genus n (if $\mathcal{E}$ is not a normal analytic space, one should perhaps consider its normal model). Namely, given any normal algebraic system of curves of genus n (by which we mean that the parameter variety is a normal variety) there should exist a natural map of the parameter variety of the nonsingular members of this system into $\mathcal{E}$.

Remark 8.1. In the above quote from [12], the second condition for coarse moduli spaces in the proof of Proposition 5.1, or [58, Definition 5.6, p. 99], was not given. But if we only consider normal spaces, then any two normal complex analytic spaces satisfying the condition in the above quote from [12] whose points are in 1-1 correspondence with the points of $\mathcal{M}_g$ are isomorphic, since any holomorphic bijective map between two normal complex analytic spaces is biholomorphic. This is an extension of a result [38, Proposition 1.1.13, p. 13] mentioned in §2.

Therefore, in conclusion, Baily *did not prove that* $\mathcal{M}_g$ *as a complex algebraic variety over* $\mathbb{C}$ *with the algebraic structure induced from the image* $\Pi(\mathcal{M}_g)$ *in* $\mathcal{A}_g$ *is a coarse moduli space for smooth projective curves of genus* g *over* $\mathbb{C}$.

To settle the question whether the algebraic subvariety $\Pi(\mathcal{M}_g)$ of $\mathcal{A}_g$ over $\mathbb{C}$ is a coarse moduli space of smooth projective algebraic curves of genus g over $\mathbb{C}$, we need to determine whether $\Pi : \mathcal{M}_g \to \mathcal{A}_g$ gives an isomorphism from $\mathcal{M}_g$ to the image $\Pi(\mathcal{M}_g)$ as *complex analytic spaces*.

To do this, we consider a lifting of the period map Π. Let

$$\tilde{\Pi} : \mathcal{T}_g \to \mathbb{H}_g$$

be the canonical map by taking the period of *marked Riemannian surfaces*. This map is equivariant with respect to a natural homomorphism $\mathrm{Mod}_g \to \mathrm{Sp}(2g,\mathbb{Z})$, and descends to the period map $\Pi : \mathcal{M}_g \to \mathcal{A}_g$ defined earlier.

Proposition 8.2. *In the above notation, when* $g \geq 2$*, the rank of the differential of* $\tilde{\Pi} : \mathcal{T}_g \to \mathbb{H}_g$ *at nonhyperelliptic Riemann surfaces in* $\mathcal{T}_g$ *is equal to* $3g-3$*; and when* $g \geq 3$*, the rank of the differential of* $\tilde{\Pi}$ *at hyperelliptic Riemann surfaces in* $\mathcal{T}_g$ *is equal to* $2g-1 < 3g-3$*. Consequently,* $\tilde{\Pi}$ *is an immersion at nonhyperelliptic Riemann surfaces, but is not an immersion at hyperelliptic Riemann surfaces.*

This follows from a result of Rauch [63, Theorem 3, p. 14].

Remark 8.3. We note that results similar to [63, [Theorem 3, p. 14] were proved in [10, p. 223] along the similar lines: "when C is a smooth nonhyperelliptic curve of genus g and $\varphi : \mathcal{C} \to (B, b_0)$ is a standard Kuranishi family for C, then the differential of the period map

$$dZ : T_{b_0}(B) \to T_{Z(b_0)}(\mathbb{H}_g)$$

is injective.... The local Torelli theorem fails at hyperelliptic curves of genus $g > 2$;... However, the local Torelli theorem fails in those directions which are tansverse to the hyperelliptic locus, not in those directions which are tangent to it."

The above result makes it non-obvious if the injective period map $\Pi : \mathcal{M}_g \to \mathcal{A}_g$ is an embedding. This problem was called the *Local Torelli Problem* in [61, p. 157], and was finally proved in [61, p. 176, Corollary

[14] Since it is not satisfactory to understand the moduli space $\mathcal{M}_g$ *only as a set* and any additional structure on it should *reflect its nature as a moduli space*, it is reasonable to expect that the additional structure on $\mathcal{M}_g$ satisfy the conditions in the definition of a *coarse moduli space*. In this sense, when we say that $\mathcal{M}_g$ has a complex analytic structure, it should satisfy the two conditions in Proposition 5.1. Similar conditions are imposed when we say that $\mathcal{M}_g$ admits the structure of an algebraic variety.

3.2] (see also the comments on [61, p. 197, §5]). We only state the special case for the moduli space $\mathcal{M}_g$ for algebraic curves over $\mathbb{C}$.

Proposition 8.4. *The period map* $\mathcal{M}_g = \mathrm{Mod}_g\backslash\mathcal{T}_g \to \mathcal{A}_g$ *is an holomorphic embedding, and* $\mathcal{M}_g$ *is biholomorphic to its image* $\Pi(\mathcal{M}_g)$.

Remark 8.5. The results in [61] are much more general. For any field k, let $\mathcal{M}_{g/k}$ be the moduli space of smooth projective curves over k. Then by the results of [58], $\mathcal{M}_{g/k}$ admits the structure of a scheme [60, p. 299]. Similarly, the moduli scheme for principally polarized abelian varieties over k exists and is denoted by $\mathcal{A}_{g/k}$. There is a Torelli morphism $j : \mathcal{M}_{g/k} \to \mathcal{A}_{g/k}$, which is equal to the period map when $k = \mathbb{C}$. When the characteristic of k is equal to 0, j is always an isomorphism from $\mathcal{M}_{g/k}$ to its image. When the characteristic of k is positive, it is not always true. See [61, Corollaries (2.8), (3.2) and (5.3)] for precise statements.

The issue whether $\Pi : \mathcal{M}_g \to A_g$ is an embedding is related to the question whether $\Pi(\mathcal{M}_g)$ is a normal subvariety, which was raised by Baily, as quoted above. Around the same time, in a letter from Grothendieck to Mumford dated May 10, 1961 [55, p. 639], Grothendieck wrote:

> In the same direction, there is the question whether the natural morphism from $\mathcal{M}_{g,1}$ into the corresponding modular space for polarized abelian varieties is really an embedding; a priori one can say only that $\mathcal{M}_{g,1}$ is the normalisation of a (nonclosed) subschema of the latter, which may not be normal.

We note that if $\Pi(\mathcal{M}_g)$ is a normal subvariety of $\mathcal{A}_g$, then $\Pi : \mathcal{M}_g \to \Pi(\mathcal{M}_g) \subset \mathcal{A}_g$ is an isomorphism. This is true for the following two reasons: (1) $\mathcal{M}_g = \mathrm{Mod}_g\backslash\mathcal{T}_g$ is a normal complex space, since $\mathcal{T}_g$ is a complex manifold and Mod_g is a discrete group acting holomorphically and properly on $\mathcal{T}_g$, and hence $\mathcal{M}_g = \mathrm{Mod}\backslash\mathcal{T}_g$ is a normal complex analytic space, (2) bijective holomorphic map between two normal spaces is an isomorphism.

As pointed out on [61, pp. 197–198], Proposition 8.4 implies the following result:

Corollary 8.6. *The subvariety* $\Pi(\mathcal{M}_g)$ *is a normal subvariety of* $\mathcal{A}_g$.

This settles the question raised by Baily and the question of Grothendieck above over the base field $\mathbb{C}$. (Note that in Grothendieck's question, $\mathcal{M}_g$ and $\mathcal{A}_g$ are schemes over $\mathrm{Spec}(\mathbb{Z})$.)

We also note that Proposition 8.4 together with Proposition 5.1 and the GAGA principle by Serre [67] imply the following result:

Proposition 8.7. *The space* $\mathcal{M}_g$ *with the algebraic variety structure induced from the image* $\Pi(\mathcal{M}_g)$ *in* $\mathcal{A}_g$ *is a coarse moduli space for projective algebraic curves of genus* g *over* $\mathbb{C}$.

This result justifies the above quote from Gieseker [54, p. 1] at the beginning of this section.

Remark 8.8. As mentioned before, there are different interpretations of the morphism $\Pi : \mathcal{M}_g \to \mathcal{A}_g$. Instead of being a coarse moduli scheme, $\mathcal{M}_g$ can be considered as the *moduli stack of curves over* $\mathrm{Spec}(\mathbb{Z})$ [51, p. 554] and denoted by $\mathbb{M}_g$. Similarly, $\mathcal{A}_g$ can be considered as a *moduli stack* of abelian varieties with principal polarizations, denoted by $\mathbb{A}_g$. Then according to [51, p. 555]:

> Let $g > 2$. The Torelli morphism $j : \mathbb{M}_g \to \mathbb{A}_g$ is ramified at the hyperelliptic locus. Outside the hyperelliptic locus it is an immersion. The picture is different for the Torelli morphism $j : \mathcal{M}_g \to \mathcal{A}_g$ on coarse moduli schemes. The morphism $j : \mathcal{M}_{g,\mathbb{Q}} \to \mathcal{A}_{g,\mathbb{Q}}$ on the characteristic zero fibers is an immersion; however, in positive characteristic this is not true in general.

The above phenomenon is related to the fact that when $\mathcal{M}_g = \mathrm{Mod}_g\backslash\mathcal{T}_g$ and $\mathcal{A}_g = \mathrm{Sp}(2g,\mathbb{Z})\backslash\mathbb{H}_g$ are considered as orbifolds, then the map $\Pi : \mathcal{M}_g \to \mathcal{A}_g$ is not an immersion at hyperelliptic Riemannian surfaces. (Note that $\mathcal{M}_g$ is a smooth orbifold but a singular complex analytic space or variety.) This follows from Proposition 8.2. We pass the degeneracy of the differential of $\tilde{\Pi}$ on the smooth covers $\mathcal{T}_g$ and $\mathbb{H}_n$ to the differential of Π on the orbifolds $\mathcal{M}_g = \mathrm{Mod}_g\backslash\mathcal{T}_g$ and $\mathcal{A}_g = \mathrm{Sp}(2g,\mathbb{Z})\backslash\mathbb{H}_g$.

Remark 8.9. Torelli type theorems are important in studying moduli spaces by relating moduli spaces to period spaces or spaces of variations of Hodge structures. See [33] [23] for overviews and introductions. There are three types of Torelli theorem: the usual Torelli Theorem (the global Torelli Theorem), the local Torelli Theorem, and the infinitesimal Torelli Theorem. The precise meaning of these names seems to be a bit confusing, at least to the author of this note. The local Torelli problem in [61] (and also in [10, p. 216]) is the infinitesimal Torelli problem, which is usually called "local Torelli problem" according to [61, p. 167]. On the other hand, according to [33, p. 247], "The *local Torelli problem* is the question of deciding when the Hodge structure on $H^*(V_s,\mathbb{C})$ separates points in the *local moduli space* (*Kuranishi space*) of V_s". According to [75, p. 258], the infinitesimal Torelli problem is "the question of whether the local period map is an immersion" from a local versal deformation space. This is also the definition in [22]. It seems that in this sense, the infinitesimal Torelli problem for compact Riemann surfaces Σ_g, $g \geq 2$, is concerned with the injectivity of the differential of the period map $\Pi : \mathcal{T}_g \to \mathbb{H}_g$, since $\mathcal{T}_g$ is the versal deformation space (or a Kuranishi family) for every Riemann surface Σ_g (see [10, Chapter XI, §4]), and $\mathbb{H}_g$ is the period domain. By the result of Rauch [63, Theorem 3, p. 14] mentioned above, the differential of Π is not injective at a point Σ_g in $\mathcal{T}_g$ when Σ_g is hyperelliptic (see also the quote from [10, p. 223] mentioned above). On the other hand, the result in [61] cited above meant something different and deals with the injectivity of the differential of $\Pi : \mathcal{M}_g \to A_g$ on the Zariski tangent spaces of the moduli space $\mathcal{M}_g$

Remark 8.10. As pointed out earlier, by [13], $\mathcal{A}_g$ admits a compactification $\mathcal{A}_g^*$ as a projective variety. As a topological compactification of $\mathcal{A}_g$, it was first constructed by Satake [66]. This compactification is often called the Satake compactification or Baily-Satake compactification of $\mathcal{A}_g$. The closure of $\Pi(\mathcal{M}_g)$ in $\mathcal{A}_g^*$ is a projective variety and is a compactification $\mathcal{M}_g^*$ of $\mathcal{M}_g$. It is called the *Satake compactification* of $\mathcal{M}_g$ in [37, p.45]. Maybe another name could be *Baily compactification*, or *Baily-Borel compactification* in view of the paper [14]. One special feature of this compactification is that the ideal boundary $\mathcal{M}_g^* \setminus \mathcal{M}_g$ is a subvariety of complex codimension 2, which implies that through every point in $\mathcal{M}_g$, there is one projective curve completely lying in $\mathcal{M}_g$ [37, p. 45]. Unfortunately, this compactification $\mathcal{M}_g^*$ is not modular like the Deligne-Mumford compactification [27] mentioned below.

9. Mumford's Work on $\mathcal{M}_g$ and Moduli Spaces

In the previous section (Proposition 8.7), we showed that the result of [11] in 1961 together with results of [61] in 1979 implies that $\mathcal{M}_g$ with the structure of an algebraic subvariety $\Pi(\mathcal{M}_g) \subset \mathcal{A}_g$ over $\mathbb{C}$ is a coarse moduli space for smooth projective curves of genus g over $\mathbb{C}$.

Right after [11], Mumford proved much more using the geometric invariant theory, the period map, and the Torelli Theorem in [58] in 1965. According to an email of Mumford [57], he proved that for any field k, if we consider the moduli space $\mathcal{M}_{g/k}$ of smooth projective curves of genus g over k, the moduli space $\mathcal{A}_{g/k}$ of principally polarized abelian varieties over k of dimension g over k, and the Torelli (or Jacobi) morphism $j : \mathcal{M}_{g/k} \to \mathcal{A}_{g/k}$, then the normalization of the image $j(\mathcal{M}_{g/k})$ is a *coarse moduli space* for smooth projective curves of genus g over k. In particular, if $k = \mathbb{C}$, then $\mathcal{M}_{g/k} = \mathcal{M}_g$, $j = \Pi$, and the *normalization* of $\Pi(\mathcal{M}_g) \subset \mathcal{A}_g$ is a *coarse moduli space for smooth projective curves of genus g over $\mathbb{C}$.*

In his book [58, p. 143, the proof of Theorem 7.13], Mumford proved more precise results. Specifically, Corollary 7.14 of [58] states:

> *The coarse moduli scheme over $\mathbb{Z}$ for curves of genus g, $(g \geq 2)$ exists, and is quasi-projective over every open subset* $\mathrm{Spec}(\mathbb{Z}) - (p)$ *in* $\mathrm{Spec}(\mathbb{Z})$.

Remark 9.1. Note that $\mathcal{M}_g = \mathrm{Mod}_g \backslash \mathcal{T}_g$ is a coarse moduli space by Proposition 5.1. Since coarse moduli spaces are unique up to isomorphisms and the period map $\Pi : \mathcal{M}_g \to \Pi(\mathcal{M}_g)$ is bijective, it follows that the map from the normalization $\Pi(\mathcal{M}_g)$ to $\Pi(\mathcal{M}_g)$ is bijective even if $\Pi(\mathcal{M}_g)$ were not normal. We note that it can happen that the canonical map from the normalization $\tilde{M}$ of a variety M to M is a holomorphic bijective map but is not an isomorphism. The simplest example of such a phenomenon is the rational curve with a cusp singularity at the origin given by $C = \{(z, w) \in \mathbb{C} \mid w^2 = z^3\}$. Its normalization $\tilde{C}$ is a smooth rational curve and hence is isomorphic to $\mathbb{C}$, and the map from $\tilde{C}$ to C is given by: for $t \in \mathbb{C}$, $t \to (t^3, t^2)$, which is clearly holomorphic and bijective, but is not an isomorphism.

In the same commentary [54, p. 1] as mentioned in the previous section, Gieseker wrote:

> Much of Mumford's work has been devoted to the study of the moduli spaces of curves.... With some historical hindsight, the main problems were to establish the existence of $\mathcal{M}_g$, to compactify $\mathcal{M}_g$ and to establish properties of $\mathcal{M}_g$. Mumford began by studying the existence of $\mathcal{M}_g$.... Mumford was the first to formulate a purely algebraic approach to the problem of the existence of $\mathcal{M}_g$ as a quasi-projective variety valid in all characteristics. Mumford actually used two quite different approaches to this existence problem, both using Geometric Invariant Theory (GIT).

From the book [58] and the first paper of Mumford [52], it is clear that the general setup of moduli spaces, and foundations and techniques in algebraic geometry such as schemes and Hilbert schemes developed by Grothendieck have played an important role in the work of Mumford on moduli spaces. In [58, p. VII], Mumford wrote:

> the tremendous contributions made by GROTHENDIECK to both the technique and the substance of algebraic geometry have not always been paralleled by their publication in permanently available form. In particular, for many of his results, we have only the barest outlines of proofs, as presented in the Bourbaki Seminar (reprinted in [13]). Nonetheless, since all the results which I want to use have been presented in detail in seminars at Harvard and will

be published before too long by GROTHENDIECK, there seems no harm in making full use of them.

We note that Grothendieck's fundamental works EGA (*Éléments de géométrie algébrique*), SGA (*Séminaire de Géométrie Algébrique*), and FGA (*Fondements de la Géometrie Algébrique*) were cited in [58], but the papers [35] at the Cartan seminar were not cited.

On the other hand, one can see from the letters of Grothendieck to Mumford in 1961 that Mumford developed his work on moduli spaces independently of the work of Grothendieck which he presented in the Cartan seminar. We quote some sentences from these letters. In a letter dated April 25, 1961, Grothendieck wrote [55, p. 636]:

> my construction of schemata of moduli for high levels (as defined axiomatically in my Cartan Seminar talks or in an older letter to Tate) resembles very much to yours, except that I did not observe that the suitably embedded polarized abelian varieties are completely determined by their sets of points of finite order n, (n big enough), which then leads you to a rather specific situation for passing to the quotient.
>
> It seems to me that because of your lack of some technical background on schemata, some proofs are rather awkward and unnatural, and the statements you give not as simple and strong as they should be.

On May 10, 1961 [55, p. 639], Grothendieck wrote: "It occurred to me that I had sent you only a copy of my Bourbaki talk on quotients, but none of my Cartan talks."

In an email to the author of this note [56], Mumford explained that the original motivation for his work on moduli spaces did not come directly from Riemann's moduli space, but rather from other moduli spaces in algebraic geometry such as Chow varieties, Picard varieties, which originated with the Italian geometers in the early 20th century (see [44]). When Grothendieck put things in the framework of categories and functors, he made everything clearer. Mumford's work also has a lot of antecedents in Hilbert's work on invariant theory, for example, the famous Hilbert-Mumford criterion for stability.

GIT, which was developed in [58], was used to construct Riemann's moduli space $\mathcal{M}_g$ in a very satisfactory way. As Gieseker wrote [54, p. 2]:

> If a moduli problem can be attacked with GIT, then not only can existence of the moduli space be established, but one also gets a compactification of the moduli space and an ample line bundle on the moduli space.

GIT also provides an elegant approach to construct and understand many other moduli spaces in algebraic geometry, as well as in symplectic geometry and differential geometry as the increased contents of the new editions of [58] indicate. The basic problems of [58] were explained by Mumford in the preface to the first edition of [58], which is [53]:

> The purpose of this book is to study two related problems: when does an orbit space of an algebraic scheme acted by an algebraic group exist? And to construct moduli schemes for various types of algebraic objects. The second problem appears to be, in essence, a special and highly non-trivial case of the first. From an Italian point of view, the crux of both problems is in passing from a birational to a biregular point of view. To construct both orbit spaces and moduli "generically" are simple exercises. The problem is whether, within the set of all models of the resulting birational class, there is one model whose geometric points classify the set of orbits in some action, or the set of algebraic objects in some moduli problem. In both cases, it is quite possible that some orbits, or some objects are so exceptional, or, as we shall say, are *not stable*, so that they must be left out of the model. The difficulty is to pin down the meaning of stability in a given case.

The key notion of *stability* in [58] allows one to get Hausdorff quotients with respect to actions of reductive algebraic groups. On the other hand, carrying out the actual construction is often difficult, as Catanese wrote [21, p. 163]:

> In my opinion geometric invariant theory, in spite of its beauty and its conceptional simplicity, but in view of its difficulty, is a foundational but not a fundamental tool in classification theory. Indeed one of the most difficulty results, due to Gieseker, is the asymptotic stability of pluricanonical images of surfaces of general type; it has as an important corollary the existence of a moduli space for canonical models of surfaces of general type, but the methods of proof do not shed light on the classification of such surfaces (indeed boundedness for the families of surfaces with given invariants had followed earlier by the results of Moishezon, Kodaira and Bombieri).

In a review of *Selected Papers of David Mumford*, Kollar wrote [47, pp. 112–113]:

> I believe that GIT did not fulfill all the early expectations. The first successes leading to the construction of the moduli spaces of curves, Abelian varieties, vector bundles and sheaves were not carried much further. For instance, GIT never came up with a good approach to compactify the moduli space of surfaces of general type, and it did not

tackle the moduli problem for higher dimensional varieties.

The first edition of the book [58] in 1965, i.e., the book [53], marked the official or formal birth of the moduli space $\mathcal{M}_g$, and it is a proper place for us to end our story. But it is only the beginning of an exciting and ongoing story about $\mathcal{M}_g$, which has become a major object of study in several branches of mathematics.

For example, one milestone further along the moduli road is the Deligne-Mumford compactification of $\mathcal{M}_g$ in [27], which is a compact orbifold. One purpose of this paper was to prove that $\mathcal{M}_g$ is irreducible for every characteristic through this compactification. It is interesting to quote a letter from Grothendieck to Mumford on April 25, 1961 [55, p. 638]:

> In your "appendix", you refer to a result of Matsusaka I did not hear of before, namely the connectedness or irreducibility of the variety of moduli for curves of genes g, in any characteristic. I did not know there was any algebraic proof for this (whatever way you state it). Yet I have some hope to prove the connectedness of the $\mathcal{M}_{g,n}$ (arbitrary levels) using the transcendental result in char. 0 and the connectedness theorem; but first one should get a natural "compactification" of $\mathcal{M}_{g,n}$ which should be simple over $\mathbb{Z}$.

The paper [27] basically followed the methods outlined above, and the first algebraic proof of irreducibility of $\mathcal{M}_g$ was given by Fulton [31]. See [60, pp. 295–296] for some related information. Note that Fulton had a different proof of the irreducibility of $\mathcal{M}_g$ in [30], and the paper [60] was written before [31].

Some more recent results on $\mathcal{M}_g$ are described in the introductory books [37] and [10]. Many results and theories on moduli spaces of higher dimensional varieties, bundles over varieties, maps from Riemann surfaces into manifolds and varieties, and other moduli problems in geometry and topology have been inspired by the work on Riemann's moduli space $\mathcal{M}_g$, or rather Riemann's moduli problem. A comprehensive survey of many kinds of moduli spaces is recently given in the three volumes of *Handbook of Moduli* [28]

Remark 9.2. After a preliminary version of this note was finished, we found a paper [60] dealing with similar topics. In the MathSciNet review of this paper, T. Oda wrote:

> At the occasion of the 100th anniversary symposium of F. Severi, to whose ideas we owe very much, the author gives a very nice survey of the moduli theory between 1857 (Riemann) and 1965 (Mumford) as well as some later developments.
>
> The author explains why it took us more than a century to define and to construct a mathematical object (the coarse moduli space) of which many properties were already established.

Our note and the paper [60] are complementary to each other. For example, Severi varieties and arithmetic geometry of moduli spaces are emphasized in [60], and the paper of Teichmüller [73] was not cited, though the papers [71] [72] were cited there. On the other hand, the papers of [73], [35] and related works on complex analytic geometry of $\mathcal{M}_g$ are emphasized in our note.

10. Ahlfors and Bers' Works on Complex Analytic Geometry on Moduli Spaces

As we emphasized right from the beginning, there are two aspects of Riemann's moduli spaces: complex analytic and algebraic geometric. The above discussion of the work of Teichmüller shows that he made substantial contributions to both aspects and his works had far reaching impact. Two heroes in the complex analytic and complex geometric structures of the moduli of Riemann surfaces and the Teichmüller space are Ahlfors and Bers.

Some major results of Teichmüller, in particular the existence of extremal quasiconformal maps between compact Riemann surfaces, were treated with caution by most people at first (compare with the quote from Weil [80, pp. 3–4] in §6), and Ahlfors and Bers filled in details of some results and theories developed and announced by Teichmüller. In the paper [4, pp. 3–4], Ahlfors wrote:

> In a systematic way the problem of extremal quasiconformal mapping was taken up by Teichmüller in a brilliant and unconventional paper [7]. He formulates the general problem and, although unable to give a binding proof, is led by heuristic arguments to a highly elegant conjectured solution. The paper contains numerous fundamental applications which clearly show the importance of the problem.
>
> In a later publication [8] Teichmüller has offered a proof of his main conjecture. In many respects this proof is an anticlimax when compared with the original article. It is based on the method of continuity, which of all classical methods is the least satisfactory because of its nature of a posteriori verification. It is also unduly complicated.
>
> The main purpose of the present paper is to give a variational proof of Teichmüller's theorem. It is not our contention that the new proof is simpler than Teichmüller's, especially if the latter would

be rewritten with greater conciseness. We claim for it merely the merit of greater directness. It relies heavily on real variable techniques, and can therefore hardly be classified as elementary.

Eventually, the major results of Teichmüller on quasiconformal maps were eventually understood and verified. For example, three decades after the above paper of Ahlfors, Bers wrote in [19, 58]:

> This note is a postscript to a paper [3] which I published many years ago and is, like that paper, essentially expository. The paper [3] had two aims: to show how the study of quasiconformal mappings can be based on the theory (due to Morrey [5]) of Beltrami equations with measurable coefficients, and to present the existence part of Teichmüller's proof [7] of his main theorem (announced in [6]) in an understandable and convincing form. At one point the two aims interfered. In proving a basic continuity assertion (Lemma 1 in §14 of [3]) I made use of a property of quasiconformal mappings (stated for the first time in [2] and also in §4F of [3]) which belongs to the theory of quasiconformal mappings with bounded measurable Beltrami coefficients (and seems not to have been known to Teichmüller). Some readers concluded that the use of that theory was indispensable for the proof of Teichmüller's theorem. This is not so, and Teichmüller's own argument is correct. This argument can be further simplified and this simplified argument will be presented below. Then we will briefly describe Teichmüller's actual argument.

It is probably fair to say that the works of Ahlfors and Bers were crucial for the Teichmüller theory to be developed into a major subject. Of course, Ahlfors and Bers also greatly expanded the complex analytic theory of the Teichmüller space $\mathcal{T}_g$ and hence of the moduli space $\mathcal{M}_g$. For example, Ahlfors proved that the Weil-Petersson metric is indeed Kähler [5] as Weil claimed in [76] (see §6),[15] Bers proved that Teichmüller space $\mathcal{T}_g$ can be realized as a bounded complex domain in $\mathbb{C}^{3g-3}$ [16], and together they proved the powerful Riemann's mapping theorem for variable metrics [9], or more precisely that the normalized solution of the Beltrami equation depends holomorphically on the Beltrami coefficient, which implies in particular the existence of a complex structure on $\mathcal{T}_g$ and hence also on $\mathcal{M}_g$, which is natural from many points of view (see for example [77]) but was not immediately clear that it is isomorphic to the natural complex structure required as a coarse moduli space, for example as in the proof of Proposition 5.1.

[15] Ahlfors did not cite the paper [76]. Instead, he said in the footnote on the first page of the paper [5, p. 171]: "According to an oral communication the fact has been known to Weil, but his proof has not been published."

The works of Ahlfors and Bers have had a tremendous impact on the complex analytic theory of Teichmüller space $\mathcal{T}_g$ and the moduli space $\mathcal{M}_g$. For example, they have built up a large school around them on the broad Teichmüller theory. For some summaries of their and others' works, see [6] [7] [5] [8] [15] [16] [17] [18]. See [2] and [41] for additional discussion and references.

It is perhaps appropriate to mention that the work of the first female Fields Medalist, Maryam Mirzakhani, deals with the geometry and dynamics of Riemann surfaces and of Riemann's moduli space $\mathcal{M}_g$. See the IMU News Release [50], for example, for a summary of her work.

11. Meanings of Moduli in Moduli Spaces

Now it seems that the word moduli is a well-established concept in mathematics. There are moduli spaces of all kinds of objects in mathematics. But what is the exact meaning of moduli and why did Riemann choose this word?

Riemann did not explain the choice of moduli in his published papers [65]. In his original paper in 1857 on abelian functions, he seemed to mean some special *non-redundant local* continuous complex parameters related to branching points of Riemann surfaces which determine general Riemann surfaces.

I have been wondering about the meaning of this word "moduli" for a long time. Only recently did I find a simple and reasonable explanation by Rauch. In [62, pp. 42–43], Rauch wrote:

> the totality of classes of conformal equivalent surfaces depends on $3p-3$ ($p \geq 2$) parameters, which, since the modulus of elliptic integral of first kind serves this function, he called "moduli." A brief résumé of his reasoning will show, however, the vagueness of this dependence....
>
> *one would like numerical moduli – a set of numbers associated with each surface whose equality would guarantee conformal equivalence between two surfaces.*
>
> Various attempts to construct *algebraic* invariants of the coefficients of the equations of the associated algebraic curves have been made. But, quite apart from the incompleteness of these attempts, they are in the wrong direction, because the modulus question arises in all of conformal mapping theory, for example, in the mapping of plane multiply connected domains, where such invariants clearly have no meaning.

For the convenience of the reader, we recall that there are three basic kinds of complete elliptic integrals: (1) *the first kind*,

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

(2) *the second kind*,

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\, d\theta = \int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}} dt,$$

(3) *the third kind*,

$$\Pi(n,k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1-n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} = \int_0^1 \frac{dt}{(1-nt^2)\sqrt{(1-t^2)(1-k^2t^2)}}.$$

In the above elliptic integrals, k is the *modulus*. It determines the elliptic integrals and can vary. (In the third kind, n is a constant and called the elliptic characteristic.) See [49], for example, for more detail and references about elliptic integrals.

At the end of the paper [73], Teichmüller discussed how to get holomorphic functions on neighborhoods of a point in $\mathcal{M}_g$ so that other holomorphic functions can be expressed holomorphically in terms of them. These functions seem to be related to moduli or invariants as described by Rauch above and in [63], and they behave quite differently near hyperelliptic Riemann surfaces.

To some mathematicians of recent generations, probably the most direct meaning of moduli in the phrase *moduli space* is like that of parameters in parameter spaces. Therefore, moduli are basically parameters, or complex parameters.

The modern and usual meaning of the word moduli in moduli spaces can be explained as follows. If we really want *one invariant* which characterizes or determines an equivalence class of Riemann surfaces of genus g, then *the equivalence class itself* is *the best or most intrinsic invariant*. By putting the equivalence classes of Riemann surfaces together, we get Riemann's moduli space $\mathcal{M}_g$. The same idea applies to other classification problems as well, and moduli spaces are usually defined this way now. Of course, putting desirable structures on them and understanding them is a different story!

Acknowledgments

I would like to thank P. Deligne, W. Fulton, D. Mumford, and J.-P. Serre for their helpful comments and correspondences related to the paper [2] with N. A'Campo and A. Papadopoulos, which motivated me to think more about the history of moduli spaces and to write this article, which could be considered as an introduction or supplement to [2].

I would also like to thank P. Deligne for several specific comments quoted in this note from two letters [24] [25] and an email [26] and many constructive suggestions, W. Fulton for emphasizing the importance of a proper definition of $\mathcal{M}_g$ as a coarse moduli space, J. Gray for emphasizing the importance of Poincaré work on the spaces discussed in this article, E. Looijenga for helpful explanations about singularities, D. Mumford for multiple emails and several constructive comments on a preliminary version of this article, F. Oort for very valuable information and references on the local Torelli problem and his emphasis on the dependence on base fields in discussing the moduli spaces and morphisms between them, and J.-P. Serre for constructive suggestions and an email [68] regarding Grothendieck. They clarified several things for me and their questions made me think more systematically and globally about the complicated moduli story and explain things better. We also like to thank L. Bieri for translating several key pages of [71] which allowed me to complete the picture presented in this note of why Teichmüller introduced and started to study Teichmüller space and Riemann's moduli space.

Finally I would like to thank an anonymous referee for reading carefully this paper and for making several valuable suggestions.

Writing this moduli story has been an educational and very challenging project for me. In spite of the kind help from many leading experts, all errors remaining are mine.

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